Higher-Dimensional Differential Geometry and Topology: A Survey Work

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Abstract

In this paper, we cover the foundations of higher-dimensional differential geometry and topology. We construct smooth manifolds, define tangent and cotangent bundles, and establish structural results like smoothness of the tangent bundle and partitions of unity. Next, we explore differential forms, prove $d^2 = 0$ to establish the exterior derivative, and arrive at the Poincaré lemma. We establish results on geodesics and completeness, prove curvature symmetries and Bianchi identities, and define the Levi-Civita connection. Lastly, we use Gauss-Bonnet and Chern-Weil theory to relate curvature to topology.

Contents

1 Smooth Manifolds

1.1 Topological preliminaries and smooth structures

Definition 1.1 (Topological manifold). *A* ***topological*** n***-manifold*** *is a Hausdorff, second countable topological space* M *such that every* $p \in M$ *has a neighborhood homeomorphic to an open subset of* $\mathbb{R}^n$.

Definition 1.2 (Charts, atlases, smooth structures). *A* ***chart*** *is a pair* (U, φ) *with* $U \subset M$ *open and* $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ *a homeomorphism. An* ***atlas*** *is a family of charts covering* M*. Two charts* (U, φ) *and* (V, ψ) *are* ***smoothly compatible*** *if the transition* $\psi \circ \varphi^{-1}$ *is* C^∞ *wherever defined. A* ***smooth structure*** *on* M *is a maximal smoothly compatible atlas. A* ***smooth manifold*** *is a pair* $(M, \mathcal{A})$ *where* $\mathcal{A}$ *is such a smooth structure.*

Proposition 1.3 (Existence and uniqueness of maximal smooth atlas). *Given any smooth atlas* $\mathcal{A}$*, there exists a unique maximal smooth atlas containing it.*

Proof. Define $\overline{\mathcal{A}}$ to be the set of all charts smoothly compatible with every chart in $\mathcal{A}$. Then $\overline{\mathcal{A}}$ is an atlas containing $\mathcal{A}$. It is maximal: if a chart is compatible with all charts in $\overline{\mathcal{A}}$, then in particular it is compatible with all charts in $\mathcal{A}$, hence belongs to $\overline{\mathcal{A}}$. For uniqueness, if $\mathcal{B}$ is another maximal atlas containing $\mathcal{A}$, then $\mathcal{B}$ must equal $\overline{\mathcal{A}}$. □

1.2 Tangent vectors: derivations and velocities

Definition 1.4 (Tangent space via derivations). *Let* $p \in M$*. A* ***derivation*** *at* p *is a linear map*

$$X : C^\infty(M) \longrightarrow \mathbb{R}$$

satisfying the Leibniz rule

$$X(fg) = f(p)X(g) + g(p)X(f), \qquad \forall f, g \in C^\infty(M).$$

The vector space of derivations at p *is called the* ***tangent space*** *and denoted* T_pM.

Definition 1.5 (Tangent bundle). *The* ***tangent bundle*** *of* M *is the disjoint union*

$$TM = \bigsqcup_{p \in M} T_pM.$$

Proposition 1.6 (Tangent vectors from curves). *Let* $\gamma : (-\varepsilon, \varepsilon) \to M$ *be a smooth curve with* $\gamma(0) = p$*. Define*

$$X_\gamma(f) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 f(\gamma(t)).$$

Then X_γ *is a derivation at* p*, hence an element of* T_pM.

Proof. Linearity follows from linearity of differentiation. For $f, g \in C^\infty(M)$,

$$X_\gamma(fg) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 f(\gamma(t))g(\gamma(t)) = f(p)\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 g(\gamma(t)) + g(p)\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 f(\gamma(t)),$$

which is the Leibniz rule. Hence X_γ is a derivation. □

Proposition 1.7 (Equivalence of definitions)**.** *Every tangent vector $X \in T_pM$ arises as X_γ for some smooth curve γ with $\gamma(0) = p$. Thus the curve-based and derivation-based definitions of tangent vectors are equivalent.*

Proof. Choose a chart (U, φ) around p with $\varphi(p) = 0$. Let $X \in T_pM$ be a derivation. Define $v^i = X(x^i)$ where x^i are coordinate functions. Define $\gamma(t) = \varphi^{-1}(tv)$. Then for any $f \in C^\infty(M)$,

$$X(f) = \sum_{i=1}^{n} v^i \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(0) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 f(\gamma(t)),$$

so $X = X_\gamma$. Hence every tangent vector arises from a curve. □

Definition 1.8 (Coordinate vector fields)**.** *In a chart (U, φ) with local coordinates $(x^1, \dots, x^n)$, define*

$$\frac{\partial}{\partial x^i}\bigg|_p (f) = \frac{\partial}{\partial x^i}(f \circ \varphi^{-1})(\varphi(p)).$$

Proposition 1.9 (Coordinate basis)**.** *The family $\left\{ \frac{\partial}{\partial x^i}\big|_p \,\middle|\, i = 1, \dots, n \right\}$ forms a basis for T_pM.*

Proof. If $\sum_{i=1}^n a^i \frac{\partial}{\partial x^i}\big|_p = 0$, apply this operator to x^j to obtain $a^j = 0$. Hence they are linearly independent. Given $X \in T_pM$, set $v^i = X(x^i)$. Then $X = \sum v^i \frac{\partial}{\partial x^i}\big|_p$. Thus they span T_pM. □

1.3 Partitions of unity

Definition 1.10 (Partition of unity)**.** *Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M. A* ***smooth partition of unity subordinate to*** *$\{U_\alpha\}$ is a family of smooth functions $\{\varphi_\alpha : M \to [0, 1]\}_{\alpha \in A}$ such that:*

1. $\operatorname{supp}(\varphi_\alpha) \subset U_\alpha$ *for each α,*
2. *the family $\{\varphi_\alpha\}$ is locally finite,*
3. $\sum_{\alpha \in A} \varphi_\alpha(x) = 1$ *for all $x \in M$.*

Theorem 1.11 (Existence of partitions of unity)**.** *Let M be a smooth manifold and $\{U_\alpha\}$ an open cover. Then there exists a smooth partition of unity subordinate to $\{U_\alpha\}$.*

Proof. Since M is paracompact and Hausdorff, the cover admits a locally finite refinement $\{V_\beta\}$ with $\overline{V_\beta} \subset U_{\alpha(\beta)}$. For each β, choose a coordinate neighborhood W_β with $\overline{W_\beta} \subset V_\beta$. In $\mathbb{R}^n$ construct a smooth bump function ψ_β supported in V_β and equal to 1 on $\overline{W_\beta}$. Pull back via the chart to M. The family $\{\psi_\beta\}$ is locally finite and nonnegative. Define

$$S(x) = \sum_\beta \psi_\beta(x),$$

which is smooth and positive everywhere. Then set $\theta_\beta = \psi_\beta / S$. Finally, for each α, define

$$\varphi_\alpha = \sum_{\alpha(\beta) = \alpha} \theta_\beta.$$

This yields the required partition of unity. □

Corollary 1.12. *Any smooth manifold admits smooth global constructions built from local data by means of partitions of unity.*

2 Tangent and Cotangent Bundles, Differential Forms, and Exterior Derivative

2.1 The tangent bundle as a smooth manifold and vector bundle

Definition 2.1 (Tangent bundle)**.** *Let M be a smooth manifold of dimension n. The* ***tangent bundle*** *is the disjoint union*

$$TM := \bigsqcup_{p\in M} T_pM,$$

together with the projection map

$$\pi : TM \to M, \qquad \pi(p, v) = p.$$

Theorem 2.2 (Smooth structure on TM)**.** *The tangent bundle TM admits a unique smooth manifold structure of dimension $2n$ such that the projection $\pi : TM \to M$ is smooth. Moreover, TM is naturally a rank-n vector bundle over M.*

Proof. Fix a chart (U, φ) on M, where $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$. For $p \in U$, let $v \in T_pM$ with coordinate representation $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\Big|_p$. Define a local trivialization map

$$\Psi_U : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n, \qquad \Psi_U(p, v) = \Big(\varphi(p), (v^1, \dots, v^n)\Big).$$

Given $(x, w) \in \varphi(U) \times \mathbb{R}^n$, set $p = \varphi^{-1}(x)$ and define $v = \sum_i w^i \frac{\partial}{\partial x^i}\Big|_p$. Then $\Psi_U(p, v) = (x, w)$. Hence Ψ_U is bijective. Let (U, φ), (V, ψ) be two charts with overlap. Consider

$$\Psi_V \circ \Psi_U^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n.$$

For (x, v) with $x \in \varphi(U \cap V)$, this transition is

$$(x, v) \mapsto \Big(\psi \circ \varphi^{-1}(x), J(x)\, v\Big),$$

where $J(x)$ is the Jacobian matrix of $\psi \circ \varphi^{-1}$ at x. Since $\psi \circ \varphi^{-1}$ is smooth, $J(x)$ depends smoothly on x. Thus the transition map is smooth. The atlas $\{\Psi_U\}$ endows TM with the structure of a $2n$-dimensional smooth manifold. In these coordinates $\pi(x, v) = x$, which is smooth with surjective differential. Hence π is a smooth submersion. On each fiber $\pi^{-1}(p) = T_pM \cong \mathbb{R}^n$, define addition and scalar multiplication coordinatewise. On overlaps, the transition functions act by multiplication with $J(x) \in GL_n(\mathbb{R})$. Thus the vector space structure is preserved across trivializations. Therefore TM is a rank-n smooth vector bundle. □

Corollary 2.3. *Local trivializations of TM are given by $\pi^{-1}(U) \cong U \times \mathbb{R}^n$. Transition maps are smooth and linear on fibers.*

2.2 The cotangent bundle and differential forms

Definition 2.4 (Cotangent space and bundle)**.** *For $p \in M$, the **cotangent space** is the dual space*

$$T_p^*M := \mathrm{Hom}(T_pM, \mathbb{R}).$$

*The **cotangent bundle** is*

$$T^*M := \bigsqcup_{p \in M} T_p^*M.$$

This is a rank-n vector bundle over M, dual to TM.

Definition 2.5 (Differential k-forms)**.** *For $k \geq 0$, define the bundle of alternating covariant k-tensors*

$$\Lambda^k T^*M.$$

*Its smooth sections are called **differential k-forms**:*

$$\Omega^k(M) := \Gamma(\Lambda^k T^*M).$$

In particular, $\Omega^0(M) = C^\infty(M)$.

Proposition 2.6 (Exterior algebra)**.** *For $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$, there exists a wedge product*

$$\alpha \wedge \beta \in \Omega^{k+\ell}(M)$$

that is bilinear, associative (up to signs), and graded-commutative:

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha.$$

Proof. At each fiber T_p^*M, the wedge product is the standard exterior product of alternating tensors. These satisfy bilinearity, associativity, and graded commutativity. Local trivializations show coefficients vary smoothly, hence the wedge product yields a smooth global form. □

2.3 Exterior derivative

Theorem 2.7 (Exterior derivative)**.** *There exists a unique sequence of $\mathbb{R}$-linear maps*

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M), \qquad k \geq 0,$$

such that:

1. *If $f \in C^\infty(M)$, then df is the usual differential.*
2. *For $\alpha \in \Omega^k(M), \beta \in \Omega^\ell(M)$,*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

3. *$d^2 = 0$.*

Proof. On a chart $(U, x^1, \dots, x^n)$, any k-form is written

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Define

$$d\omega = \sum_{i_1 < \cdots < i_k} \sum_{j=1}^{n} \frac{\partial \omega_{i_1 \cdots i_k}}{\partial x^j}(x)\, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

(1) For $k = 0$, $\omega = f$, this recovers the usual differential.
(2) For wedge products, expand coefficients and check the sign rules; the Leibniz identity holds.
(3) Applying d twice yields terms with second partial derivatives $\partial_j \partial_k \omega_I$ multiplying $dx^j \wedge dx^k$, which vanish by symmetry of partials and antisymmetry of wedge. Thus $d^2 = 0$.

