Poincaré–Hopf Theorem

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Introduction

In the late 19th century, mathematicians began to expand differential geometry beyond the classical study of curves and surfaces by exploring the properties of smooth manifolds and tangent bundles. Henri Poincaré investigated vector fields on these manifolds, particularly paying attention to zeros, points where a vector field vanishes. He observed that these zeros were not random, but instead suggested insight into a manifold's structure.

Poincaré introduced the concept of an index for an isolated zero of a vector field, generalizing the notion of winding numbers from planar vector fields. This index measures how the surrounding vectors "wrap around" the zero and can be formalized as the degree of a map from a small sphere in the tangent space to the unit sphere. His work suggested that the sum of the indices of all zeros might be connected to a global topological invariant, the Euler characteristic of the manifold.

Building on Poincaré's insights, in the early 20th century, Heinz Hopf provided a precise formulation of the Euler characteristic. He proved that for any smooth vector field with isolated zeros on a compact, oriented manifold, the sum of the indices equals the manifold's Euler characteristic. This came to be what is now known as the Poincaré–Hopf theorem, which aligned with earlier observations such as the Hairy Ball theorem and established a connection between local differential behavior and global topological properties.

The discovery of the theorem marked a major milestone in differential geometry. By generalizing classical results and pointing toward later advances in characteristic classes and index theory, the Poincaré–Hopf theorem highlights a fundamental principle of modern geometry: how local analytic behavior of vector fields is linked to information about the manifold's structure.

Preliminaries

An **n-dimensional manifold** is a topological space M that is Hausdorff (meaning any two distinct points can be separated by disjoint open sets) and second-countable (meaning there exists a countable collection of open sets such that every open set in M can be expressed as a union of them), and in which every point $p \in M$ has a neighborhood $U \subseteq M$ homeomorphic to an open subset of $\mathbb{R}^n$.

A **smooth manifold** is an n-dimensional manifold that has a maximal atlas of charts $\varphi : U \subset M \to \mathbb{R}^n$, whose transition maps are C^∞-smooth. If M has a boundary, the charts map neighborhoods in M to open subsets of the closed half-space $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n \geq 0\}$. Unless stated otherwise, all manifolds and maps are assumed C^∞-smooth.

For the following definitions, let M be a smooth, compact, oriented manifold of dimension n, possibly with boundary ∂M. Throughout, we assume all manifolds and maps are C^∞-smooth.

A **vector field** on M is a smooth section $X : M \to TM$ of the tangent bundle TM, assigning to each point $p \in M$ a tangent vector $X(p) \in T_pM$. A point $p \in M$ is called a **zero** (or singularity) of X if $X(p) = 0 \in T_pM$. We say X has an *isolated zero* at p if there exists a neighborhood $U \subseteq M$ of p such that p is the only zero of X in U.

To each isolated zero p of X, there's an associated integer called the **index** of X at p, denoted $\mathrm{ind}_p(X)$. The **index** measures the local behavior of the vector field near p, describing how the vectors "wrap around" the zero. Formally, in a local coordinate chart (U, φ) with $\varphi(p) = 0 \in \mathbb{R}^n$, the map

$$\frac{X}{\|X\|} : \partial B_\varepsilon(0) \to S^{n-1}$$

from the boundary of a small ball around zero to the unit sphere is well-defined and continuous, and the index is defined as the topological degree of this map. The index is independent of choices made and is stable under homotopies of the vector field that do not create or destroy zeros.

The **Euler characteristic** $\chi(M)$ is a topological invariant of the manifold M, defined via any finite triangulation or cellular decomposition by the alternating sum

$$\chi(M) = \sum_{i=0}^{n} (-1)^i c_i,$$

where c_i is the number of i-dimensional cells. Equivalently, in algebraic topology, it can be computed as

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim H^i(M; \mathbb{Q}),$$

the alternating sum of the dimensions of the rational cohomology groups of M.

For oriented surfaces, the Euler characteristic relates with curvature through the Gauss–Bonnet theorem:

$$\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\pi\chi(M),$$

where K is the Gaussian curvature of the surface, k_g is the geodesic curvature of the boundary, and dA, ds denote area and arc-length measures respectively.

