DERHAM COHOMOLOGY

AARAV SHAH

ABSTRACT. This paper presents a detailed exposition of de Rham cohomology, an important bridge between differential geometry and algebraic topology. Beginning with the fundamentals of exterior derivatives and differential forms, we construct the de Rham complex and introduce the notions of closed and exact forms. We then generalize to cohomology in the context of differential complexes and establish the smooth homotopy invariance of de Rham cohomology groups. Key results, including Poincaré's lemma, the Mayer–Vietoris sequence, Stokes' theorem, and de Rham's theorem, are stated and proved.

1. Introduction

De Rham cohomology occupies a central position in modern mathematics, providing a unifying framework that connects smooth differential forms on a manifold with topological invariants. The theory demonstrates that smooth data-captured by the calculus of differential forms-can encode purely topological properties of a space.

The study begins with the algebraic structure of the space of smooth differential forms on a manifold [3–5, 8], equipped with the exterior derivative. This structure forms the de Rham complex, whose cohomology groups [1, 2, 4, 9] measure the failure of closed forms to be exact. These groups are finite-dimensional for compact manifolds and reveal deep relationships between geometry and topology.

From this foundation, one can investigate invariance properties under smooth homotopy [7], understand the local-to-global structure via Mayer–Vietoris sequences [3, 7], and relate integration on manifolds to cohomological properties through Stokes' theorem [6, 7]. The culmination of the theory is de Rham's theorem [7], which asserts an isomorphism between de Rham cohomology and singular cohomology with real coefficients—a profound statement bridging analysis and topology.

The purpose of this paper is to provide a systematic development of these ideas. We aim to present de Rham cohomology [3, 7, 9] as a conceptual cornerstone of differential geometry.

2. DeRham Theory

To start of with studying DeRham theory, we need a notion of exterior derivatives:-

Definition 2.1. Let $x_1, x_2, ..$ be the linear coordinates on $\mathbb{R}^n$. We define Ω^* as the algebra over $\mathbb{R}$ generated by $dx_1, dx_2, ...dx_n$ with the relations $(dx_i)^2 = 0$ and $dx_i dx_j = -dx_j dx_i$. The operation d is called an exterior derivative.

Following the notion of an exterior derivative, we have the following definition of *n-forms*:-

Definition 2.2. ω is said to be an *n-form* if it can be written as $\sum_{i_1,...1_n} f_{i_1,...,i_n} dx_{i_1}...dx_{i_n}$ where $f_{i_1,...i_n}$ are C^∞ functions.

Date: August 16, 2025.

Now Ω^* has a basis $1, dx_i, dx_i dx_j, dx_i dx_j dx_k, ..., dx_1 dx_2 ... dx_n$ and thus we can decompose it as $\Omega^*(\mathbb{R}^n) = \oplus_{q-=0}^{n} \Omega^q(\mathbb{R}^n)$ where $\Omega^q(\mathbb{R}^n)$ consists of the C^∞ q-forms on $\mathbb{R}^n$. Now, the exterior derivative $d : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n)$ is defined as follows:-
(i)if $f \in \Omega^0(\mathbb{R}^n)$ then $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$
(ii) if $\omega = \sum f_I dx_I$, then $d\omega = \sum df_I dx_I$ We now illustrate the above definitions by considering the following examples:-

Example. In 1 dimension, if $\omega = xdy$ then $d\omega = dxdy$

A particularly interesting case is that of 3 Dimensions,

Example.
- **Functions/0-forms:-**$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$
- **1 forms:-** $d(f_1 dx + f_2 dy + f_3 dz) = \frac{\partial f_3}{\partial y} dydz + \frac{\partial f_2}{\partial z} dzdy + \frac{\partial f_1}{\partial z} dzdx + \frac{\partial f_3}{\partial x} dxdz + \frac{\partial f_2}{\partial x} dxdy + \frac{\partial f_1}{\partial y} dydx = (\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}) dydz - (\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}) dxdz + (\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}) dxdy$ where the first step follows from the expansion of f and the fact that $d^2 x = 0$ and the second step follows from the fact that $dx_i dx_j = -dx_j dx_i$
- **2 forms:-** $d(f_1 dydz - f_2 dxdz + f_3 dxdy) = (\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}) dxdydz$