Under a change of coordinates $\tilde{x} = \tilde{x}(x)$, the local expression transforms tensorially. Transition Jacobians cancel precisely because the formula is alternating in indices. Hence the definition is globally consistent.By partitions of unity, this operator extends to all of M. Any operator satisfying (1)–(3) agrees with this local definition, hence is unique. □

2.4 Interior product, Lie derivative, and Cartan's formula

Definition 2.8 (Interior product)**.** *Let $X \in \Gamma(TM)$ be a smooth vector field. The **interior product** (or contraction) $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is defined by*

$$(\iota_X \omega)(X_1, \dots, X_{k-1}) := \omega(X, X_1, \dots, X_{k-1}),$$

for $\omega \in \Omega^k(M)$ and $X_1, \dots, X_{k-1} \in \Gamma(TM)$.

Proposition 2.9. *For $X \in \Gamma(TM)$, the interior product ι_X is $\mathbb{R}$-linear and satisfies the graded Leibniz rule:*

$$\iota_X(\alpha \wedge \beta) = (\iota_X \alpha) \wedge \beta + (-1)^k \alpha \wedge (\iota_X \beta),$$

for $\alpha \in \Omega^k(M), \beta \in \Omega^\ell(M)$.

Proof. This follows by direct evaluation on vector fields, expanding both sides and comparing signs from antisymmetry. □

Definition 2.10 (Lie derivative of forms)**.** *The **Lie derivative** of a form $\omega \in \Omega^k(M)$ along a vector field $X \in \Gamma(TM)$ is defined by*

$$\mathcal{L}_X \omega := \frac{d}{dt}\Big|_{t=0} (F_t)^* \omega,$$

where F_t is the flow of X.

Proposition 2.11 (Properties of $\mathcal{L}_X$)**.** *The operator $\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$ satisfies:*

1. *$\mathcal{L}_X$ is $\mathbb{R}$-linear.*
2. *$\mathcal{L}_X$ commutes with d: $\mathcal{L}_X d = d\mathcal{L}_X$.*

3. $\mathcal{L}_X$ *is a graded derivation of degree* 0*:*

$$\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X\alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X\beta).$$

Theorem 2.12 (Cartan's magic formula)**.** *For any vector field* X *and form* ω*,*

$$\mathcal{L}_X\omega = d(\iota_X\omega) + \iota_X(d\omega).$$

Proof. It suffices to check on functions (0-forms) and 1-forms, and then extend by the derivation property.

(1) For $f \in C^\infty(M)$, $\mathcal{L}_X f = X(f)$, while $d(\iota_X f) = 0$ and $\iota_X(df) = df(X) = X(f)$.

(2) For $\alpha \in \Omega^1(M)$, $\mathcal{L}_X\alpha = \frac{d}{dt}|_0(F_t^*\alpha)$. Expanding, one obtains

$$\mathcal{L}_X\alpha(Y) = X(\alpha(Y)) - \alpha([X,Y]).$$

On the other hand,

$$\Big(d(\iota_X\alpha) + \iota_X(d\alpha)\Big)(Y) = Y(\alpha(X)) - d\alpha(X,Y).$$

Using the coordinate expression for $d\alpha$, one checks the two are equal.

(3) By derivation properties, equality extends to all k-forms. □

2.5 Poincaré lemma

Theorem 2.13 (Poincaré lemma)**.** *Let* $U \subset \mathbb{R}^n$ *be open and star-shaped with respect to the origin. If* $\omega \in \Omega^k(U)$ *is closed (*$d\omega = 0$*) and* $k \geq 1$*, then* ω *is exact: there exists* $\eta \in \Omega^{k-1}(U)$ *with* $d\eta = \omega$*.*

Proof. Define the radial vector field $X = \sum_{i=1}^n x^i \frac{\partial}{\partial x^i}$, and the homothety $H_t : U \to U$, $H_t(x) = tx$. Define the operator

$$(K\omega)_x = \int_0^1 t^{k-1}\, \iota_X\Big((H_t)^*\omega\Big)_x \,\mathrm{d}t.$$

Then $K : \Omega^k(U) \to \Omega^{k-1}(U)$ is linear.

We compute:

$$(dK + Kd)(\omega) = \omega - (H_0)^*\omega.$$

Indeed, differentiating $(H_t)^*\omega$ with respect to t and using Cartan's formula $\mathcal{L}_X = d\iota_X + \iota_X d$ yields

$$\frac{d}{dt}(H_t^*\omega) = \frac{1}{t} H_t^*(\mathcal{L}_X\omega).$$

Integrating from $t = 0$ to $t = 1$ and rearranging gives the stated identity.

Now if $d\omega = 0$, then

$$\omega = d(K\omega) + K(d\omega) = d(K\omega).$$

Moreover, $(H_0)^*\omega = 0$ for $k \geq 1$ because pullback by a constant map annihilates positive-degree forms.

Thus $\eta := K\omega$ satisfies $d\eta = \omega$. □

3 Riemannian Geometry

3.1 Riemannian metrics and the Levi-Civita connection

Definition 3.1 (Riemannian metric)**.** *A **Riemannian metric** on a smooth manifold M is a smooth section $g \in \Gamma(\mathrm{Sym}^2 T^*M)$ such that g_p is an inner product on T_pM for each $p \in M$. We write (M, g) for a Riemannian manifold.*

Definition 3.2 (Connection on TM)**.** *A **connection** on TM is an $\mathbb{R}$-bilinear map*

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM), \qquad (X, Y) \mapsto \nabla_X Y,$$

such that for $f \in C^\infty(M)$ and $X, Y \in \Gamma(TM)$:

1. $\nabla_{fX} Y = f\, \nabla_X Y$,
2. $\nabla_X(fY) = X(f)\, Y + f\, \nabla_X Y$.

*The **torsion** is $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$.*

Definition 3.3 (Metric-compatibility)**.** *A connection ∇ is **metric-compatible** with g if*

$$X\Big(g(Y, Z)\Big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for all $X, Y, Z \in \Gamma(TM)$. Equivalently, $\nabla g = 0$.

Theorem 3.4 (Fundamental theorem of Riemannian geometry)**.** *For every Riemannian manifold (M, g) there exists a unique connection ∇ on TM that is torsion-free and metric-compatible. It is characterized by the Koszul formula: for all $X, Y, Z \in \Gamma(TM)$,*

$$\begin{aligned} 2\, g(\nabla_X Y, Z) = {}& X\, g(Y, Z) + Y\, g(Z, X) - Z\, g(X, Y) \\ & - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]). \end{aligned} \tag{3.1}$$

Proof. **Existence.** Define ∇ pointwise by (3.1): for fixed X, Y, the right-hand side is C^∞-linear in Z and g-nondegeneracy determines a unique $\nabla_X Y$. Check the connection axioms by bilinearity and tensoriality from (3.1). Metric-compatibility follows by setting $Z = Y$ and polarizing; torsion-freeness follows by antisymmetrizing (3.1) in (X, Y), which yields $2\, g(\nabla_X Y - \nabla_Y X - [X, Y], Z) = 0$ for all Z.

Uniqueness. If $\widetilde{\nabla}$ is another torsion-free, metric-compatible connection, subtract the two Koszul identities to obtain $g\Big((\nabla - \widetilde{\nabla})_X Y, Z\Big) = 0$ for all X, Y, Z, hence $\nabla = \widetilde{\nabla}$. □

3.2 Local expressions and Christoffel symbols

Let $(x^1, \dots, x^n)$ be local coordinates. Write $\partial_i := \frac{\partial}{\partial x^i}$, $g_{ij} := g(\partial_i, \partial_j)$, and (g^{ij}) the inverse matrix.

Definition 3.5 (Christoffel symbols)**.** *The Levi-Civita connection is locally determined by*

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\,\partial_k,$$

where

$$\Gamma^k_{ij} = \tfrac{1}{2}\,g^{k\ell}\Big(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\Big).$$

The symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$ *holds by torsion-freeness.*

Lemma 3.6 (Covariant derivatives of tensors)**.** *If* $Y = Y^k\partial_k$*, then*

$$(\nabla_{\partial_i}Y)^k = \partial_i Y^k + \Gamma^k_{i\ell}Y^\ell.$$

If $\alpha = \alpha_j\,dx^j$*, then*

$$(\nabla_{\partial_i}\alpha)_j = \partial_i\alpha_j - \Gamma^\ell_{ij}\,\alpha_\ell.$$

These extend uniquely to arbitrary tensor fields by the Leibniz rule and contraction invariance.

3.3 Geodesics, exponential map, and normal coordinates

Definition 3.7 (Geodesic)**.** *A smooth curve* $\gamma : I \to M$ *is a* ***geodesic*** *if* $\nabla_{\dot\gamma}\dot\gamma = 0$*. In coordinates,*

$$\ddot\gamma^k + \Gamma^k_{ij}(\gamma)\,\dot\gamma^i\dot\gamma^j = 0.$$

Theorem 3.8 (Local existence, uniqueness, smooth dependence)**.** *For each* $(p, v) \in TM$ *there exists* $\varepsilon > 0$ *and a unique geodesic* $\gamma : (-\varepsilon, \varepsilon) \to M$ *with* $\gamma(0) = p$, $\dot\gamma(0) = v$*. The solution depends smoothly on* (p, v)*.*

Proof. The geodesic equation is a smooth first-order system on TM after rewriting as $\dot x = y$, $\dot y = -\Gamma(x)(y, y)$. Apply Picard-Lindelöf and smooth dependence on parameters. □

Definition 3.9 (Exponential map)**.** *For* $p \in M$*, define* $\exp_p$ *on a neighborhood of* $0 \in T_pM$ *by* $\exp_p(v) = \gamma_v(1)$*, where* γ_v *is the geodesic with* $\gamma_v(0) = p$, $\dot\gamma_v(0) = v$*. Then* $d(\exp_p)_0 = \mathrm{id}_{T_pM}$*, hence* $\exp_p$ *is a local diffeomorphism near* 0*.*

Proposition 3.10 (Normal coordinates and Gauss lemma)**.** *There exists a coordinate chart* $(U; x^1, \dots, x^n)$ *centered at* p *such that*

1. $x = \exp_p^{-1}$ *on* U *(identify* $T_pM \simeq \mathbb{R}^n$ *via an orthonormal basis),*
2. $g_{ij}(p) = \delta_{ij}$ *and* $\partial_k g_{ij}(p) = 0$*,*
3. $\Gamma^k_{ij}(p) = 0$*,*
4. $g(\partial_r, \partial_\theta) = 0$ *in polar normal coordinates (radial orthogonality).*

Proof. Items (1)-(3) follow from $\exp_p$ and Taylor expansion; (4) is Gauss lemma, obtained by differentiating $g(\dot\gamma_v, \dot\gamma_w)$ along radial geodesics and using $\nabla g = 0$. □

3.4 Completeness and Hopf-Rinow

Definition 3.11 (Metric completeness)**.** *Let d_g be the distance induced by g. A Riemannian manifold (M, g) is **complete** if (M, d_g) is a complete metric space.*

Theorem 3.12 (Hopf-Rinow)**.** *For a connected Riemannian manifold (M, g) the following are equivalent:*

1. *(M, d_g) is complete.*
2. *Every geodesic can be extended to a domain $\mathbb{R}$ (geodesic completeness).*
3. *Closed and bounded subsets of (M, d_g) are compact.*

Moreover, for any $p, q \in M$ there exists a minimizing geodesic joining p to q.

Proof. (1) ⇒ (2): If a maximal geodesic $\gamma : [0, b) \to M$ has finite b, then $\{\gamma(t)\}$ is Cauchy as $t \uparrow b$ (bounded speed in normal coordinates), hence converges to $x \in M$ by completeness. Solve the geodesic IVP at $(x, \lim \dot\gamma)$ to extend γ, contradiction.

(2) ⇒ (1): If (M, d_g) is not complete, there is a Cauchy sequence without limit. By local compactness, extract a limit curve of piecewise geodesics with finite length accumulating in finite time; extend geodesically to obtain a limit point, contradiction.

(1) ⇔ (3): In a proper length space completeness implies Heine-Borel and conversely (use Arzelà-Ascoli for minimizing sequences and Hopf-Rinow's original argument on closed balls).

The minimizing geodesic follows by existence of length-minimizing curves between points in proper length spaces and regularity of length minimizers, which satisfy the geodesic equation. □

3.5 Curvature: definitions, symmetries, and contractions

Definition 3.13 (Curvature tensor)**.** *For a connection ∇ on TM, the **curvature** is the $(1, 3)$-tensor*

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The $(0, 4)$-tensor is $R(W, X, Y, Z) := g(R(Y, Z)X, W)$.

