On Minimal Surfaces

Aleksei Lopatin

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Abstract

In this paper we will discuss minimal surfaces giving a few examples and interesting theorems surrounding the topic.

1 Acknowledgements

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2 Background

We assume that the reader is familiar with surface patches and the first and second fundamental forms. Recall the definition of the *GauB* map and so forth the *Weingarten map*.

Definition 2.1. Let S be a surface. The *GauB* map is the map $G : S \to S^2$ (where S^2 is the unit sphere) sending a point $p \in S$ to the unit normal vector N. If the choice of surface is ambiguous, we write G_S.

Definition 2.2. Let S be a surface, and let $p \in S$. The *Weingarten* map W_p of S at p is defined to be

$$W_p = -D_pG$$

where D_p refers to the derivative at p and G is the *GauB* map.

Let W be the *Weingarten* map of a surface S at a point $p \in S$. Recall that W is a linear map from $T_p(S)$ to itself (where $T_p(s)$ is the set of all tangent vectors to S at p). We now define the mean curvature.

Definition 2.3. The mean curvature of S at p is $H = \frac{1}{2}\text{Tr}(W)$ where Tr is the trace.

From the first and second fundamental forms $Edu^2 + 2Fdudv + Gdv^2$ and $Ldu^2 + 2Mdudv + Ndv^2$ respectively, we define symmetric 2×2 matrices F_1 and F_{11} by $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ and $\begin{bmatrix} L & M \\ M & N \end{bmatrix}$.

We offer this following proposition as a way of simplifying the $\text{Tr}(W)$ expression.

Proposition 2.4. *The Weingarten map* W *with respect to the basis* σ_u, σ_v *of* T_pS *is* $F_1^{-1}F_{11}$.

Proof. consult either simon's proof or Pressley massey ch8 ■

We now show a simpler form for H.

Lemma 2.5.

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}$$

Proof. To compute H, we need the trace of the matrix $F_1^{-1}F_{11}$. Noting the general formula for an inverted matrix, we calculate.

$$F_1^{-1}F_{11} = \frac{1}{EG - F^2}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix}\begin{bmatrix} L & M \\ M & N \end{bmatrix} = \frac{1}{EG - F^2}\begin{bmatrix} LG - MF & MG - NF \\ ME - LF & NE - MF \end{bmatrix}$$

Therefore,

$$H = \frac{1}{2}\mathrm{Tr}(F_1^{-1}F_{11}) = \frac{LG - 2MF + NE}{2(EG - F^2)}$$

Consult

■

If we have $z = f(x, y)$, for a surface $\sigma(x, y) = (x, y, f(x, y))$, we have

$$H = \frac{(1 + f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1 + f_x^2)f_{yy}}{2(1 + f_x^2 + f_y^2)^{\frac{3}{2}}}$$

Some more information on this formula can be found in [dC16, Chapter 3] We now define the minimal surface and a corollary to go along.

Definition 2.6. A minimal surface is a surface whose mean curvature is zero everywhere.

Corollary 2.7. *If a surface* S *has least area among all surfaces with the same boundary curve, then* S *is a minimal surface.*

We will also give the definition for a ruled surface as we will make use of it in another section.

Definition 2.8. A ruled surface is a surface that is a union of straight lines. We call the lines the rulings of the surface.

3 Motivation

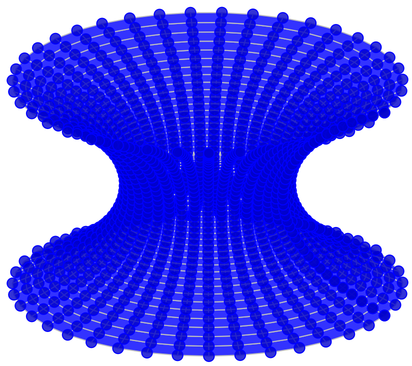
Minimal surfaces have a curious interpretation in soap films. Soap films are thin layers of liquid surrounded by air. A soap film has energy due to the surface tension on it. Those with physics knowledge will recognize that this energy is directly proportional to its area ($W = \gamma dS$). A soap film spanning a wire in the shape of a curve would adopt the shape of a surface of least area with a boundary curve. By the previous corollary, this will be a minimal surface.

4 Examples

The most common examples of a minimal surface are the catenoid and the helicoid. We check that both of these are indeed minimal surfaces.