And hence, we can make the following identification:-
- d(0-forms)=gradient
- d(1-forms)=curl
- d(2-forms)=divergence

Now a natural thing to do is to have a notion of products of differential forms;

Definition 2.3. Let $\tau = \sum f_I dx_I$ and $\omega = \sum g_J dx_J$, then the *wedge product* of τ and ω $(\tau \wedge \omega)$ is defined as $\tau \wedge \omega = \sum f_I g_J dx_I dx_J$.

Proposition 2.4. *If τ is an n-form and ω is an m-form then,* $\tau \wedge \omega = (-1)^{n+m} \omega \wedge \tau$

We would naturally like to explore the product rule:-

Proposition 2.5. $d(\tau \wedge \omega) = (d\tau) \wedge \omega + (-1)^n \tau \wedge d\omega$. *This property is called **anti-derivation***

Proof. This follows from the product rule of functions and the anti-symmetry of $dx_i dx_j$ since $d(\tau \wedge \omega) = \sum d(f_I g_I) dx_I dx_J = \sum df_I g_I dx_I dx_J + \sum f_I dg_I dx_i dx_j$ now since dg is a 1-form, $d(\tau \wedge \omega) = \sum df_I g_I dx_I dx_J + (-1)^n \sum f_I dx_I dg_I dx_J = (d\tau) \wedge \omega + (-1)^n \tau d\omega$ ∎

Proposition 2.6. $d^2 = 0$

Proof. Now for functions, $d^2 f = d(\sum_i \frac{\partial f}{\partial x_i} dx_i) = \sum_i \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j = 0$. On forms, $\omega = f_I dx_I$, $d^2 \omega = d(df_I dx_I) = d^2 f_I dx_I - df_I d^2 x_I = 0$. And thus, we conclude that $d^2 = 0$ ∎

Putting it all together, we have

Definition 2.7. The complex $\Omega^*(\mathbb{R}^n)$ together with the differential operator d is called the **de Rham complex** on $\mathbb{R}^n$.

Recall the following definition of the kernel:-

Definition 2.8. Let $d : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n)$ the kernel of $\Omega^q(\mathbb{R}^n)$ is the elements of $\Omega^q(\mathbb{R}^n)$ such that $d(\Omega^q(\mathbb{R}^n)) = 0$

Now we have the following notion of closed and exact forms:-

Definition 2.9. A *closed* q-form is the kernel of $\Omega^q(\mathbb{R}^n)$

and,

Definition 2.10. A *exact* q-form is the image of $\Omega^{q-1}(\mathbb{R}^n)$

We now have

Proposition 2.11. *All exact forms are closed forms*

Proof. This follows from the fact that $d^2 = 0$ because that implies $d(d(exactform)) = 0$ ■

3. Co homology, DeRham Co homology, Differential Complexes and Smooth Homotopy Invariance

Now we have the following notion of a general differentiation complex by relaxing $dx_i dx_j = -dx_i dx_j$ differential complex:-

Definition 3.1. A direct sum of vector spaces$C = \oplus_{q \in Z} H^q(C)$, indexed by integers is called a *differential complex* if there are homomorphisms ... $\to C^{q-1} \to^d C^q \to^d C^{q+1}$ such that $d^2 = 0$

We thus have the following notion of a *co homology* of C by the fact that all exact forms are closed forms:-

Definition 3.2. The co homology of C is the direct sum of vector spaces $H(C) = \oplus_{q \in Z} H^q(C)$, where $H^q(C) = kerC^q / imC^{q-1}$

And we special to the case of a De-Rham Cohomolgy as:-

Definition 3.3. The $q - th$ de Rham Co homology of $\mathbb{R}^n$ is the vector space $H^q_{DR}(\mathbb{R}^n) = \{closedq - forms\}/\{exactq - forms\}$

Note that all definitions so far work equally well for any open subset U of $\mathbb{R}^n$. Dealing with chains and complexes, we make the following notion of a special class of functions:-

Definition 3.4. A map $f : A \to B$ between two differential complexes is called a *chain* map if it commutes with differential operators of A and B; $fd_A = d_B f$.