Proposition 3.14 (Curvature symmetries)**.** *For all $W, X, Y, Z \in \Gamma(TM)$,*

1. *$R(W, X, Y, Z) = -R(X, W, Y, Z)$,*
2. *$R(W, X, Y, Z) = -R(W, X, Z, Y)$,*
3. *$R(W, X, Y, Z) = R(Y, Z, W, X)$,*
4. *$R(W, X, Y, Z) + R(W, Y, Z, X) + R(W, Z, X, Y) = 0$ (first Bianchi identity).*

Proof. (1) and (2) follow from skew-symmetry of R in its first two and last two slots, using torsion-freeness. (3) is obtained by swapping pairs and using metric-compatibility. (4) follows from the Jacobi identity and bilinearity of ∇. □

Proposition 3.15 (Second Bianchi identity)**.** *The covariant derivative of curvature satisfies*

$$(\nabla_W R)(X,Y) + (\nabla_X R)(Y,W) + (\nabla_Y R)(W,X) = 0.$$

Proof. Compute ∇R in local coordinates, antisymmetrize in (W,X,Y), and use $\nabla g = 0$ and torsion-freeness to cancel terms. □

Definition 3.16 (Ricci and scalar curvature)**.** *The* ***Ricci tensor*** *is the trace* $\mathrm{Ric}(X,Y) := \mathrm{tr}\Big(Z \mapsto R(Z,X)Y\Big)$*. The* ***scalar curvature*** *is* $S := \mathrm{tr}_g(\mathrm{Ric}) = g^{ij}\mathrm{Ric}_{ij}$*.*

Definition 3.17 (Sectional curvature)**.** *For a* 2*-plane* $\sigma \subset T_pM$ *spanned by* u, v*, the* ***sectional curvature*** *is*

$$K(\sigma) := \frac{R(u,v,v,u)}{g(u,u)g(v,v) - g(u,v)^2}.$$

This is independent of the choice of basis $\{u,v\}$ *of* σ*.*

Lemma 3.18 (Coordinate expressions)**.** *In local coordinates,*

$$R^m{}_{ijk} = \partial_i \Gamma^m_{jk} - \partial_j \Gamma^m_{ik} + \Gamma^m_{i\ell}\Gamma^\ell{}_{jk} - \Gamma^m_{j\ell}\Gamma^\ell{}_{ik},$$

and $R_{ijkl} = g_{mi}R^m{}_{jkl}$*. In normal coordinates at* p*,* $\Gamma^k_{ij}(p) = 0$ *and*

$$R_{ijkl}(p) = \tfrac{1}{2}\Big(\partial_{ik}g_{jl} + \partial_{jl}g_{ik} - \partial_{il}g_{jk} - \partial_{jk}g_{il}\Big)\Big|_p.$$

Proof. The first formula is standard from the definition of R. The second uses normal coordinates, where first derivatives of g vanish at p, yielding the stated symmetrized second-derivative identity. □

Proposition 3.19 (Metric expansion in normal coordinates)**.** *In normal coordinates centered at* p*,*

$$g_{ij}(x) = \delta_{ij} - \tfrac{1}{3}\, R_{ikj\ell}(p)\, x^k x^\ell + O(|x|^3).$$

Proof. Differentiate the geodesic equation and use $\Gamma^k_{ij}(p) = 0$, $\partial_m \Gamma^k_{ij}(p)$ expressed via $R_{ikj\ell}(p)$; integrate the resulting Taylor expansions for g_{ij}. □

4 The Gauss-Bonnet Theorem in Dimension 2

4.1 Orthogonal frames and connection 1-forms

Let (Σ^2, g) be an oriented Riemannian surface. Choose an oriented orthonormal frame of vector fields (e_1, e_2) on an open set $U \subset \Sigma$. Let (ω^1, ω^2) be the dual coframe: $\omega^i(e_j) = \delta^i_j$.

Definition 4.1 (Connection form)**.** *There exists a unique* 1*-form* ω^1_2 *on* U *such that*

$$\nabla e_1 = \omega^1_2\, e_2, \qquad \nabla e_2 = -\omega^1_2\, e_1.$$

The form ω^1_2 *is called the* ***connection*** 1***-form****.*

Lemma 4.2 (Structure equations in dimension 2)**.** *With notation as above,*

$$d\omega^1 = -\omega^1{}_2 \wedge \omega^2,$$
$$d\omega^2 = \omega^1{}_2 \wedge \omega^1.$$

Moreover, there exists a smooth function K such that

$$d\omega^1{}_2 = K\,\omega^1 \wedge \omega^2,$$

where K is the Gaussian curvature.

Proof. The first two equalities are Cartan's first structure equations specialized to dimension 2. The third is Cartan's second structure equation: curvature 2-form $\Omega^1{}_2 = d\omega^1{}_2$, which must be a multiple of $\omega^1 \wedge \omega^2$ since this spans $\Lambda^2 T^*\Sigma$. The coefficient is K by definition of Gaussian curvature. □

4.2 Statement and proof of Gauss-Bonnet

Theorem 4.3 (Gauss-Bonnet, compact surfaces)**.** *Let (Σ, g) be a compact oriented Riemannian surface without boundary. Then*

$$\int_\Sigma K\,dA = 2\pi\,\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

Proof. On Σ, cover by finitely many domains admitting orthonormal frames. On each domain, $d\omega^1{}_2 = K\,\omega^1 \wedge \omega^2$. Thus globally,

$$\int_\Sigma K\,dA = \int_\Sigma d\omega^1{}_2.$$

Triangulate Σ by geodesic triangles. By Stokes' theorem,

$$\int_\Sigma d\omega^1{}_2 = \sum_{\text{faces } F} \int_{\partial F} \omega^1{}_2.$$

Each geodesic edge contributes no geodesic curvature, and integrals reduce to turning angles at vertices. At each vertex v, the contribution is $2\pi - \sum(\text{angles at } v)$. Summing over all vertices gives

$$\sum_v (2\pi - \text{angle sum at } v) = 2\pi(V - E + F) = 2\pi\chi(\Sigma),$$

by the standard angle-defect count for a geodesic triangulation, which yields $\sum_v (2\pi - \text{angle sum at } v) = 2\pi\,\chi(\Sigma)$. Hence $\int_\Sigma K\,dA = 2\pi\chi(\Sigma)$. □

4.3 Extension to surfaces with boundary

Theorem 4.4 (Gauss-Bonnet with boundary)**.** *Let (Σ, g) be a compact oriented Riemannian surface with smooth boundary $\partial\Sigma$. Then*

$$\int_\Sigma K\, dA + \int_{\partial\Sigma} k_g\, ds = 2\pi\chi(\Sigma),$$

where k_g is the geodesic curvature of $\partial\Sigma$ with respect to the inward normal.

Proof. Repeat the proof of Theorem 4.3, but now each boundary edge remains in the final sum. For a boundary component γ, the integral $\int_\gamma \omega^1_2$ equals $\int_\gamma k_g\, ds$. Summing gives the extra term $\int_{\partial\Sigma} k_g\, ds$. □

Corollary 4.5 (Sphere and torus)**.** *On S^2 with the round metric, $K = 1$ and $\mathrm{Area}(S^2) = 4\pi$, so $\int_{S^2} K\, dA = 4\pi = 2\pi\chi(S^2)$ with $\chi(S^2) = 2$. On T^2, $\chi(T^2) = 0$, hence $\int_{T^2} K\, dA = 0$ for any Riemannian metric.*

5 Characteristic Classes and the Chern-Gauss-Bonnet Theorem

5.1 Connections on vector bundles

Definition 5.1 (Smooth vector bundle)**.** *A* ***smooth real (resp. complex) vector bundle*** *of rank r over M is a smooth manifold E with a smooth surjection $\pi : E \to M$, such that for every $p \in M$ there exists an open $U \ni p$ and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ (resp. $\mathbb{C}^r$) commuting with the projections and linear on fibers. A choice of such (U, Φ) is a* ***local trivialization****.*

Definition 5.2 (Connection on a bundle)**.** *Let $E \to M$ be a smooth real or complex vector bundle of rank r. A* ***connection*** *on E is an $\mathbb{R}$-linear map*

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s, \qquad f \in C^\infty(M),\ s \in \Gamma(E).$$

Lemma 5.3 (Local expression)**.** *In a local trivialization $\{s_1, \ldots, s_r\}$ of E over $U \subset M$, each section $s = \sum f^i s_i$ has*

$$\nabla s = \sum_i df^i \otimes s_i + \sum_{i,j} f^i A^j_{\ i} \otimes s_j,$$

where $(A^j_{\ i})$ is a matrix of 1-forms on U. Equivalently, $\nabla s = (df + Af)$ in matrix notation, where $f = (f^i)$.

Proof. Apply ∇ to each basis section s_i. Define $A^j_{\ i}$ by $\nabla s_i = \sum_j A^j_{\ i} \otimes s_j$. Then extend to general s by linearity and Leibniz. □

Definition 5.4 (Connection matrix)**.** *In a chosen local frame, the matrix $A = (A^j_{\ i})$ of 1-forms is called the* ***connection matrix****.*

5.2 Curvature of a connection

Definition 5.5 (Curvature). *The* ***curvature*** *of a connection* ∇ *is the bundle endomorphism*

$$F_\nabla := \nabla^2 : \Gamma(E) \to \Gamma(\Lambda^2 T^*M \otimes E).$$

Lemma 5.6 (Curvature in a local frame). *In a local trivialization with connection matrix* A,

$$F_\nabla = dA + A \wedge A,$$

where the wedge includes matrix multiplication: $(A \wedge A)^j{}_i = \sum_k A^j{}_k \wedge A^k{}_i$.

Proof. Compute $\nabla^2 s$ for $s = \sum f^i s_i$, using $\nabla s = df + Af$. A direct expansion gives

$$\nabla^2 s = (dA + A \wedge A)f,$$

hence $F_\nabla = dA + A \wedge A$. □

Proposition 5.7 (Gauge transformation). *If the local frame changes by* $g : U \to GL_r(\mathbb{R})$, *then*

$$A' = g^{-1}Ag + g^{-1}dg, \qquad F' = g^{-1}Fg.$$

Thus F_∇ *transforms by conjugation under change of frame.*

Theorem 5.8 (Bianchi identity). *The curvature satisfies*

$$\nabla F_\nabla = 0.$$

In a local frame, this is

$$dF_\nabla + A \wedge F_\nabla - F_\nabla \wedge A = 0.$$

Proof. Compute $\nabla^3 = 0$ and expand in terms of A and F. Alternatively, observe that $F_\nabla = \nabla^2$ and $[\nabla, \nabla^2] = 0$ as graded derivations. □

5.3 Invariant polynomials and the Chern-Weil homomorphism

Definition 5.9 (Invariant polynomial). *Let* $r \geq 1$. *A homogeneous polynomial* P *of degree* k *on* $\mathfrak{gl}_r(\mathbb{C})$ *is called* ***invariant*** *if*

$$P(gAg^{-1}) = P(A), \qquad \forall g \in GL_r(\mathbb{C}),\ A \in \mathfrak{gl}_r(\mathbb{C}).$$

Equivalently, P *is constant on conjugacy classes.*

Example 5.10.

1. *The trace* $\mathrm{tr}(A)$ *is invariant.*
2. *The determinant* $\det(A)$ *is invariant.*
3. *More generally, the elementary symmetric polynomials in eigenvalues are invariant.*
4. *For degree* k, $P(A) = \mathrm{tr}(A^k)$ *is invariant.*

Definition 5.11 (Chern-Weil form)**.** *Let $E \to M$ be a complex vector bundle with connection ∇ and curvature F_∇. For an invariant polynomial P of degree k on $\mathfrak{gl}_r(\mathbb{C})$, define*

$$\omega_P := P\Big(\frac{i}{2\pi} F_\nabla\Big) \in \Omega^{2k}(M).$$

Theorem 5.12 (Chern-Weil)**.** *For any invariant polynomial P:*

1. *ω_P is closed: $d\omega_P = 0$,*
2. *the cohomology class $[\omega_P] \in H^{2k}_{\mathrm{dR}}(M)$ is independent of the choice of connection ∇.*

Proof. **Step 1 (Closedness).** By the Bianchi identity, $\nabla F_\nabla = 0$. In a local frame, F transforms by conjugation. Since P is invariant, we have

$$d\,P(F_\nabla) = P(dF_\nabla + [A, F_\nabla]) = 0.$$

Hence $d\omega_P = 0$.

