Calculus on Manifolds: From Differential Forms to Stokes' Theorem

Adhit Mandal

Abstract

A unifying framework for multivariable calculus, differential geometry, and mathematical physics is offered by calculus on manifolds. In order to get to the general Stokes' Theorem, this paper presents the fundamentals of differential forms, tangent spaces, smooth manifolds, and integration theory. The fundamental theorem of calculus, Green's theorem, the divergence theorem, and Stokes' theorem in three dimensions are among the traditional results of vector calculus that are subsumed by the general theorem. By doing this, it illustrates the mathematical power of abstraction by revealing that conclusions that are recognizable from basic calculus are actually shadows of a more profound geometric truth.

1 Introduction

Functions on the real line $\mathbb{R}$, limits, derivatives, and integrals are the first concepts in classical calculus. These concepts are extended to functions on subsets of $\mathbb{R}^n$ in the passage to multivariable calculus, which also introduces partial derivatives, gradients, multiple integrals, and vector fields. However, the breadth of physics and mathematics necessitates an even more expansive framework. Although spaces like spheres, tori, and Grassmannians have local Euclidean structure, they cannot be simply represented globally as subsets of $\mathbb{R}^n$. We refer to these areas as manifolds.

Calculus on manifolds allows one to describe and study geometric and physical processes in their natural environment by generalizing the tools of differential and integral calculus to these spaces. Stokes' theorem, which subsumes a family of vector calculus theorems and connects integrating over a manifold to integration over its border, is a key unifying conclusion.

2 Smooth Manifolds

Definition 2.1. *A* topological manifold *of dimension n is a Hausdorff, second-countable topological space M such that every point $p \in M$ has a neighborhood U homeomorphic to an open subset of $\mathbb{R}^n$. A* smooth manifold *is a topological manifold equipped with a maximal smooth atlas, meaning that transition maps between overlapping charts are smooth.*

The concept of smoothness allows one to define differentiability of functions on M and construct the framework for differential geometry.

Example 2.1. *The circle* $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ *is a one-dimensional smooth manifold. Locally, neighbourhoods on* S^1 *resemble open intervals in* $\mathbb{R}$*. However, globally,* S^1 *is compact and has nontrivial topology.*

Example 2.2. *The real projective plane* $\mathbb{RP}^2$ *is the set of lines through the origin in* $\mathbb{R}^3$*. Although it cannot be embedded in* $\mathbb{R}^3$ *without self-intersection, it is a smooth 2-manifold.*

Charts and atlases provide local coordinate systems, enabling us to perform calculus as if we were in $\mathbb{R}^n$, while smooth transition functions guarantee global consistency.

3 Tangent Space and Cotangent Space

Let M be a smooth n-dimensional manifold. At each point $p \in M$, we define:

- The **tangent space** T_pM, the vector space of directional derivatives at p.
- The **cotangent space** T_p^*M, the dual space of T_pM, consisting of linear maps:

$$\omega_p : T_pM \to \mathbb{R}.$$

  Elements of T_p^*M are called **covectors** (or **dual vectors**).

4 Differential Forms

At each point of a manifold, one can define a vector space that represents all possible directions of motion.

Definition 4.1. *The* tangent space T_pM *at* $p \in M$ *is the vector space of derivations at* p*, i.e., linear maps* $v : C^\infty(M) \to \mathbb{R}$ *satisfying the Leibniz rule* $v(fg) = v(f)g(p) + f(p)v(g)$*.*

An alternative characterization uses equivalence classes of smooth curves through p. This intuition aligns with the notion of velocity vectors.

The dual space T_p^*M is the *cotangent space*, whose elements are linear maps $T_pM \to \mathbb{R}$. A *differential 1-form* at p is an element of T_p^*M.

Definition 4.2. *A* differential k-form *on* M *is a smooth section of* $\Lambda^k T^*M$*, i.e., a smooth assignment to each* $p \in M$ *of an alternating multilinear map* $(T_pM)^k \to \mathbb{R}$*.*

Differential forms can be combined using the wedge product, which satisfies skew-symmetry and bilinearity. This structure provides the algebraic backbone for integration theory.

5 Integration on Manifolds

To generalize integration, orientation must be defined.

Definition 5.1. *An* orientation *on an n-manifold M is a choice of equivalence class of atlases where all transition functions have positive Jacobian determinant. A manifold with a chosen orientation is called an* oriented manifold.