We now define a special sequence of vector spaces such that the co homology of that sequence is the empty set:-

Definition 3.5. A sequence of vector spaces ... $\to V_{i-1} \to^{f_{i-1}} V_i \to^{f_i} V_{i+1} \to$ ... is said to be *exact* if for all i the kernel of f_i is equal to the image of the predecessor f_{i-1} (or, the co homology of the Vector spaces V_i is the empty set).

As a special case, we have:-

Definition 3.6. An exact sequence of the form $0 \to A \to B \to C \to 0$ is called a *short exact sequence.*

Proposition 3.7. *Given a short exact sequence of differential complexes,* $0 \to A \to^f B \to^g C \to 0$ *in which the maps* f *and* g *are chain maps, there exists a long sequence groups*$H^q(A) \to^{f^*} H^q(B) \to^{g^*} H^q(C) \to^{d^*} H^{q+1}(C)$ *and such a sequence is exact*

Proof. This follows since f and g are chain maps and hence such maps between kernels (minus a trivial part) are well defined and are exact. ■

Having defined chain maps, we now define the pull back map as:-

Definition 3.8. Given a map $f : M \to N$, the pull back map $f^* : \Omega^k(N) \to \Omega^k(M)$ is defined by $f^*(w)|_p(\nu_1, ...\nu_k) = w|_{f(p}(Df^{-1}\nu_1, ..., Df^{-1}\nu_k)$

Now the pull back map is indeed a chain map:-

Proposition 3.9. *The pullback commutes with the exterior derivative*

Additionally,

Proposition 3.10. *The pullback commutes with the wedge product*

Now we have a rather interesting theorem for which require the following definition:-

Definition 3.11. We say a map $h : M \times I \to N$ is a smooth homotopy if it is smooth. We say that two maps are smoothly homo topic, $f : M \to N$ and $g : M \to N$, if there exists a smooth homotopy h with $h(0, x) = f(x)$ and $h(x, 1) = g(x)$. We say that two manifolds M and N are smoothly equivalent if there exist smooth maps $f : M \to N$ and $g : N \to M$ such that $f \circ g$ and $g \circ f$ are smoothly homo topic to the identity maps on M and N

Finally, we have the following interesting theorem:-

Theorem 3.12. *If two manifolds M and N are smoothly homo topically equivalent then their k^{th} co homology groups are isomorphic for all k.*

Proof. Suppose $f : M \to N$ and $g : M \to N$ as above, with $h : M \times I \to M$ such that $h(x, 0) = x$ and $h(x, 1) = g(f(x))$. Let $\omega \in H^k((g \circ f)(M))$. Then there exists a pullback $h^*(\omega) = \eta + dt \wedge \alpha$ where $\eta \in H^k(M \times I)$, $\alpha \in H^k(M \times I)$, neither η or α contain a dt term t is the coordinate in I. Furthermore, for all $t_0 \in I$,$h(, t_0) : M \to M$ induces the pullback $h^*(., t_0)(\omega) = \eta|_{t=t_0}$. Since η varies smoothly with time, we can apply the fundamental theorem of calculus , $h^*(1, \omega) - h^*(0, \omega) = \eta|_{t=1} - \eta|_{t=t_0} = \int \frac{\partial \eta}{\partial t} dt$. We have two exterior derivative operators which we will temporarily distinguish, d_M and $d_{M \times I}$. Because ω is closed and the pullback commutes with the exterior derivative. $0 = h^*(d_M(\omega)) = d_{M\times I}(h^*(\omega)) = dt \wedge (\frac{\partial \eta}{\partial t} - d_M(\alpha))$. Combining these last two equations yields $h^*(., 1)(\omega) - h^*(., 0)(\omega) = \int_0^1 d_M(\alpha)dt = d_M(\alpha)(\int_0^1 \alpha dt)$ which is an exact form .Therefore, $(g \circ f)^* = h^*(., 1)$ is the identity map on H^k so f^* and g^* are inverses so they are isomorphisms. ∎