Step 2 (Independence of ∇). Let ∇^0, ∇^1 be two connections with curvatures F^0, F^1. Define the affine family $\nabla^t = (1-t)\nabla^0 + t\nabla^1$, $t \in [0,1]$, with curvature F^t. Differentiate:

$$\frac{d}{dt} P(F^t) = d\,\eta_t,$$

for some explicitly defined **transgression form** η_t depending polynomially on $A^1 - A^0$ and F^t. Concretely, in a local frame, let $A^t = (1-t)A^0 + tA^1$, then

$$\frac{d}{dt} F^t = d(A^1 - A^0) + [A^t, A^1 - A^0].$$

By multilinearity of P, this gives

$$\frac{d}{dt} P(F^t) = k\,P\Big(\tfrac{d}{dt}F^t, (F^t)^{k-1}\Big).$$

This expression is exact: $\frac{d}{dt}P(F^t) = d\,\eta_t$. Integrating from 0 to 1:

$$P(F^1) - P(F^0) = d\Big(\int_0^1 \eta_t\,dt\Big).$$

Hence $P(F^1)$ and $P(F^0)$ differ by an exact form. Therefore the de Rham class $[P(F_\nabla)]$ is independent of ∇. □

Definition 5.13 (Chern-Weil homomorphism)**.** *Assign to each invariant polynomial P the cohomology class*

$$w_P(E) := \Big[P\Big(\frac{i}{2\pi} F_\nabla\Big)\Big] \in H^{2k}_{\mathrm{dR}}(M).$$

This is the ***Chern-Weil homomorphism****. It depends only on the isomorphism class of E.*

Corollary 5.14. *Characteristic classes arise as images of invariant polynomials under the Chern-Weil homomorphism.*

5.4 Chern classes and characteristic forms

Definition 5.15 (Total Chern class)**.** *Let $E \to M$ be a complex vector bundle of rank r with connection ∇ and curvature F_∇. The **total Chern class** is defined by*

$$c(E) := \det\Big(I + \frac{i}{2\pi} F_\nabla\Big) \in \Omega^{\text{even}}(M).$$

Expanding,

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E),$$

*where $c_k(E) \in \Omega^{2k}(M)$ is a closed $2k$-form whose de Rham class is independent of ∇. The cohomology class $[c_k(E)] \in H^{2k}(M;\mathbb{R})$ is called the **k-th Chern class** of E.*

Proposition 5.16 (First Chern class for line bundles)**.** *If $L \to M$ is a complex line bundle with connection ∇ and curvature F_∇, then*

$$c_1(L) = \Big[\frac{i}{2\pi} F_\nabla\Big] \in H^2(M;\mathbb{R}).$$

Proof. Here F_∇ is a 2-form valued in $\mathfrak{gl}_1(\mathbb{C}) = \mathbb{C}$. Thus

$$c(L) = 1 + \frac{i}{2\pi} F_\nabla,$$

so $c_1(L) = [\frac{i}{2\pi} F_\nabla]$. □

Example 5.17 (Hopf bundle on S^2)**.** *Let $L \to S^2$ be the tautological line bundle over $\mathbb{CP}^1 \simeq S^2$. With the Fubini–Study metric, the curvature is $F_\nabla = -i\,\omega_{FS}$, where ω_{FS} is the Kähler form normalized so that $\int_{S^2} \omega_{FS} = 2\pi$. Hence*

$$\int_{S^2} c_1(L) = \frac{i}{2\pi} \int_{S^2} F_\nabla = 1.$$

Thus $c_1(L)$ generates $H^2(S^2;\mathbb{Z})$.

Proposition 5.18 (Whitney sum formula)**.** *If E, F are complex vector bundles over M, then*

$$c(E \oplus F) = c(E) \wedge c(F).$$

Proof. In a block-diagonal trivialization, the curvature is $F_{E\oplus F} = \begin{pmatrix} F_E & 0 \\ 0 & F_F \end{pmatrix}$. Hence

$$c(E \oplus F) = \det\Big(I + \tfrac{i}{2\pi} F_{E\oplus F}\Big) = \det\Big(I + \tfrac{i}{2\pi} F_E\Big)\det\Big(I + \tfrac{i}{2\pi} F_F\Big).$$

□

Example 5.19 (Chern classes of $\mathbb{CP}^n$)**.** *Let $h \in H^2(\mathbb{CP}^n;\mathbb{Z})$ be the hyperplane class. The Euler sequence gives*

$$c(T\mathbb{CP}^n) = (1+h)^{n+1}.$$

Thus

$$c_k(T\mathbb{CP}^n) = \binom{n+1}{k} h^k.$$

In particular, $c_n(T\mathbb{CP}^n) = (n+1)h^n$, whose integral equals $n+1$, the Euler characteristic.

5.5 Euler class and the Pfaffian

Definition 5.20 (Euler class of an oriented real bundle)**.** *Let $E \to M$ be an oriented real vector bundle of rank $2n$ with connection ∇. Its* ***Euler class*** *is*

$$e(E) := \left[\mathrm{Pf}\Big(\frac{1}{2\pi}F_\nabla\Big)\right] \in H^{2n}(M;\mathbb{R}),$$

where Pf *denotes the Pfaffian.*

Definition 5.21 (Pfaffian)**.** *For a skew-symmetric $2n \times 2n$ matrix $A = (a_{ij})$, the* ***Pfaffian*** *is*

$$\mathrm{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{k=1}^{n} a_{\sigma(2k-1),\sigma(2k)}.$$

It satisfies $\mathrm{Pf}(A)^2 = \det(A)$.

Example 5.22 (Rank 2 case)**.** *If E has rank 2, then $F_\nabla = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$ for some 2-form α. Then*

$$\mathrm{Pf}(F_\nabla) = \alpha, \qquad e(E) = \left[\frac{1}{2\pi}\alpha\right].$$

Example 5.23 (Tangent bundle of S^2)**.** *For the round sphere S^2, $e(TS^2) = [\frac{1}{2\pi}K\,dA]$, where $K = 1$. Thus $\int_{S^2} e(TS^2) = \frac{1}{2\pi}\int_{S^2} dA = 2$, which equals $\chi(S^2)$.*

Proposition 5.24 (Properties of Euler class)**.**

1. *$e(E \oplus F) = e(E) \wedge e(F)$ when* $\mathrm{rank}(E), \mathrm{rank}(F)$ *are even.*
2. *If E admits a nowhere-vanishing section, then $e(E) = 0$.*
3. *For $E = TM$, $\int_M e(TM) = \chi(M)$ when* $\dim M$ *is even (proved later).*

5.6 The Chern-Gauss-Bonnet theorem

We now establish the precise relationship between the Euler class of the tangent bundle and the Euler characteristic of a compact, oriented even-dimensional manifold.

Theorem 5.25 (Chern-Gauss-Bonnet)**.** *Let M be a compact, oriented smooth manifold of dimension $2n$. Then*

$$\int_M e(TM) = \chi(M),$$

where $e(TM) \in H^{2n}(M;\mathbb{R})$ is the Euler class of the tangent bundle and $\chi(M)$ is the Euler characteristic.

Sketch of proof. **Step 1. Local expression for the Euler form.** Choose a Riemannian metric g on M with Levi-Civita connection ∇. Let F_∇ be its curvature 2-form, viewed as a skew-symmetric matrix of 2-forms in $\mathfrak{so}(2n)$. Define the Euler form by

$$\mathrm{Eul}(g) := \mathrm{Pf}\Big(\frac{1}{2\pi}F_\nabla\Big) \in \Omega^{2n}(M).$$

By Chern-Weil theory, $[\mathrm{Eul}(g)] = e(TM)$ in cohomology, and $\mathrm{Eul}(g)$ is closed.

Step 2. Independence of g. If g_t is a smooth family of metrics with Levi-Civita connections ∇^t, then the corresponding Euler forms differ by an exact form. Thus their integrals over M coincide. Hence $\int_M \mathrm{Eul}(g)$ depends only on the topology of M.

Step 3. Evaluation via Morse theory. Choose g to be generic so that a Morse function $f : M \to \mathbb{R}$ has non-degenerate critical points. Locally, the Euler form integrates to ± 1 in a neighborhood of each critical point, with the sign given by the Morse index parity. Summing over all critical points, one finds

$$\int_M \mathrm{Eul}(g) = \sum_{p \in \mathrm{Crit}(f)} (-1)^{\mathrm{ind}(p)} = \chi(M).$$

Step 4. Conclusion. Therefore

$$\int_M e(TM) = \int_M \mathrm{Eul}(g) = \chi(M).$$

□

Example 5.26 (Dimension 2)**.** *If* $\dim M = 2$*, then* $F_\nabla = \begin{pmatrix} 0 & K\,dA \\ -K\,dA & 0 \end{pmatrix}$*, where* K *is the Gaussian curvature and* dA *the area form. Thus*

$$\mathrm{Eul}(g) = \frac{1}{2\pi} K\,dA.$$

Hence the Chern-Gauss-Bonnet theorem reduces to the classical Gauss-Bonnet theorem:

$$\frac{1}{2\pi} \int_M K\,dA = \chi(M).$$

6 Differential Topology

We now turn to global results in smooth topology, emphasizing the role of transversality and critical point theory.

6.1 The Regular Value Theorem

Definition 6.1 (Regular and critical values)**.** *Let* $f : M \to N$ *be a smooth map between smooth manifolds. A point* $p \in M$ *is called a* ***critical point*** *of* f *if the differential*

$$df_p : T_pM \to T_{f(p)}N$$

fails to be surjective. Otherwise, p *is called a* ***regular point****. A point* $q \in N$ *is a* ***regular value*** *if all* $p \in f^{-1}(q)$ *are regular points.*

Theorem 6.2 (Regular Value Theorem)**.** *Let* $f : M^m \to N^n$ *be a smooth map between smooth manifolds, and let* $q \in N$ *be a regular value. Then the preimage*

$$f^{-1}(q) \subset M$$

is a smooth submanifold of M *of codimension* n *(dimension* $m - n$*).*

Proof. Let $p \in f^{-1}(q)$. Since q is regular, $df_p : T_pM \to T_qN$ is surjective. By the constant rank theorem, there exist coordinate charts

$$(x_1, \dots, x_m) \text{ near } p \quad \text{and} \quad (y_1, \dots, y_n) \text{ near } q$$

such that in these coordinates,

$$f(x_1, \dots, x_m) = (x_1, \dots, x_n).$$

Then

$$f^{-1}(q) = \{(0, \dots, 0, x_{n+1}, \dots, x_m)\},$$

which is locally a smooth submanifold of dimension $m - n$. Since this description is invariant under coordinate changes, the submanifold structure is global. □

Example 6.3 (Level sets in $\mathbb{R}^3$). *Let $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = x^2 + y^2 + z^2$. The differential $df_{(x,y,z)} = (2x, 2y, 2z)$. At any point except $(0, 0, 0)$, this is surjective, so every nonzero $r^2 \in \mathbb{R}$ is a regular value. Thus*

$$f^{-1}(r^2) = \{x^2 + y^2 + z^2 = r^2\} \cong S^2$$

is a smooth 2-dimensional submanifold of $\mathbb{R}^3$.

Example 6.4 (Rank-deficient value). *Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 + y^2$. Here $df_{(x,y)} = (2x, 2y)$. At $(0, 0)$ this vanishes, so 0 is a **critical value**. The preimage $f^{-1}(0) = \{(0, 0)\}$ is not a 1-dimensional submanifold, but a point.*

6.2 Sard's Theorem

Definition 6.5 (Critical values). *Let $f : M^m \to N^n$ be smooth. The set of **critical points** is*

$$\mathrm{Crit}(f) = \{p \in M \mid df_p \text{ is not surjective}\}.$$

*The set of **critical values** is*

$$f(\mathrm{Crit}(f)) \subset N.$$

Theorem 6.6 (Sard's Theorem). *Let $f : M^m \to N^n$ be a smooth map. Then the set of critical values of f has measure zero in N. Equivalently, almost every point $q \in N$ is a regular value.*

Sketch of proof. **Step 1. Reduction to Euclidean case.** Using charts, it suffices to prove the result for $f : \mathbb{R}^m \to \mathbb{R}^n$.