Given an oriented n-manifold M, one can integrate compactly supported n-forms by pulling them back to coordinate charts, using partitions of unity, and summing over the atlas.

Example 5.1. *On the 2-sphere S^2, in spherical coordinates (θ, ϕ), the area form is $\omega = \sin(\theta) d\theta \wedge d\phi$. Its integral over S^2 is $\int_{S^2} \omega = 4\pi$, which is the total surface area.*

Integrating over curved spaces without embedding them in Euclidean space is made possible by the machinery of integration on manifolds, which generalizes classical multiple integrals.

6 The Exterior Derivative

The exterior derivative extends the notions of gradient, curl, and divergence.

Definition 6.1. *The* exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ *is the unique linear operator satisfying:*

1. $d^2 = 0$ *(nilpotency),*
2. $d(f) = df$ *for smooth functions f (the differential),*
3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ *for $\alpha \in \Omega^k(M)$.*

Example 6.1. *In $\mathbb{R}^3$, a 1-form $\omega = f dx + g dy + h dz$ has exterior derivative*

$$d\omega = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) dz \wedge dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy,$$

which corresponds precisely to the curl of (f, g, h).

Thus, d captures geometric operations from vector calculus in an abstract framework.

7 Stokes' Theorem

Theorem 7.1 (Stokes' Theorem)**.** *Let M be an oriented smooth n-dimensional manifold with boundary ∂M. For any $(n-1)$-form ω with compact support,*

$$\int_M d\omega = \int_{\partial M} \omega.$$

This theorem subsumes classical results:

- For $n = 1$, it is the fundamental theorem of calculus.
- For $n = 2$, it recovers Green's theorem in the plane.
- For $n = 3$, it yields both the divergence theorem and the classical Stokes' theorem.

Example 7.1. *Let M be a region in $\mathbb{R}^3$ with smooth boundary ∂M. If ω corresponds to a vector field $\mathbf{F}$, then $\int_M d\omega$ becomes $\int_M (\nabla \cdot \mathbf{F})\, dV$, while $\int_{\partial M} \omega$ becomes $\int_{\partial M} \mathbf{F} \cdot d\mathbf{S}$. This is the divergence theorem.*

7.1 Fundamental Theorem of Calculus (FTC)

The FTC relates the integral of a derivative over an interval to the values of the function at the boundary:

$$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a).$$

This is a special case of the general Stokes' theorem for 0-forms (functions) on a 1-dimensional manifold.

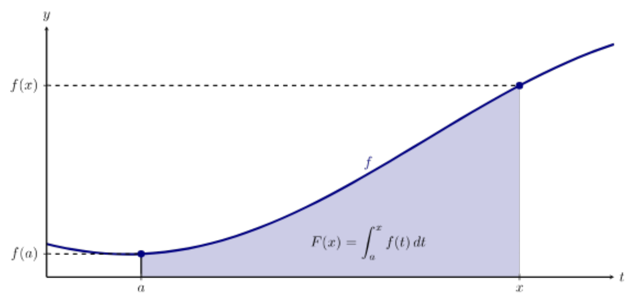

Figure 1: Diagrammatic representation of FTC

7.2 Green's Theorem

Green's theorem relates a line integral around a closed curve C to a double integral over the plane region D it encloses:

$$\oint_C (P\, dx + Q\, dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy.$$

This is a special case of Stokes' theorem for 1-forms in $\mathbb{R}^2$.

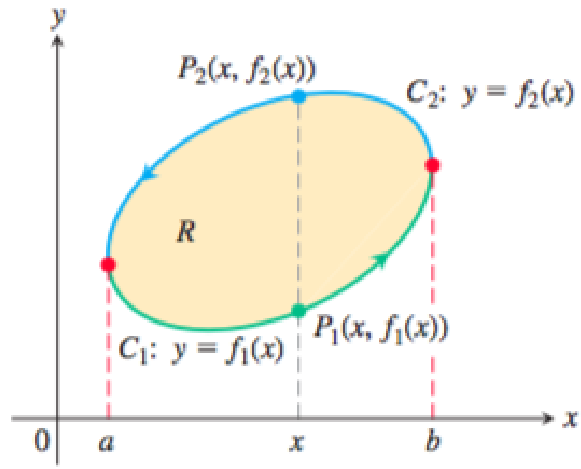

Figure 2: Diagrammatic representation of Green's Theorem

7.3 Divergence Theorem (Gauss's Theorem)

The divergence theorem relates the flux of a vector field through a closed surface S to the divergence over the volume V it encloses:

$$\oint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V (\nabla \cdot \mathbf{F})\, dV.$$

This is a special case of the general Stokes' theorem for $(n-1)$-forms in $\mathbb{R}^n$ (where $n = 3$ here).