4. Poincare's Lemma

We now turn to an important lemma in the study of **De Rham Co homology**, namely, the so-called **Poincare's lemma**.
To begin with, we notice the following fact:-

Proposition 4.1. *$H^q_{DR}(\{x\})$ and $H^q_{DR}(\{x\})$ are isomorphic to $\mathbb{R}$ when $q = 0$ and vanish when $q \neq 0$*

Proof. First lets compute the DeRham Cohomology groups. The one point space is a $0-dimensional$ smooth manifold and all maps are smooth on it. Since it has $0-$dimensions $\Omega^k(\{x\}) = 0$ for $k > 0$ and so only the differentials are the set of all functions of the point x into $\mathbb{R}$ Therefore, $H^0_{DR}(\{x\}) \simeq \mathbb{R}$. To compute the singular co homology groups, we note that there is only one $k - simplex$ into the space $\{x\}$ which is the constant function. So the set of k-chains is then isomorphic to $\mathbb{R}$ i.e, the set of all scalar multiples of this simplex.

This then implies that $S^k(\{x\}, \mathbb{R}) \simeq \mathbb{R}$. Now, consider a k-complex σ where $k > 0$. Now cosnider $d\sigma =$ each σ^i must also be the same since there is only one possible map they can be. Therefore d on odd k-chains is the zero map. This then forces $H_0(\{x\}; \mathbb{R}) \simeq \mathbb{R}$ and $H_0(\{x\}; \mathbb{R}) = 0$ when k is both odd and even. ■

This implies

Theorem 4.2. ***Poincare's lemma:-*** *Every closed form is locally exact.*

Proof. Take a ball around any point p. Every ball is contractible. Now $H^k(R^{n-1}) \simeq H^k(R^{n-1} \times I)$ and thus the theorem follows from the fact that $H^n(R^n) = 0$. ■

5. Mayer-Vietoris

To start of we shall define 3 special types of maps:-

Definition 5.1. Let U and V be open sets whose union forms a manifold. Define the restriction map $res : H^k(U \cup V) \to H^k(U) \oplus H^k(V)$ by $res(w) \equiv (w|_U, w|_V)$

Definition 5.2. Let U and V be open sets whose union forms a manifold.Define the difference map $diff : H^k(U) \oplus H^k(V) \to H^k(U \cap V)$ by $diff(\omega, \eta) \equiv \omega - \eta$

Definition 5.3. Let U and V be open sets whose union forms a manifold. Define the coboundary map $cobd : H^k(U \cap V) \to H^{k+1}(U \cap V)$ by $cobd(\omega)_p \equiv \begin{cases} d(\alpha)_p : p \in U \\ d(\beta)_p : p \in V \end{cases}$ where α is a k-form on U, β is a k-form on V, and $\alpha - \beta = \omega$ on $U \cap V$. Since ω is closed, $d\omega = 0 = d\alpha - d\beta$ on $U \cap V$. Notice, this map is obtained by extending ω to the whole manifold, the applying the co boundary map.