4.1 Catenoid

A picture of the catenoid is given below.



Lemma 4.1. *A catenoid is a minimal surface.*

Proof. The parametrization for a catenoid is $\sigma(u,v) = (\cosh(u)\cos(v), \cosh(u)\sin(v), u)$. We straightforwardly calculate the fundamental forms.

$$\sigma_u = (\sinh(u)\cos(v), \sinh(u)\sin(v), 1)$$
$$\sigma_v = (-\cosh(u)\sin(v), \cosh(u)\cos(v), 0)$$
$$\sigma_u \times \sigma_v = (-\cosh(u)\cos(v), -\cosh(u)\sin(v), \sinh(u)\cosh(u))$$
$$N = \frac{\sigma_u \times s\sigma_v}{\|\sigma_u \times \sigma_v\|} = (-\operatorname{sech}(u)\cos(v), -\operatorname{sech}(u)\sin(v), \tanh(u))$$
$$\sigma_{uu} = (\cosh(u)\cos(v), \cosh(u)\sin(v), 0)$$
$$\sigma_{uv} = (-\sinh(u)\sin(v), \sinh(u)\cos(v), 0)$$
$$\sigma_{vv} = (-\cosh(u)\cos(v), -\cosh(u)\sin(v), 0)$$
$$E = \|\sigma_u\|^2 = \cosh^2(u)$$
$$F = \sigma_u \cdot \sigma_v = 0$$
$$G = \|\sigma_v\|^2 = \cosh^2(u)$$
$$L = N \cdot \sigma_{uu} = -1$$
$$M = N \cdot \sigma_{uv} = 0$$
$$N = N\sigma_{vv} = 1$$

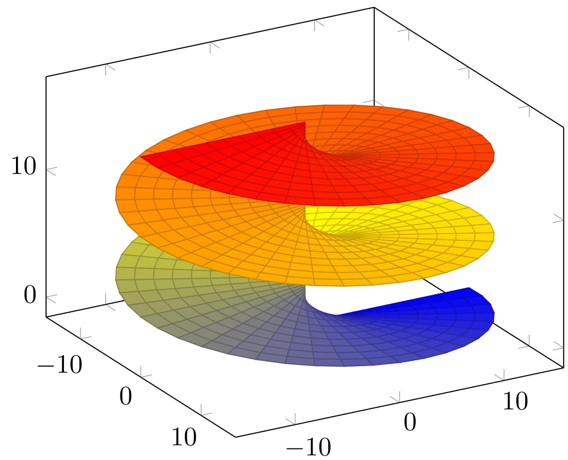
We now calculate H.

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{-\cosh^2(u) + \cosh^2(u)}{2\cosh^4(u)} = 0$$

Thus, the catenoid is a minimal surface. ■

4.2 Helicoid

A picture of the helicoid is given below.



Lemma 4.2. *The helicoid is a minimal surface.*

Proof. The parametrization of a helicoid is $\sigma(u,v) = (v\cos(u), v\sin(u), \lambda u)$. From here, we just do the calculations.

$$\sigma_u = (-v\sin(u), v\cos(u), \lambda)$$

$$\sigma_v = (\cos(u), \sin(u), 0)$$

$$\sigma_u \times \sigma_v = (-\lambda\sin(u), \lambda\cos(u), -v)$$

$$N = \frac{(\sigma_u \times \sigma_v)}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\lambda^2 + v^2}(-\lambda\sin(u), \lambda\cos(u), -v)$$

$$\sigma_{uu} = (-v\cos(u), -v\sin(u), 0)$$

$$\sigma_{uv} = (-\sin(u), \cos(u), 0)$$

$$\sigma_{vv} = (0,0,0)$$

$$E = v^2 + \lambda^2$$

$$F = 0$$

$$G = 1$$

$$L = \sigma_{uu} \cdot N = 0$$

$$M = \sigma_{uv} \cdot N = \frac{\lambda}{\sqrt{\lambda^2 + v^2}}$$

$$N = \sigma_{vv} \cdot N = 0$$

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = 0$$

■

5 Theorems with Minimal Surfaces

We will now get into some relationships between minimal surfaces and surfaces of revolution and ruled surfaces. Recall that the surfaces of revolution are constructed by taking a plane curve, called the *profile* curve, and rotate it around a line in the plane, thus forming a surface in $\mathbb{R}^3$.