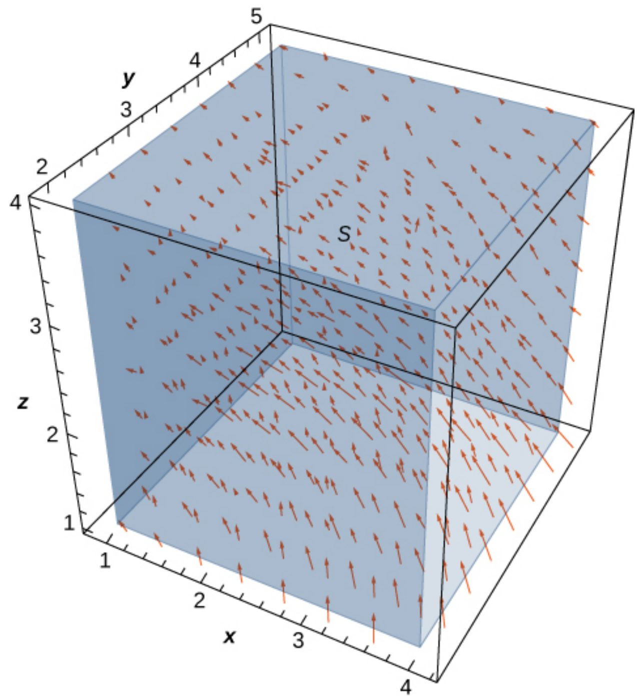

Figure 3: Diagrammatic representation of Divergence Theorem

7.4 Classical Stokes' Theorem

The classical Stokes' theorem relates a line integral around a closed curve C to a surface integral over a surface S bounded by C:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

This is a special case of the general Stokes' theorem for 1-forms in $\mathbb{R}^3$.

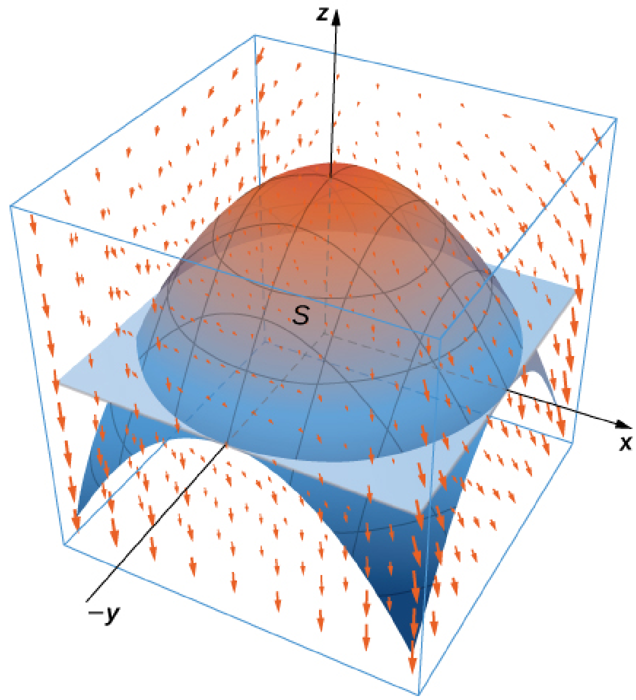

Figure 4: Diagrammatic representation of Classical Stokes' Theorem

8 Applications and Conclusion

Stokes' theorem is a cornerstone of modern geometry and physics. In Maxwell's equations, the curl and divergence laws are succinctly expressed using differential forms and Stokes' theorem. In topology, the theorem connects local differential properties to global invariants, leading to de Rham cohomology.

Calculus on manifolds reveals that the seemingly disparate results of vector calculus are special cases of a single elegant principle. The framework not only provides computational tools but also insight into the structure of space itself, highlighting the unity of mathematics across algebra, analysis, and geometry.