Step 2. Taylor expansion. Fix $p \in \mathbb{R}^m$. If df_p has rank $< n$, then in local coordinates,

$$f(x) = f(p) + P_k(x - p) + o(|x - p|^k),$$

where P_k is the degree-k Taylor polynomial for some k depending on f.

Step 3. Volume estimates. By estimating the measure of images of small cubes under f using higher-order terms, one shows that the Hausdorff dimension of the set of critical values is at most $n - 1$.

Step 4. Conclusion. Therefore, the set of critical values has Lebesgue measure zero. □

Remark 6.7. *The proof requires care: smoothness ensures sufficiently high differentiability to control the remainder term in the Taylor expansion. In fact, the theorem is valid for C^k maps whenever $k > \max(m-n, 0)$.*

Example 6.8 (Quadratic map $\mathbb{R}^2 \to \mathbb{R}$)**.** *Let $f(x,y) = x^2 + y^2$. Critical points occur where $\nabla f = (0,0)$, i.e. only at $(0,0)$. Thus the set of critical values is $\{0\}$, which has measure zero in $\mathbb{R}$. Hence almost every value $r^2 > 0$ is regular.*

Example 6.9 (Projection map)**.** *Let $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x,y,z) = (x,y)$. Then df has rank 2 everywhere, so there are no critical points. Thus every value in $\mathbb{R}^2$ is regular. This is consistent with Sard's theorem since the critical set is empty.*

6.3 Transversality

Definition 6.10 (Transversality to a submanifold)**.** *Let $f : M \to N$ be smooth and let $S \subset N$ be an embedded submanifold. We say that f **is transverse to** S **at** $p \in M$ (written $f \pitchfork S$ at p) if either $f(p) \notin S$, or else*

$$df_p(T_pM) + T_{f(p)}S \;=\; T_{f(p)}N.$$

*We say f **is transverse to** S (written $f \pitchfork S$) if this holds at every $p \in M$.*

Lemma 6.11 (Equivalent characterization)**.** *Suppose $f(p) \in S$. Then $f \pitchfork S$ at p iff the composite*

$$T_pM \xrightarrow{df_p} T_{f(p)}N \twoheadrightarrow T_{f(p)}N/T_{f(p)}S$$

is surjective.

Proof. By definition, $df_p(T_pM) + T_{f(p)}S = T_{f(p)}N$. Modding out by $T_{f(p)}S$ gives the stated surjectivity. ☐

Theorem 6.12 (Preimage theorem under transversality)**.** *Let $f : M^m \to N^n$ be smooth, $S^k \subset N$ an embedded submanifold, and assume $f \pitchfork S$. Then $f^{-1}(S)$ is an embedded submanifold of M of codimension $n-k$ (hence of dimension $m-n+k$). Moreover, for all $p \in f^{-1}(S)$,*

$$T_p\Big(f^{-1}(S)\Big) \;=\; \{\, v \in T_pM \mid df_p(v) \in T_{f(p)}S \,\}.$$

Proof. Fix $p \in f^{-1}(S)$. Choose charts $\phi : U \to \mathbb{R}^n$ on N with $\phi(f(p)) = 0$ and such that

$$\phi(S \cap U) = \{(y', y'') \in \mathbb{R}^k \times \mathbb{R}^{n-k} \mid y'' = 0\}.$$

Choose a chart $\psi : V \to \mathbb{R}^m$ on M with $\psi(p) = 0$ and $f(V) \subset U$. Write $\tilde{f} := \phi \circ f \circ \psi^{-1} : \psi(V) \subset \mathbb{R}^m \to \mathbb{R}^k \times \mathbb{R}^{n-k}$. Decompose $\tilde{f} = (g, h)$ with $g : \mathbb{R}^m \to \mathbb{R}^k$, $h : \mathbb{R}^m \to \mathbb{R}^{n-k}$. Then $f^{-1}(S) \cap V$ corresponds to $h^{-1}(0)$.

The transversality condition at p is equivalent to $dh_0 : \mathbb{R}^m \to \mathbb{R}^{n-k}$ being surjective (previous lemma). By the (Euclidean) regular value theorem, $h^{-1}(0)$ is a submanifold of $\mathbb{R}^m$ of codimension $n-k$. The tangent space description follows from differentiating $h = 0$ at 0. Finally, pass back through charts. ☐

Proposition 6.13 (Stability and functoriality)**.** *Let $f : M \to N$, $g : N \to P$, and $S \subset P$ be submanifolds.*

1. *If $f \pitchfork S$, then transversality holds on a neighborhood of each p (openness in the C^1 topology).*

2. *If $f \pitchfork S$ and g is a submersion near $f(p)$, then $g \circ f \pitchfork S$ at p.*

3. *If $f \pitchfork S$ and $T \subset S$ is a submanifold, then (for p with $f(p) \in T$) $f \pitchfork T$ iff*

$$df_p(T_pM) + T_{f(p)}T = T_{f(p)}N.$$

4. *If $f_i : M_i \to N$ are transverse to S ($i = 1, 2$), then $f_1 \times f_2 : M_1 \times M_2 \to N \times N$ is transverse to $S \times S$.*

Proof. (1) follows from upper semicontinuity of rank of dh in the proof of Theorem 6.12. (2) If $dg_{f(p)}$ is surjective, then $d(g \circ f)_p$ composed with the quotient by $T_{g(f(p))}S$ is surjective. (3) is immediate from the definition. (4) is a direct computation on differentials. □

Definition 6.14 (Transversal intersection of submanifolds)**.** *Let A^a, B^b be embedded submanifolds of M^m. They* ***intersect transversely at*** *$p \in A \cap B$ if*

$$T_pA + T_pB = T_pM.$$

We write $A \pitchfork B$ if this holds at every point of $A \cap B$.

Theorem 6.15 (Intersection submanifold)**.** *If $A, B \subset M$ are embedded submanifolds with $A \pitchfork B$, then $A \cap B$ is an embedded submanifold of M of dimension $a + b - m$ (the* ***expected*** *dimension). Moreover,*

$$T_p(A \cap B) = T_pA \cap T_pB.$$

Proof. Let $\iota_A : A \hookrightarrow M$ be the inclusion. Consider $f := \iota_A : A \to M$ and $S := B$. Then $f \pitchfork S$ is exactly $T_pA + T_pB = T_pM$. Apply Theorem 6.12 to f to obtain that $f^{-1}(B) = A \cap B$ is a submanifold of A of codimension $m - b$, hence of dimension $a - (m - b) = a + b - m$. The tangent space formula is immediate. □

Definition 6.16 (Orientation conventions)**.** *Assume M is oriented. If $A, B \subset M$ are oriented, transversal submanifolds with $\dim A + \dim B = \dim M$, then $A \cap B$ is a finite set. Each point $p \in A \cap B$ is assigned a sign ± 1 as follows: choose oriented bases $(\mathcal{B}_A, \mathcal{B}_B)$ for T_pA and T_pB. The concatenation $(\mathcal{B}_A, \mathcal{B}_B)$ projects to a basis of T_pM. The sign is $+1$ if this agrees with the orientation of M, and -1 otherwise.*

Definition 6.17 (Intersection number)**.** *If $A, B \subset M$ are compact oriented transversal submanifolds with complementary dimensions, define*

$$A \cdot B := \sum_{p \in A \cap B} \operatorname{sign}(p) \in \mathbb{Z}.$$

Proposition 6.18 (Invariance of intersection number)**.** *If A_t, B_t are smooth families of compact oriented submanifolds in M of complementary dimensions such that $A_t \pitchfork B_t$ for all $t \in [0,1]$, then $A_t \cdot B_t$ is independent of t. In particular, the intersection number is invariant under small perturbations and under homology.*

Proof. Consider the cobordism $W_A = \bigcup_{t\in[0,1]} A_t \times \{t\} \subset M \times [0,1]$ and similarly W_B. Transversality implies $W_A \pitchfork W_B$, and $W_A \cap W_B$ is a compact 1-manifold with boundary $(A_0 \cap B_0) \sqcup (A_1 \cap B_1)$ with appropriate signs. Thus the signed count of boundary points vanishes, yielding $A_0 \cdot B_0 = A_1 \cdot B_1$. □

Theorem 6.19 (Thom transversality theorem (non-parametric))**.** *Let M, N be smooth manifolds, $S \subset N$ an embedded submanifold. Then the set*

$$\mathcal{T} := \{\, f \in C^\infty(M,N) \mid f \pitchfork S \,\}$$

is residual (countable intersection of open dense sets) in the C^∞ Whitney topology. If M is compact, $\mathcal{T}$ is open and dense in each C^r topology ($1 \le r \le \infty$).

Proof sketch. Use Sard's theorem on the evaluation map $\mathrm{ev} : C^\infty(M,N)\times M \to N$, $\mathrm{ev}(f,p) = f(p)$, and consider the jet extension $j^1 f$ and the submanifold of 1-jets transverse to S. Parametric transversality plus Baire category yields residuality; compactness of M gives openness. □

Theorem 6.20 (Parametric transversality)**.** *Let $F : P \times M \to N$ be smooth, $S \subset N$ a submanifold, and suppose $F \pitchfork S$. Then for a residual subset of parameters $p \in P$, the partial map $F_p := F(p,\cdot) : M \to N$ is transverse to S.*

Proof. Apply Sard's theorem to the projection of the transverse set $F^{-1}(S) \subset P \times M$ to P, using that transversality implies that projection is a submersion on a dense set of parameters. □

Corollary 6.21 (Generic regular values and intersections)**.**

1. *For a smooth $f : M \to N$, a residual set of $q \in N$ are regular values; thus $f^{-1}(q)$ is a submanifold of the expected codimension.*
2. *If $A, B \subset M$ are submanifolds, then after an arbitrarily small perturbation (or by generic ambient isotopy) they can be made transverse; if $\dim A + \dim B = \dim M$, the intersection is a finite set counted with signs.*

Example 6.22 (Graphs intersecting a horizontal plane)**.** *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be smooth, and consider the graph $\Gamma = \{(x, y, u(x, y))\} \subset \mathbb{R}^3$ and the plane $z = c$. For generic c, the intersection $\Gamma \cap \{z = c\}$ is a smooth 1-dimensional submanifold (a collection of curves) because the height map has c as a regular value.*

Example 6.23 (Curves in the plane)**.** *Two embedded C^∞ curves in $\mathbb{R}^2$ can be perturbed slightly to intersect transversely; intersections are isolated and counted with signs $+1$ or -1 according to orientation.*

Remark 6.24 (Manifolds with boundary)**.** *If M has boundary and $S \subset N$ is a submanifold, the statement of Theorem 6.12 holds with $f^{-1}(S)$ a manifold with boundary when $f(\partial M) \pitchfork S$ and the interior transversality conditions hold.*

6.4 Degree Theory

Definition 6.25 (Orientation of manifolds)**.** *A smooth n-manifold M is **oriented** if its tangent bundle admits a continuous choice of orientation of each T_pM. Equivalently, the structure group of TM reduces to $\mathrm{GL}^+(n,\mathbb{R})$. If M is oriented, so is each tangent space T_pM, consistently.*

Definition 6.26 (Degree of a smooth map)**.** *Let M^n, N^n be compact, connected, oriented n-manifolds without boundary, and let $f : M \to N$ be smooth. Choose a regular value $q \in N$ of f (guaranteed by Sard). Then $f^{-1}(q)$ is a finite set, and we define*

$$\deg(f) := \sum_{p\in f^{-1}(q)} \mathrm{sign}\Big(\det(df_p)\Big),$$

where $\mathrm{sign}(\det(df_p)) = +1$ if $df_p : T_pM \to T_qN$ preserves orientation, and -1 otherwise.