Theorem 5.1. *Any minimal surface of revolution S is an open subset of a plane or a catenoid.*

Proof. By applying an isometry of $\mathbb{R}^3$, we can assume that the axis of the surface S is the z-axis and the profile curve lies in the xz-plane. We parametrize S as $\sigma(u,v) = (f(u)\cos(v), f(u)\sin(v), g(u))$ where we assume the profile curve $u \to (f(u), 0, g(u))$ is assumed to be unit-speed and $f > 0$. We omit the calculations for the first and second fundamental forms. Consult [Pre10] for more information.

$$du^2 + f(u)^2 dv^2 \text{ and } (\dot{f}\ddot{g} - \ddot{f}\dot{g})du^2 + f\dot{g}dv^2$$

where a dot denotes $\frac{d}{du}$. Applying the lemma for the simpler form of H, we find that the mean curvature is:

$$H = \frac{1}{2}(\dot{f}\ddot{g} - \ddot{f}\dot{g} + \frac{\dot{g}}{f})$$

We suppose now that, for some value of u, say $u = u_0$, we have $\dot{g}(u_0) \neq 0$. Since $\dot{g}$ is continuous (which follows from the fact that g is smooth), we have that $\dot{g}(u) \neq 0$ for u in some open interval containing u_0. Let (α, β) be the largest interval. Assuming that $u \in (\alpha, \beta)$, we differentiate the unit speed condition $\dot{f}^2 + \dot{g} = 1$ to get $\dot{f}\ddot{f} + \dot{g}\ddot{g} = 0$. Thus we have

$$(\dot{f}\ddot{g} - \ddot{f}\dot{g})\dot{g} = -\dot{f}^2\ddot{f} - \ddot{f}\dot{g}^2 = -\ddot{f}(\dot{f}^2 + \dot{g}) = -\ddot{f}$$

$$\dot{f}\ddot{g} - \ddot{f}\dot{g} = -\frac{\ddot{f}}{\dot{g}}$$

We substitute this in for our previous expression for H:

$$H = \frac{1}{2}(\frac{\dot{g}}{f} - \frac{\ddot{f}}{\dot{g}})$$

From the unit speed condition $\dot{g}^2 = 1 - \dot{f}^2$, S is minimal if and only if $H = 0$:

$$H = \frac{1}{2}\frac{\dot{g}^2 - f\ddot{f}}{f\dot{g}} \implies f\ddot{f} = 1 - \dot{f}^2$$

To solve this differential equation, we substitute $h = \dot{f}$, and notice the following chain rule application:

$$\ddot{f} = \frac{dh}{dt} = \frac{dh}{df}\frac{df}{dt} = h\frac{dh}{df}$$

Hence, we have:

$$fh\frac{dh}{df} = 1 - h^2$$

Since we assumed that $\dot{g} \neq 0$, we have $h^2 \neq 1$, so we integrate:

$$\int \frac{hdh}{1-h^2} = \int \frac{df}{f}$$

$$h = \frac{\sqrt{a^2f^2-1}}{af}$$

where a is a non-zero constant and we ommited signs. Noting that $h = \frac{df}{du}$ and integrating again:

$$\int \frac{afdf}{\sqrt{a^2f^2-1}} = \int du$$

$$f = \frac{1}{a}\sqrt{1+a^2(u+b)^2}$$

where b is a constant. By a change of parameter from $u \to u+b$, we can assume that $b = 0$. Therefore, $f = \frac{1}{a}\sqrt{1+a^2u^2}$. To compute g, we recall that $\dot{g}^2 = 1 - \dot{f}^2$.

$$\dot{g}^2 = 1 - \dot{f}^2 = 1 - h^2 = \frac{1}{a^2f^2}$$

$$\frac{dg}{du} = +\frac{1}{\sqrt{1+a^2u^2}}$$

$$g = \pm\frac{1}{a}\sinh^{-1}(au) + c(\text{ where c is a constant})$$

$$au = \pm\sinh(a(g-c))$$

$$f = \frac{1}{a}\cosh(a(g-c))$$

Therefore the profile curve of S is $x = \frac{1}{a}\cosh(a(z-c))$. The surface S is obtained by applying to the catenoid S_a a translation along the z-axis. However, we are not done. We have shown that the open subset of S corresponding to $u \in (\alpha, \beta)$ is part of the catenoid. Hence, we have excluded the possibility that S is a plane.