Let M be a smooth manifold. Two vector fields $X, Y \in \mathfrak{X}(M)$ with isolated zeros are said to be **homotopic** through vector fields with isolated zeros if there exists a smooth map

$$H : M \times [0,1] \to TM$$

such that for each $t \in [0,1]$, the map $H_t(x) := H(x,t)$ defines a smooth vector field on M with isolated zeros, and

$$H_0 = X, \qquad H_1 = Y.$$

Theorem (Poincaré–Hopf)

Let M be a compact, oriented, smooth manifold, and let X be a smooth vector field on M with only finitely many isolated zeros. Then

$$\sum_{p \in \mathrm{Zero}(X)} \mathrm{ind}_p(X) = \chi(M),$$

where the sum is taken over all isolated zeros p of the vector field X, $\text{ind}_p(X)$ denotes the index of X at p, and $\chi(M)$ is the Euler characteristic of M.

Example 1 ($\mathbb{S}^2$)

The unit sphere $\mathbb{S}^2$, defined as

$$\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\},$$

is among the most classical and comprehensible examples of a smooth, compact, oriented 2-dimensional manifold. For this reason, we will use it to explore and show the Poincaré–Hopf theorem and its implications.

According to the theorem, for any smooth vector field X on a compact oriented manifold M with isolated zeros, the sum of the indices of these zeros equals the Euler characteristic $\chi(M)$. In the case of the 2-sphere, the Euler characteristic is known to be

$$\chi(\mathbb{S}^2) = 2.$$

Therefore, any vector field X on $\mathbb{S}^2$ must have zeros whose indices sum to 2.

A standard example illustrating the Poincaré–Hopf theorem on $\mathbb{S}^2$ is constructed from the radial vector field on $\mathbb{R}^3$:

$$X_{\text{rad}}(x) = x, \quad x \in \mathbb{R}^3.$$

This vector field points directly outward from the origin, where the magnitude of the vector is proportional to the distance from the origin. While X_{rad} is not tangent to $\mathbb{S}^2$, we can project it orthogonally onto the tangent space $T_x\mathbb{S}^2$ at each point $x \in \mathbb{S}^2$.

The tangent space $T_x\mathbb{S}^2$ consists of all vectors perpendicular to x, since $\mathbb{S}^2$ is defined by the constraint $\|x\| = 1$. The orthogonal projection $P_x : \mathbb{R}^3 \to T_x\mathbb{S}^2$ is given by

$$P_x(v) = v - \langle v, x \rangle x,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product.

We then apply this to a radial vector field:

$$X(x) = P_x(x) = x - \langle x, x \rangle x = x - (1)x = 0.$$

At first glance, this appears to be zero everywhere, but we want to manipulate the radial vector field to generate a more useful tangent vector field. Instead, consider the constant vector $e_3 = (0, 0, 1)$ in $\mathbb{R}^3$. Projecting e_3 onto $T_x\mathbb{S}^2$ yields the vector field:

$$X(x) = e_3 - \langle e_3, x \rangle x.$$

This vector field is tangent to the sphere at every point.

The zeros of X are the points $x \in \mathbb{S}^2$ where

$$X(x) = 0 \implies e_3 = \langle e_3, x \rangle x.$$

Taking the inner product of both sides with x, we find

$$\langle e_3, x \rangle = \langle e_3, x \rangle \langle x, x \rangle = \langle e_3, x \rangle.$$

The vector e_3 lies in the direction of x scaled by the factor $\langle e_3, x\rangle$, so for $X(x) = 0$, x must be parallel to e_3, i.e.,

$$x = \pm e_3.$$

Therefore, the zeros of X occur precisely at the north pole $(0, 0, 1)$ and the south pole $(0, 0, -1)$.

Each of these zero is isolated. We can also compute the index at each zero by observing the behavior of X near these points. Both zeros correspond to simple "sources" or "sinks" of the vector field, and each has index $+1$.

More precisely, the index can be computed by considering the map from a small circle around the zero into the unit circle in the tangent space, induced by normalizing the vector field. Around the north pole, the vector field looks like a radial outward pointing field on the tangent plane, which has degree $+1$. This is also the case at the south pole.