Proposition 6.27 (Independence of choice of regular value)**.** *The integer $\deg(f)$ does not depend on the choice of regular value q.*

Proof. Let $q_0, q_1 \in N$ be two regular values. By Sard's theorem, almost every q is regular, so we can connect q_0 and q_1 by a path $\gamma : [0,1] \to N$ intersecting only regular values. Consider the set $Z = \{(p,t) \in M \times [0,1] \mid f(p) = \gamma(t)\}$. Transversality ensures Z is a compact 1-manifold with boundary $f^{-1}(q_0) \times \{0\} \sqcup f^{-1}(q_1) \times \{1\}$. Each boundary point inherits a sign from orientation. Since the oriented boundary of a compact 1-manifold has total signed count zero, the two sums agree. □

Theorem 6.28 (Homotopy invariance of degree)**.** *If $f, g : M \to N$ are homotopic smooth maps between compact oriented n-manifolds, then*

$$\deg(f) = \deg(g).$$

Proof. Let $H : M \times [0,1] \to N$ be a homotopy between f and g. For a regular value q of H (again by Sard's theorem), the preimage $H^{-1}(q)$ is a compact oriented 1-manifold with boundary

$$f^{-1}(q) \times \{0\} \sqcup g^{-1}(q) \times \{1\}.$$

The boundary orientation argument shows the sums of signs agree, hence $\deg(f) = \deg(g)$. □

Example 6.29 (Maps between spheres)**.** *For $f : S^n \to S^n$, the degree $\deg(f)$ is an integer classifying $[S^n, S^n] \cong \mathbb{Z}$.*

1. *The identity map id_{S^n} has degree $+1$.*
2. *The antipodal map $x \mapsto -x$ has degree $(-1)^{n+1}$.*
3. *Any reflection $S^n \to S^n$ reversing orientation has degree -1.*

Example 6.30 (Covering maps)**.** *Let $\pi : S^1 \to S^1$ be $\pi(z) = z^k$ for $z \in \mathbb{C}$, $|z| = 1$. Then $\deg(\pi) = k$ because each point has exactly k preimages, all orientation-preserving.*

Theorem 6.31 (Properties of degree)**.** *For smooth maps between compact oriented n-manifolds:*

1. ***Normalization:*** $\deg(\mathrm{id}_M) = 1$.
2. ***Multiplicativity:*** $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.
3. ***Homotopy invariance:*** *If* $f \simeq g$*, then* $\deg(f) = \deg(g)$.
4. ***Surjectivity criterion:*** *If* $\deg(f) \neq 0$*, then* f *is surjective.*

Sketch. (1) is immediate. (2) follows from the chain rule and counting preimages. (3) was proved in Theorem 6.28. (4) If $\deg(f) \neq 0$, then every regular value has a nonempty preimage, hence f must hit every point of N. □

Remark 6.32 (Degree and homology)**.** *The degree is characterized homologically as follows: for* $f : M^n \to N^n$ *between compact oriented manifolds,*

$$f_*[M] = \deg(f)\,[N] \in H_n(N; \mathbb{Z}),$$

where $[M]$ *and* $[N]$ *are fundamental classes. This gives a topological definition independent of smoothness, extending the theory to continuous maps.*

6.5 The Poincaré-Hopf Theorem

Definition 6.33 (Index of a zero of a vector field)**.** *Let* X *be a smooth vector field on an oriented* n*-manifold* M*, and let* $p \in M$ *be an isolated zero of* X*. Choose a small neighborhood* U *of* p *diffeomorphic to* $B^n \subset \mathbb{R}^n$*. Then* $X|_{\partial U}$ *is a map* $\partial U \cong S^{n-1} \to \mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$*. The* ***index*** *of* X *at* p *is defined as*

$$\mathrm{ind}_p(X) := \deg\Big(\tfrac{X}{\|X\|} : \partial U \to S^{n-1}\Big).$$

Remark 6.34. *The index is an integer independent of the choice of neighborhood* U*. It measures how the vector field "winds" around the zero* p*.*

Definition 6.35 (Index sum)**.** *If* X *has isolated zeros* $\{p_1, \ldots, p_k\}$*, the* ***total index*** *of* X *is*

$$\mathrm{ind}(X) = \sum_{i=1}^{k} \mathrm{ind}_{p_i}(X).$$

Theorem 6.36 (Poincaré-Hopf)**.** *Let* M *be a compact, oriented* n*-manifold, and let* X *be a smooth vector field on* M *with isolated zeros. Then*

$$\sum_{p \in \mathrm{Zero}(X)} \mathrm{ind}_p(X) = \chi(M).$$

Sketch of proof. **Step 1. Local description.** Near each zero, the index is defined via the degree of a normalized map on a small sphere.

Step 2. Global construction. Patch together neighborhoods of zeros to cover M. Outside of these, X never vanishes and defines a nowhere-vanishing vector field.

Step 3. Euler class relation. The Euler class $e(TM)$ of the tangent bundle is represented by a Thom form whose pullback via X yields a differential form supported near the zeros of X. Integrating gives

$$\int_M e(TM) = \sum_p \operatorname{ind}_p(X).$$

Step 4. Chern-Gauss-Bonnet. By Theorem 5.25, $\int_M e(TM) = \chi(M)$. Thus the index sum equals $\chi(M)$. □

Example 6.37 (Sphere S^2)**.** *On S^2, every smooth vector field must vanish somewhere. Take the height function $h(x, y, z) = z$; its gradient vector field has exactly two zeros (north and south poles), each of index $+1$. Thus* $\operatorname{ind}(X) = 2 = \chi(S^2)$*.*

Example 6.38 (Torus T^2)**.** *On T^2, consider the constant vector field induced from $\mathbb{R}^2$. This has no zeros, hence* $\operatorname{ind}(X) = 0$*. Since $\chi(T^2) = 0$, the theorem holds.*

Remark 6.39 (Existence of nowhere-vanishing vector fields)**.** *The theorem shows that a compact oriented manifold admits a nowhere-vanishing vector field iff $\chi(M) = 0$. For example, odd-dimensional spheres S^{2n+1} admit such fields, but even-dimensional spheres S^{2n} do not.*

6.6 Morse Theory

Definition 6.40 (Morse function)**.** *Let M be a smooth manifold. A smooth function $f : M \to \mathbb{R}$ is a **Morse function** if all its critical points are non-degenerate, i.e.*

$$\det\Big(\operatorname{Hess}_p f\Big) \neq 0 \quad \textit{for all } p \in \operatorname{Crit}(f),$$

where $\operatorname{Hess}_p f$ *is the Hessian of f at p with respect to local coordinates.*

Definition 6.41 (Morse index)**.** *For a non-degenerate critical point p of f, the **Morse index** $\lambda(p)$ is the number of negative eigenvalues of* $\operatorname{Hess}_p f$*. Equivalently, $\lambda(p)$ is the dimension of the maximal subspace on which the quadratic form is negative definite.*

Theorem 6.42 (Morse lemma)**.** *Let p be a non-degenerate critical point of $f : M \to \mathbb{R}$. Then there exist local coordinates $(x_1, \ldots, x_n)$ centered at p such that*

$$f(x) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2.$$

Thus near p, f looks like a quadratic form determined by the index.

Sketch. Taylor expand f at p. Since $\nabla f(p) = 0$, the leading nontrivial term is the quadratic form $\frac{1}{2}\operatorname{Hess}_p f$. By a linear change of coordinates (Sylvester's law of inertia), this quadratic form can be diagonalized with ± 1 coefficients. Higher-order terms can be absorbed by a smooth coordinate change, yielding the normal form. □

Theorem 6.43 (Morse inequalities)**.** *Let M be a compact smooth manifold and $f : M \to \mathbb{R}$ a Morse function. Let C_k denote the number of critical points of index k, and let $b_k = \dim H_k(M;\mathbb{R})$ be the k-th Betti number. Then:*

$$\begin{aligned} C_k &\geq b_k \quad \text{for all } k, \\ \sum_{i=0}^{k}(-1)^{k-i}C_i &\geq \sum_{i=0}^{k}(-1)^{k-i}b_i \quad \text{for each } k, \\ \sum_{i=0}^{n}(-1)^{i}C_i &= \sum_{i=0}^{n}(-1)^{i}b_i = \chi(M). \end{aligned}$$

Idea of proof. The sublevel sets $M_a = \{x \in M \mid f(x) \leq a\}$ change topology only when a passes a critical value. Crossing a critical value of index λ corresponds to attaching a λ-cell. Hence the chain complex generated by critical points computes the homology of M. Comparing with singular homology yields the inequalities. □

Example 6.44 (Height function on S^2)**.** *Take $f(x, y, z) = z$ on $S^2 \subset \mathbb{R}^3$. There are two critical points: north pole (index 2) and south pole (index 0). Thus $C_0 = C_2 = 1$, $C_1 = 0$. The Betti numbers are $b_0 = b_2 = 1$, $b_1 = 0$. So $C_k = b_k$, and $\chi(S^2) = 2$ matches.*

Example 6.45 (Height function on T^2)**.** *Embed the torus T^2 in $\mathbb{R}^3$ as a surface of revolution. The height function has 4 critical points: a minimum (index 0), two saddles (index 1), and a maximum (index 2). So $(C_0, C_1, C_2) = (1, 2, 1)$. The Betti numbers are $(b_0, b_1, b_2) = (1, 2, 1)$. Again the Morse inequalities are sharp.*

Remark 6.46. *Morse theory provides a powerful method to compute the topology of manifolds by analyzing critical points of generic smooth functions. It is also the foundation of Floer theory and many modern developments in symplectic geometry and topology.*

7 Characteristic Classes

Characteristic classes provide algebraic invariants associated to vector bundles. They are cohomology classes constructed from curvature or obstruction theory, and they measure the "twisting" of bundles. Characteristic classes unify geometry and topology: locally a vector bundle is trivial, but globally it may fail to be, and characteristic classes detect this failure.

7.1 Vector bundles and structure groups

Definition 7.1 (Vector bundle)**.** *A* ***real vector bundle*** *of rank k over a smooth manifold M is a smooth manifold E together with a smooth surjection $\pi : E \to M$ such that each fiber $\pi^{-1}(p)$ is a k-dimensional vector space, and for each $p \in M$ there exists a neighborhood U of p and a diffeomorphism*

$$\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$$

that restricts fiberwise to linear isomorphisms. The maps ϕ are called local trivializations.

Example 7.2. *The tangent bundle TM is a rank n vector bundle over an n-manifold M. Similarly, the cotangent bundle T^*M and tensor bundles $\otimes^r T^*M \otimes^s TM$ are natural examples.*

Definition 7.3 (Transition functions and structure group)**.** *Given a vector bundle $E \to M$ with trivializations over $\{U_i\}$, on overlaps $U_i \cap U_j$ we obtain transition maps*

$$g_{ij} : U_i \cap U_j \to GL(k, \mathbb{R})$$

satisfying the cocycle condition $g_{ij} g_{jk} = g_{ik}$. The group $GL(k, \mathbb{R})$ is called the ***structure group*** *of the bundle.*

Remark 7.4. *Reduction of structure group (e.g. to $O(k)$ or $SO(k)$) corresponds to additional geometric structures: metrics and orientations. Characteristic classes will be defined relative to such reductions.*

7.2 Definition of characteristic classes

Characteristic classes are cohomology classes $c(E) \in H^*(M)$ assigned functorially to vector bundles $E \to M$, with the following key properties:

1. **Naturality:** For $f : N \to M$, we have $c(f^*E) = f^*c(E)$.
2. **Whitney sum formula:** For bundles E, F over M,
$$c(E \oplus F) = c(E) \smile c(F).$$
3. **Normalization:** On trivial bundles, c reduces to 1 (or a fixed known value).

Different cohomology theories give rise to different classes:

- Stiefel–Whitney classes $w_i(E) \in H^i(M; \mathbb{Z}/2)$ for real bundles.
- Chern classes $c_i(E) \in H^{2i}(M; \mathbb{Z})$ for complex bundles.
- Pontryagin classes $p_i(E) \in H^{4i}(M; \mathbb{Z})$ for real bundles.
- The Euler class $e(E) \in H^n(M; \mathbb{Z})$ for oriented rank n real bundles.

7.3 Stiefel-Whitney Classes

Stiefel-Whitney classes are the fundamental characteristic classes for real vector bundles. They live in mod 2 cohomology and capture subtle obstructions to orientability and to the existence of nonvanishing sections.