To finish up, assume that $\beta < \infty$. Then, if the profile curve is defined for values of $u \geq \beta$, we must have $\dot{g}(\beta) = 0$, otherwise $\dot{g}$ would be non-zero on an open interval containing β, which would contradict our assumption that (α, β) is the largest open interval containing u_0 on which $\dot{g} \neq 0$. The equations written above show that $\dot{g}^2 = \frac{1}{1+a^2u^2}$ if $u \in (\alpha, \beta)$ and so, since $\dot{g}$ is a continuous function of u, $\dot{g}(\beta) = \pm(1+a^2\beta^2)^{-\frac{1}{2}} \neq 0$. This contradiction shows that the profile curve is not defined for values of $u \geq \beta$. A similar argument applies to α, and shows that (α, β) is the entire domain of definition of the profile curve. Hence, the whole of S is an open subset of a catenoid.

The last case is if $\dot{g}(u) = 0$ for all values of u on which the profile curve is defined. Then, $g(u)$ is a constant, for example C, and then S is an open subset of the plane $z = C$. ■

We might wonder about the relationship between ruled surfaces and minimal surfaces. The following theorem sheds some light on it.

Theorem 5.2. *Any ruled minimal surface is an open subset of a plane or helicoid.*

Proof. We take the usual parametrization of a ruled surface as $\sigma(u,v) = \gamma(u) + v\delta(u)$, where γ is a curve that meets each of the rulings and $\delta(u)$ is a vector parallel to the ruling through $\gamma(u)$.

We make some assumptions. Firstly,, assume that $\|\delta(u)\| = 1$ for all values of u. Secondly, we assume that $\dot{\delta}$ is never zero, where the dot refers to $\frac{d}{du}$. We also assume that $\dot{\gamma} \cdot \dot{\delta} = 0$.

Note that $\sigma_u = \dot{\gamma} + v\dot{\delta}$ and $\sigma_v = \delta$.

$$E = \|\dot{\gamma} + v\dot{\delta}\|^2 \text{ and } F = (\dot{\gamma} + v\dot{\delta}) \cdot \delta = \dot{\gamma} \cdot \delta \text{ and } G = 1$$

For simplicty, let $A = \sqrt{EG - F^2}$. We can then calculate N:

$$N = A^{-1}(\dot{\gamma} + v\dot{\delta}) \times \delta$$

Notice that we have $\sigma_{uu} = \ddot{\gamma} + v\ddot{\delta}$, $\sigma_{uv} = \dot{\delta}$, and $\sigma_{vv} = 0$, so:

$$L = A^{-1}(\ddot{\gamma} + v\ddot{\delta}) \cdot ((\dot{\gamma} + v\dot{\delta}) \times \delta)$$

$$M = A^{-1}\dot{\delta} \cdot ((\dot{\gamma} + v\dot{\delta}) \times \delta) = A^{-1}\dot{\delta} \cdot (\dot{\gamma} \times \delta)$$

$$N = 0$$

Recall that $H = \frac{LG - 2MF + NE}{2A^2} = 0$, so we have:

$$(\ddot{\gamma} + v\ddot{\delta}) \cdot ((\dot{\gamma} + v\dot{\delta}) \times \delta) = 2(\delta \cdot \dot{\gamma})(\dot{\delta} \cdot (\dot{\gamma} \times \delta))$$

This has to hold for all values of (u, v). Equating coefficients of powers of v gives

$$\ddot{\gamma} \cdot (\dot{\gamma} \times \delta) = 2(\delta \cdot \dot{\gamma})(\dot{\delta} \cdot (\dot{\gamma} \times \delta))$$

$$\ddot{\gamma} \cdot (\dot{\delta} \times \delta) + \ddot{\delta} \cdot (\dot{\gamma} \times \delta) = 0$$

$$\ddot{\delta} \cdot (\dot{\delta} \times \delta) = 0$$

where the last line stems from an inner product identity. The last line shows that δ, $\dot{\delta}$, and $\ddot{\delta}$ are linearly dependent. Since δ and $\dot{\delta}$ are perpendicular unit vectors (by assumption), there are smooth functions $\alpha(u)$ and $\beta(u)$ such that