Thus, summing the indices,

$$\mathrm{ind}_{\text{north pole}}(X) + \mathrm{ind}_{\text{south pole}}(X) = 1 + 1 = 2,$$

which matches the Euler characteristic $\chi(\mathbb{S}^2) = 2$.

Example 2: The Torus

The Poincaré–Hopf theorem states that for any smooth vector field X on a compact, orientable surface M with isolated zeros, the sum of the indices of those zeros equals the Euler characteristic:

$$\sum_{p \in Z(X)} \mathrm{ind}_p(X) \; = \; \chi(M).$$

To confirm this for the torus T^2, we'll check this identity in this section. We start with the computations where:

- Using a CW-decomposition, T^2 has one 0-cell, two 1-cells, and one 2-cell, giving

  $$\chi(T^2) = 1 - 2 + 1 = 0.$$

- Using the product formula, since $T^2 \cong S^1 \times S^1$,

  $$\chi(T^2) = \chi(S^1)\,\chi(S^1) = 0 \cdot 0 = 0.$$

Now we consider the standard embedding

$$\sigma(\theta, \phi) = \big((a + b\cos\theta)\cos\phi,\; (a + b\cos\theta)\sin\phi,\; b\sin\theta\big), \qquad 0 < b < a.$$

The coordinate vector field σ_ϕ is tangent to the torus and nowhere zero. Define

$$X(\theta, \phi) := \sigma_\phi(\theta, \phi).$$

Thus X is a smooth, nowhere-vanishing vector field, so its zero set is empty: $Z(X) = \varnothing$.
Since X has no zeros,

$$\sum_{p \in Z(X)} \mathrm{ind}_p(X) = 0,$$

which agrees with $\chi(T^2) = 0$. This proves the Poincaré–Hopf theorem in the case of the torus.

Example 3: The Klein Bottle

The Klein bottle K is a compact, non-orientable surface with

$$\chi(K) = 0.$$

By the Poincaré–Hopf theorem, for any vector field X on K with isolated zeros,

$$\sum_{p \in \mathrm{Zero}(X)} \mathrm{ind}_p(X) = 0.$$

Since K admits a nowhere-vanishing vector field (e.g. a horizontal constant field on the fundamental rectangle compatible with identifications), one obtains

$$\mathrm{Zero}(X) = \varnothing \quad \Longrightarrow \quad \sum \mathrm{ind}_p(X) = 0,$$

matching $\chi(K)$.

If zeros are present, they must occur in pairs with opposite index:

$$\mathrm{ind}_{p_1}(X) + \mathrm{ind}_{p_2}(X) = 0,$$

so that the global index sum remains 0.

Application to Physics

The Poincaré–Hopf theorem application on the Klein bottle has interesting applications in physics, particularly in magnetic confinement fusion. Surfaces with Klein bottle topology can support smooth, nowhere-vanishing vector fields, which appear in certain magnetic surfaces of tokamaks and stellarators. These configurations influence the behavior of magnetic field lines and are associated with phenomena like abnormal sawtooth crashes. This results with the theorem playing an integral role in understanding and designing plasma confinement systems.

The Hairy Ball Theorem

The Hairy Ball theorem states that there does not exist a continuous, nowhere-vanishing tangent vector field on an even-dimensional sphere, in particular on $\mathbb{S}^2$. Equivalently, every continuous tangent vector field X on $\mathbb{S}^2$ must vanish at least at one point. This follows from the Poincaré–Hopf theorem: since

$$\chi(\mathbb{S}^2) = 2,$$

any smooth vector field X on $\mathbb{S}^2$ satisfies

$$\sum_{p \in Z(X)} \mathrm{ind}_p(X) \;=\; 2,$$

so $Z(X) \neq \varnothing$. In particular, no continuous tangent field on $\mathbb{S}^2$ can be nonvanishing.

This theorem gets its name from the fact that it is impossible to "comb" the hair on a sphere smoothly without creating at least one "cowlick" or zero where the hair stands straight.

It is also applicable to the real world where, for example, it is present in meteorology: there must always be at least one calm point or vortex in any continuous wind pattern on the Earth's surface.

References

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