Theorem 7.5 (Existence)**.** *To every real vector bundle $E \to M$ of rank k there is associated a sequence of cohomology classes*

$$w_i(E) \in H^i(M; \mathbb{Z}/2), \quad 0 \leq i \leq k,$$

called the ***Stiefel-Whitney classes****, with $w_0(E) = 1$. They satisfy:*

1. *$w(E \oplus F) = w(E) \smile w(F)$ (Whitney sum formula),*
2. *$w(E)$ is natural under pullbacks: $w(f^*E) = f^*w(E)$,*
3. *If E is trivial, then $w(E) = 1$.*

Definition 7.6 (Total class)**.** *The* ***total Stiefel-Whitney class*** *of E is*

$$w(E) = 1 + w_1(E) + w_2(E) + \cdots + w_k(E).$$

Remark 7.7. *$w_1(E)$ measures orientability: $w_1(E) = 0$ if and only if E is orientable. In particular, $w_1(TM) = 0$ if and only if M is an orientable manifold.*

Example 7.8 (The Möbius bundle)**.** *The Möbius line bundle $L \to S^1$ is nontrivial. It has $w_1(L) \neq 0 \in H^1(S^1; \mathbb{Z}/2)$. This exactly captures the failure of orientability of L.*

Example 7.9 (Tangent bundle of real projective space)**.** *The tangent bundle of $\mathbb{R}P^n$ has total Stiefel-Whitney class*

$$w(T\mathbb{R}P^n) = (1 + a)^{n+1}, \quad a \in H^1(\mathbb{R}P^n; \mathbb{Z}/2) \text{ generator.}$$

This formula follows from the splitting principle and the short exact sequence relating the tautological line bundle γ^1 and its orthogonal complement.

Proposition 7.10 (Obstructions to nonvanishing sections)**.** *Let $E \to M$ be a rank k vector bundle. Then $w_k(E) \in H^k(M; \mathbb{Z}/2)$ is the obstruction to the existence of a nowhere-vanishing section. In particular, for the tangent bundle TM, the Euler class mod 2 coincides with $w_n(TM)$.*

Remark 7.11. *The Stiefel-Whitney classes give nontrivial information even when ordinary cohomology vanishes over $\mathbb{Z}$, since they live in $\mathbb{Z}/2$-coefficients. For example, they detect the nontriviality of vector bundles over spheres.*

7.4 Chern Classes

Chern classes are the primary characteristic classes for complex vector bundles. They live in integral cohomology and satisfy axioms analogous to those of Stiefel-Whitney classes, together with stronger functoriality properties.

Theorem 7.12 (Existence and uniqueness (axiomatic characterization))**.** *There exists a unique assignment to each complex vector bundle $E \to M$ of cohomology classes*

$$c_k(E) \in H^{2k}(M; \mathbb{Z}), \qquad 0 \le k \le \operatorname{rank}(E),$$

*called the **Chern classes** of E, such that:*

1. ***Naturality:*** *For any smooth map $f : N \to M$, $c_k(f^*E) = f^*c_k(E)$.*
2. ***Whitney sum:*** *$c(E \oplus F) = c(E) \smile c(F)$, where $c(E) = 1 + c_1(E) + \cdots$ is the total class.*
3. ***Normalization:*** *If $L \to \mathbb{CP}^1$ is the tautological line bundle $\mathcal{O}(-1)$ and $x \in H^2(\mathbb{CP}^1; \mathbb{Z})$ is the positive generator, then $c_1(L) = -x$.*

Moreover $c_k(E) = 0$ for $k > \operatorname{rank}(E)$ and $c_0(E) = 1$.

Definition 7.13 (Total Chern class and Chern roots)**.** *The **total Chern class** is $c(E) = 1 + c_1(E) + \cdots + c_r(E) \in H^{\text{even}}(M; \mathbb{Z})$ for $r = \operatorname{rank}(E)$. On a space where E splits as a direct sum of line bundles (splitting principle), write $E \cong \bigoplus_{j=1}^r L_j$ and set $x_j := c_1(L_j) \in H^2(\cdot; \mathbb{Z})$. Then*

$$c(E) = \prod_{j=1}^{r}(1 + x_j), \qquad c_k(E) = e_k(x_1, \dots, x_r),$$

*where e_k is the k-th elementary symmetric polynomial. The x_j are formal **Chern roots**.*

Theorem 7.14 (Splitting principle)**.** *For any complex bundle $E \to M$ there exists a map $\pi : \widetilde{M} \to M$ such that $\pi^* : H^*(M; \mathbb{Z}) \hookrightarrow H^*(\widetilde{M}; \mathbb{Z})$ is injective and π^*E splits as a direct sum of complex line bundles. Any identity between universal polynomials in Chern classes that holds for split bundles holds for all bundles.*

Proof idea. Construct the full flag bundle of E, whose fibers parametrize complete flags in E_p. Over this bundle the pullback of E admits a tautological filtration with line-bundle quotients. Injectivity of π^* on cohomology follows from Leray-Hirsch. □

Proposition 7.15 (Basic properties)**.** *Let E, F be complex bundles.*

1. *$c_1(E^*) = -c_1(E)$ and $c(E^*) = \prod_j (1 - x_j)$.*
2. *$c(E \otimes F) = \prod_{i,j}(1 + x_i + y_j)$ under the splitting principle.*
3. *If L is a line bundle, then $c(L) = 1 + c_1(L)$ and $c_k(L) = 0$ for $k \ge 2$.*
4. *If E is trivial, then $c(E) = 1$.*

Proof. Under the splitting principle $E = \bigoplus L_i$, $F = \bigoplus M_j$. Duals, tensor products, and sums reduce to line-bundle operations, where the identities are immediate from $c(L) = 1 + c_1(L)$ and $c_1(L^*) = -c_1(L)$. □

Proposition 7.16 (Relation with Stiefel-Whitney classes)**.** *For a complex vector bundle E, regard it as a real bundle $E_{\mathbb{R}}$ of rank $2r$. Then*

$$w_{2k}(E_{\mathbb{R}}) \equiv c_k(E) \pmod 2, \qquad w_{2k+1}(E_{\mathbb{R}}) = 0.$$

Proof. This follows from the identification of the complex structure with a reduction of structure group $U(r) \subset SO(2r)$ and the naturality of the Bockstein homomorphism; see standard arguments via classifying spaces $BU(r) \to BSO(2r)$. □

Example 7.17 (Line bundles on $\mathbb{CP}^n$)**.** *Let $\mathcal{O}(1) \to \mathbb{CP}^n$ be the hyperplane line bundle and $h := c_1(\mathcal{O}(1)) \in H^2(\mathbb{CP}^n; \mathbb{Z})$ the hyperplane class. Then $c(\mathcal{O}(1)) = 1 + h$ and $c(\mathcal{O}(-1)) = 1 - h$.*

Example 7.18 (Tangent bundle of $\mathbb{CP}^n$)**.** *The Euler (short exact) sequence*

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus(n+1)} \longrightarrow T\mathbb{CP}^n \longrightarrow 0$$

implies, by Whitney multiplicativity,

$$c(T\mathbb{CP}^n) = \frac{c(\mathcal{O}(1)^{\oplus(n+1)})}{c(\mathcal{O})} = (1+h)^{n+1}.$$

Hence $c_k(T\mathbb{CP}^n) = \binom{n+1}{k} h^k$ and in particular $c_n(T\mathbb{CP}^n) = (n+1)h^n$.

Proposition 7.19 (First Chern class via connections)**.** *If $L \to M$ is a complex line bundle with unitary connection ∇ and curvature F_∇, then*

$$c_1(L) = \left[\frac{i}{2\pi} F_\nabla\right] \in H^2(M; \mathbb{Z}),$$

and this class is integral (periods lie in $\mathbb{Z}$).

Proof. This is the $r = 1$ case of Chern-Weil theory (curvature form represents c_1) together with integrality of its periods via the holonomy/transition-function cocycle. □

Example 7.20 (Product manifolds)**.** *For complex manifolds X, Y, the tangent bundle satisfies $T(X \times Y) \cong \pi_X^* TX \oplus \pi_Y^* TY$, hence*

$$c\Big(T(X \times Y)\Big) = \pi_X^* c(TX) \smile \pi_Y^* c(TY).$$

Remark 7.21 (Chern character)**.** *The* ***Chern character*** *is the rational class*

$$\mathrm{ch}(E) := \sum_{j=1}^{r} e^{x_j} \in H^{\mathrm{even}}(M;)$$

under the splitting principle. It satisfies $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$ and $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \smile \mathrm{ch}(F)$. In terms of Chern classes,

$$\mathrm{ch}(E) = r + c_1(E) + \tfrac{1}{2}\Big(c_1(E)^2 - 2c_2(E)\Big) + \cdots.$$

Corollary 7.22 (Top Chern class and Euler class)**.** *If E is a complex rank-n vector bundle, then the Euler class of the underlying real oriented bundle satisfies*

$$e(E_{\mathbb{R}}) = c_n(E) \in H^{2n}(M;\mathbb{Z}).$$

Proof. Under a unitary connection, the real curvature lies in $\mathfrak{u}(n) \subset \mathfrak{so}(2n)$; the Pfaffian representing $e(E_{\mathbb{R}})$ equals the top Chern form, hence the integral classes coincide. □

7.5 Pontryagin Classes

Pontryagin classes are characteristic classes of real vector bundles, defined in integral cohomology, but detected via Chern classes of their complexifications.

Definition 7.23 (Pontryagin classes)**.** *Let $E \to M$ be a real vector bundle of rank k. Its* ***Pontryagin classes*** *are elements*

$$p_i(E) \in H^{4i}(M;\mathbb{Z}), \qquad 0 \le i \le \lfloor k/2 \rfloor,$$

defined by

$$p_i(E) := (-1)^i\, c_{2i}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(M;\mathbb{Z}),$$

where $E \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of E.

Definition 7.24 (Total Pontryagin class)**.** *The total Pontryagin class is*

$$p(E) = 1 + p_1(E) + p_2(E) + \cdots \in H^{4*}(M;\mathbb{Z}).$$

Proposition 7.25 (Basic properties)**.** *Pontryagin classes satisfy:*

1. ***Naturality:*** *$p_i(f^*E) = f^*p_i(E)$ for smooth $f : N \to M$.*
2. ***Whitney sum:*** *$p(E \oplus F) = p(E) \smile p(F)$.*
3. *For a complex bundle E, with Chern roots x_j, one has $p(E_{\mathbb{R}}) = \prod_j (1 + x_j^2)$; equivalently, $p_i(E_{\mathbb{R}}) = (-1)^i\, c_{2i}(E \oplus \overline{E})$ (so $c(E)\, c(\overline{E}) = 1 - p_1(E_{\mathbb{R}}) + p_2(E_{\mathbb{R}}) - \cdots$).*

Example 7.26 (Trivial bundle)**.** *If E is trivial, then $p(E) = 1$.*

Example 7.27 (Tangent bundle of S^n)**.** *All Pontryagin classes vanish for the tangent bundle of spheres S^n (because $H^{4i}(S^n) = 0$ for $i \ge 1$).*

Example 7.28 (Tangent bundle of $\mathbb{C}P^n$)**.** *For $\mathbb{C}P^n$, using Example 7.18, the tangent bundle has*

$$c(T\mathbb{C}P^n) = (1+h)^{n+1}, \quad h \in H^2(\mathbb{C}P^n;\mathbb{Z}).$$

From the relation $p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{R}\mathbb{C})$, one obtains explicit formulas for $p_1(T\mathbb{C}P^n)$ and higher classes.

Remark 7.29 (Relation with Stiefel-Whitney classes)**.** *Modulo 2, Pontryagin classes reduce to squares of Stiefel-Whitney classes:*

$$p_i(E) \equiv w_{2i}(E)^2 \quad \in H^{4i}(M;\mathbb{Z}/2).$$

Theorem 7.30 (Hirzebruch signature theorem, preview)**.** *For a closed oriented $4k$-dimensional manifold M, the signature of the quadratic form on $H^{2k}(M;\mathbb{R})$ is given by*

$$\sigma(M) = \langle L(p_1, \ldots, p_k), [M] \rangle,$$

where L is the Hirzebruch L-polynomial in Pontryagin classes. This deep theorem links topology, geometry, and analysis.

7.6 Euler Class

The Euler class is the fundamental characteristic class associated to real, oriented vector bundles of even rank. It generalizes the notion of the Euler characteristic of a manifold.