$$\ddot{\delta} = \alpha\delta + \beta\dot{\delta}.$$

But, since δ is unit speed, $\dot{\delta} \cdot \ddot{\delta} = 0$. Differentiating this gives $\delta \cdot \ddot{\delta} = -\dot{\delta} \cdot \dot{\delta} = -1$. Hence, $\alpha = -1$ and $\beta = 0$, so

$$\ddot{\delta} = -\delta$$

This equation shows that the curvature of the curve δ is 1, and that its principal normalis $-\delta$. Hence, the binormal is $\dot{\delta} \times (-\delta)$, and noting the following

$$\frac{d}{du}(\dot{\delta} \times \delta) = \ddot{\delta} \times \delta + \dot{\delta} \times \dot{\delta} = -\delta \times \delta = 0$$

it follows that the torsion of δ is 0. Therefore, δ parametrizes a circle of radius 1. By applying an isometry of $\mathbb{R}^3$, we can assume that δ is the circle with radius 1 and centre the origin in the $xy-$ plane, so that

$$\delta(u) = (\cos(u), \sin(u), 0)$$

From $\ddot{\delta} = -\delta$, we get $\ddot{\delta} \cdot (\dot{\gamma} \times \delta) = -\delta \cdot (\dot{\gamma} \times \delta) = 0$, so by $\ddot{\gamma} \cdot (\dot{\delta} \times \delta) + \ddot{\delta} \cdot (\dot{\gamma} \times \delta) = 0$, we have

$$\ddot{\gamma} \cdot (\dot{\delta} \times \delta) = 0$$

Therefore, $\ddot{\gamma}$ is parallel to the xy plane, and hence $\gamma(u) = (f(u), g(u), au + b)$ where f and g are smooth functions and a and b are constants. If $a = 0$, the surface is an open subset of the plane $z = b$. Otherwise, from a previous equation, we have

$$\ddot{g} \cos(u) - \ddot{f} \sin(u) = 2(\dot{f} \cos(u) + \dot{g} \sin(u))$$

Making use of the condition that $\dot{\gamma} \cdot \dot{\gamma} = 0$, we have

$$\dot{f} \sin(u) = \dot{g} \cos(u)$$

Differentiating this leaves:

$$\ddot{f} \sin(u) + \dot{f} \cos(u) = \ddot{g} \cos(u) - \dot{g} \sin(u)$$

Therefore, we have

$$\dot{f} \cos(u) + \dot{g} \sin(u) = 0$$

and using a previous equation we have $\dot{f} = \dot{g} = 0$. Moreover, f and g are constants. By a translation of the surface, we can assume that the constants f,g, and b are zero, so then $\gamma(u) = (0, 0, au)$ and hence $\sigma(u, v) = (v \cos(u), v \sin(u), au)$ which is a helicoid.

We assumed at the start that $\dot{\delta}$ is never zero. If $\dot{\delta}$ is always zero, then δ is a constant vector, and the surface is a generalized cylinder. However, a generalized cylinder is a minimal surface only if the cylinder is an open subset of a plane. Using a similar argument at the end of the previous theorem, we are done. ■

6 Open Questions

We leave the reader with several open questions on the topic of Minimal Surfaces.

- Plateau's problem involves finding a minimal surface from a fixed boundary curve. Is it possible for a single smooth boundary curve to bound infinitely many minimal surfaces.
- What is the number of solutions for a given boundary configuration. Can a single boundary bound an infinite number of minimal surfaces?
- What is the overall structure of the space of all minimal surfaces as their boundaries vary? The point of this question is to look at these surfaces globally.

The reader should consult [DHS10] to learn more about these open questions.

References

[dC16] Manfredo P. do Carmo. *Differential Geometry of Curves and Surfaces.* Dover Publications, Mineola, NY, revised and updated second edition, 2016.

[DHS10] Ulrich Dierkes, Stefan Hildebrandt, and Friedrich Sauvigny. *Minimal surfaces. With assistance and contributions by A. Küster and R. Jakob*, volume 339 of *Grundlehren Math. Wiss.* Dordrecht: Springer, 2nd revised and enlarged ed. edition, 2010.

[Pre10] Andrew Pressley. *Elementary Differential Geometry.* Springer Undergraduate Mathematics Series. Springer, 2 edition, 2010.