Definition 7.31 (Euler class of an oriented bundle)**.** *Let $E \to M$ be a real, oriented vector bundle of rank $n = 2k$. The* ***Euler class***

$$e(E) \in H^n(M;\mathbb{Z})$$

is the unique characteristic class such that:

1. *$e(E)$ is natural under pullback: $e(f^*E) = f^*e(E)$.*
2. *If E admits a nowhere-vanishing section, then $e(E) = 0$.*
3. *If E is the tangent bundle of a closed oriented n-manifold M, then $\langle e(TM), [M] \rangle = \chi(M)$, the Euler characteristic of M.*

Remark 7.32. *The Euler class can be defined abstractly using the Thom isomorphism: $e(E)$ is the pullback of the Thom class under the zero section $M \to E$.*

Proposition 7.33 (Top Chern class)**.** *If E is a complex vector bundle of rank r, then the Euler class of the underlying real bundle $E_{\mathbb{R}}$ satisfies*

$$e(E_{\mathbb{R}}) = c_r(E).$$

Example 7.34 (Sphere tangent bundle)**.** *For S^2, $e(TS^2)$ is a generator of $H^2(S^2;\mathbb{Z}) \cong \mathbb{Z}$, and $\langle e(TS^2), [S^2] \rangle = 2$, in agreement with $\chi(S^2) = 2$. For S^{2k+1} there exists a nowhere-vanishing vector field, so $e(TS^{2k+1}) = 0$ (and indeed $\chi(S^{2k+1}) = 0$), even though $H^{2k+1}(S^{2k+1};\mathbb{Z}) \cong \mathbb{Z}$.*

Theorem 7.35 (Gauss-Bonnet)**.** *Let M be a closed, oriented Riemannian manifold of dimension $2k$. Then*

$$\chi(M) = \int_M e(TM),$$

where $e(TM)$ is represented by the Pfaffian of the curvature form of the Levi-Civita connection. This is the differential-geometric incarnation of the Euler class.

Remark 7.36. *This result bridges algebraic topology and differential geometry: the Euler class is purely topological, but its de Rham representative is the Pfaffian polynomial in curvature.*

8 Chern-Weil Theory via Principal Bundles

8.1 Principal bundles and connections

Definition 8.1 (Principal G-bundle)**.** *Let G be a Lie group. A **principal G-bundle** over a smooth manifold M is a smooth manifold P with a smooth right action $R : P \times G \to P$ that is free and proper, together with a smooth surjective submersion $\pi : P \to M$ whose fibers are the G-orbits. Locally, there exist trivializations $\Phi : \pi^{-1}(U) \to U \times G$ intertwining the G-action by right multiplication on G.*

Definition 8.2 (Connection 1-form)**.** *A **connection** on a principal G-bundle $\pi : P \to M$ is a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P; \mathfrak{g})$ such that:*

1. *For each $\xi \in \mathfrak{g}$, letting $\xi^\sharp$ be the fundamental vertical vector field on P, one has $\omega(\xi^\sharp) = \xi$.*
2. *$R_g^*\omega = \mathrm{Ad}_{g^{-1}}\omega$ for all $g \in G$.*

*The **horizontal subspace** at $u \in P$ is $H_uP := \ker \omega_u \subset T_uP$, giving a G-equivariant splitting $T_uP = H_uP \oplus V_uP$ with $V_uP = \ker d\pi_u$.*

Definition 8.3 (Curvature 2-form)**.** *The **curvature** of a connection ω is the $\mathfrak{g}$-valued 2-form*

$$\Omega := d\omega + \tfrac{1}{2}[\omega, \omega] \in \Omega^2(P; \mathfrak{g}),$$

where $[\,,\,]$ is the Lie bracket extended fiberwise and wedge-multiplied.

Lemma 8.4 (Equivariance and horizontality)**.** *For a connection ω with curvature Ω:*

1. *$R_g^*\Omega = \mathrm{Ad}_{g^{-1}}\Omega$ for all $g \in G$,*
2. *$\iota_{\xi^\sharp}\Omega = 0$ for all $\xi \in \mathfrak{g}$ (horizontality).*

Proof. Differentiate $R_g^*\omega = \mathrm{Ad}_{g^{-1}}\omega$ and use the definition of Ω. For horizontality, compute $\iota_{\xi^\sharp}\Omega = \iota_{\xi^\sharp}d\omega + \frac{1}{2}\iota_{\xi^\sharp}[\omega, \omega]$ and use Cartan's formula together with $\omega(\xi^\sharp) = \xi$ and G-equivariance. □

Theorem 8.5 (Bianchi identity on principal bundles)**.** *For any connection ω with curvature Ω,*

$$D\Omega := d\Omega + [\omega, \Omega] = 0,$$

where D is the covariant exterior derivative.

Proof. Compute $D\Omega = d(d\omega + \frac{1}{2}[\omega, \omega]) + [\omega, d\omega + \frac{1}{2}[\omega, \omega]]$ and apply $d^2 = 0$, the graded Jacobi identity, and bilinearity of the bracket. □

8.2 Invariant polynomials and basic forms

Definition 8.6 (Invariant polynomial)**.** *A symmetric k-linear form $P : \mathfrak{g}^k \to \mathbb{R}$ (or $\mathbb{C}$) is* ***Ad-invariant*** *if*

$$P(\mathrm{Ad}_g X_1, \ldots, \mathrm{Ad}_g X_k) = P(X_1, \ldots, X_k) \quad \forall g \in G,\ X_i \in \mathfrak{g}.$$

Equivalently, $P \circ (\mathrm{Ad}_g)^{\otimes k} = P$ for all g.

Definition 8.7 (Chern-Weil form on P)**.** *Given an Ad-invariant polynomial P of degree k and the curvature Ω, define the $\mathbb{R}$-valued $2k$-form on P*

$$P(\Omega) := P\Big(\underbrace{\Omega, \ldots, \Omega}_{k \ \textit{times}}\Big) \in \Omega^{2k}(P).$$

Lemma 8.8 (Basicness)**.** *$P(\Omega)$ is G-invariant and horizontal; hence it is* ***basic*** *and descends to a unique form $\omega_P \in \Omega^{2k}(M)$ satisfying $\pi^*\omega_P = P(\Omega)$.*

Proof. G-invariance follows from equivariance $R_g^*\Omega = \mathrm{Ad}_{g^{-1}}\Omega$ and Ad-invariance of P. Horizontality follows from $\iota_{\xi^\sharp}\Omega = 0$ and multilinearity. Basic forms are exactly the invariant horizontal forms; thus there is a unique descendant on M. □

Theorem 8.9 (Chern-Weil)**.** *Let P be an Ad-invariant degree-k polynomial on $\mathfrak{g}$ and $\omega_P \in \Omega^{2k}(M)$ its descendant. Then:*

1. *$d\omega_P = 0$,*
2. *The de Rham class $[\omega_P] \in H^{2k}_{\mathrm{dR}}(M)$ is independent of the choice of connection ω,*
3. *The assignment $P \mapsto [\omega_P]$ is natural with respect to pullback of bundles and bundle maps.*

Proof. (1) $dP(\Omega) = DP(\Omega)$ on P because P is constant coefficient. Using Bianchi $D\Omega = 0$ and Ad-invariance of P, $DP(\Omega) = 0$. Hence $d\omega_P = 0$.

(2) Let ω^0, ω^1 be connections with curvatures Ω^0, Ω^1 and consider $\omega^t = (1-t)\omega^0 + t\omega^1$ on $P \times [0,1]$. A standard transgression computation yields

$$\frac{d}{dt} P(\Omega^t) = d\,\mathrm{CS}_P(\omega^t; \dot{\omega}^t),$$

where CS_P is an explicit $(2k-1)$-form (Chern-Simons transgression). Pulling back to M and integrating in t shows the difference of the descendants is exact.

(3) Naturality follows from functoriality of pullback of principal bundles, connections, and curvatures together with Ad-invariance of P. □

8.3 Chern-Simons transgression forms

Definition 8.10 (Relative transgression). *Given two connections ω^0, ω^1 with difference $\alpha := \omega^1 - \omega^0$, define*

$$\mathrm{CS}_P(\omega^0, \omega^1) := k \int_0^1 P\Big(\alpha, \Omega^t, \ldots, \Omega^t\Big)\, dt \ \in \Omega^{2k-1}(P),$$

where Ω^t is the curvature of $\omega^t = \omega^0 + t\alpha$. It is G-basic up to exact terms and descends locally.

Proposition 8.11 (Transgression identity).

$$P(\Omega^1) - P(\Omega^0) = d\, \mathrm{CS}_P(\omega^0, \omega^1).$$

Consequently, the associated base forms on M differ by an exact form.

Proof. Differentiate $P(\Omega^t)$ with respect to t, use $D^t \dot{\omega}^t = \frac{d}{dt}\Omega^t$ and Bianchi, then integrate from 0 to 1. □

8.4 Associated bundles and classical classes

Definition 8.12 (Associated vector bundle). *Let $\pi : P \to M$ be a principal G-bundle and $\rho : G \to GL(V)$ a representation. The* ***associated bundle*** *is $E := P \times_\rho V \to M$. A principal connection ω induces a covariant derivative ∇ on E whose local connection matrix is $\rho_*(\omega)$, with curvature $F_\nabla = \rho_*(\Omega)$.*

Example 8.13 (Chern classes from $U(r)$). *Take $G = U(r)$, ρ the defining representation on $\mathbb{C}^r$. For the invariant polynomials $P_k(A) = \frac{1}{k!}\mathrm{tr}\, A^k$ on $\mathfrak{u}(r)$, the normalized forms*

$$c_k(E) := \left[\frac{1}{(2\pi i)^k}\, P_k(F_\nabla)\right] \in H^{2k}(M; \mathbb{R})$$

recover the Chern classes (and are integral). The total Chern class is $\det\Big(I + \frac{i}{2\pi}F_\nabla\Big)$.

Example 8.14 (Pontryagin from $SO(n)$). *For a real bundle with structure group $SO(n)$, complexify and use $P_{2k}(A) = \mathrm{tr}(A^{2k})$ with normalization to obtain $p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C})$.*

Example 8.15 (Euler class from $SO(2n)$). *On an oriented real rank-2n bundle, the invariant polynomial* Pf *(Pfaffian) on $\mathfrak{so}(2n)$ yields*

$$e(E) = \left[\mathrm{Pf}\Big(\frac{1}{2\pi}\Omega\Big)\right] \in H^{2n}(M; \mathbb{R}).$$

8.5 Functoriality, Whitney sum, and block structure

Proposition 8.16 (Naturality). *If $f : N \to M$ and f^*P is the pulled-back principal G-bundle with pulled-back connection, then for any invariant polynomial P,*

$$f^*\omega_P = \omega_P(f^*P).$$

Proposition 8.17 (Whitney sum via block-diagonal)**.** *Let E, F be complex vector bundles with unitary connections ∇^E, ∇^F and curvatures F^E, F^F. On $E \oplus F$ with block-diagonal connection, the curvature is $F^{E\oplus F} = \left(\begin{smallmatrix} F^E & 0 \\ 0 & F^F \end{smallmatrix}\right)$. Hence*

$$\det\Big(I + \tfrac{i}{2\pi} F^{E\oplus F}\Big) = \det\Big(I + \tfrac{i}{2\pi} F^E\Big) \cdot \det\Big(I + \tfrac{i}{2\pi} F^F\Big),$$

so $c(E \oplus F) = c(E) \smile c(F)$.

8.6 Normalization and integrality

Proposition 8.18 (Normalization constants)**.** *With the normalizations $\frac{i}{2\pi}$ for unitary cases and $\frac{1}{2\pi}$ for orthogonal Pfaffians, the Chern–Weil forms represent integral cohomology classes when the structure group is compact and connected, coinciding with topological characteristic classes defined via classifying spaces.*

Proof idea. Use the classifying map $M \to BG$ and identify de Rham representatives with pullbacks of universal forms on BG whose periods are integral, together with the de Rham isomorphism. □

8.7 Summary

Chern-Weil theory assigns to each Ad-invariant polynomial P on the Lie algebra of the structure group a canonical closed form on the base whose cohomology class is independent of the connection. For classical structure groups and canonical polynomials (trace powers, Pfaffian, elementary symmetric functions), these recover the standard characteristic classes (Chern, Pontryagin, Euler), and satisfy functoriality and Whitney sum via curvature block-diagonalization. The difference of representatives for different connections is exact, measured by Chern-Simons transgression forms.

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