Proposition 5.4. *The co boundary map is well defined up to exact forms.*

With the above notions, we have the following three theorems:-

Theorem 5.5. *The sequence $H^k(U \cup V) \to^{res} H^k(U) \oplus H^k(V) \to^{diff} H^k(U \cap V)$ is exact.*

Proof. Let $\omega \in H^k(U \cup V)$. Then $res(\omega) = (\omega|_U, \omega|_V)$ and therefore, $diff(res(\omega)) = 0$. Thus, $im(res) \subset ker(diff)$.Let $(\omega, \eta) \in ker(diff)$, so $\omega = \eta$ on $U \cap V$. Then define

$$\alpha_p = \begin{cases} \omega_p : p \in U \\ \eta_p : p \in V. \end{cases}$$

Then $res(\alpha) = (\omega, \eta)$ so $ker(diff) \subset im(res)$ ■

Theorem 5.6. *The sequence $H^k(U \cup V) \to^{res} H^k(U) \oplus H^k(V) \to^{diff} H^k(U \cap V) \to^{cobd} H^{k+1}(U \cap V)$ is exact.*

Proof. Let $(\omega, \eta) \in H^k(U) \oplus H^k(V)$. Then $cobd(diff(\omega, \eta)) = cobd(\omega - \eta)$. We can chose $\alpha = \omega$, $\beta = \eta$ with α and β defined as in the definition of the co-boundary map. Therefore ,

$cobd(\omega)_p = \begin{cases} d(\omega)_p : p \in U \\ d(\eta)_p : p \in V. \end{cases}$ Now since ω and η are closed,$cobd(\omega) = 0$. Thus,$im(diff) \subset ker(cobd)$. Next, let $\omega \in ker(cobd)$. Since $\omega \in ker(cobd)$, $d\alpha = 0$ and $d\beta = 0$. Therefore, $(\alpha, \beta) \in H^k(U) \oplus H^k(V)$. Then $diff(\alpha, \beta) = \alpha - \beta = \omega$. Thus, $ker(cobd) \subset im(diff)$ ■

Theorem 5.7. *The sequence $H^k(U \cap V) \to^{cobd} H^{k+1}(U \cup V) \to^{res} H^{k+1}(U) \oplus H^{k+1}(V)$ is exact.*

Proof. Let $\omega \in U \cap V$,α, β, as in the definition for the co boundary map. Then $res = (cobd(\omega)) = (d\alpha, d\beta)$ which is exact be definition. Then, $im(cobd) \subset ker(res)$. Finally , let $\omega \in ker(res)$. Then $\alpha - \beta \in H^k(U \cap V)$. By definition, $cobd(\alpha - \beta)_p = \begin{cases} (d\alpha)_p : p \in U \\ (d\beta)_p : p \in V \end{cases}$
But $d(\alpha) = \omega$ and $d(\beta) = \omega|_V$ so $ker \subset im(cobd)$. ■

Finally, we have

Definition 5.8. The Mayer-Vietrois sequence is the sequence $0 \to H^0(U \cup V) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(U \cup V) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V)$

Theorem 5.9. *The Mayer-Vietoris sequence is exact.*

6. Stokes' Theorem

We now focus on an important notion of mathematics namely- Stokes' theorem. We shall start of with a notion of a boundary:-

Definition 6.1. The standard k-simplex Δ^k is defined to be
$\Delta^k = \begin{cases} \{(x_1, ..., x_k) \in \mathbb{R}^k : \sum_{i=1}^k x_i \leq 1, \forall x_i \geq 0\} \\ \{0\}. \end{cases}$ For a differentiable manifold M, a differentiable (or smooth) singular k-simplex is a map $\sigma : \Delta^k \to M$ which extends to a C^∞-mapping on a neighborhood of Δ^k. We will simply call this a k-simplex.

Definition 6.2. For a differentiable manifold M , a differentiable(or smooth) singular $k-$simplex which we will denote by $S_k(M; \mathbb{R})$. An element of this vector space is of the form $\sum_{i=1}^m a_i \sigma_i$ and is called a k-chain. The set of all chains is denoted by $S_*(M; \mathbb{R})$.

Definition 6.3. We can define a boundary operator ∂ on its simplexes. Suppose we have some k-simplex σ. We define σ^i to be its i-th face i.e, we restrict σ onto the i-th face of the standard k-simplex. Then we define the boundary operator as $\partial\sigma = \sum(-1)^i\sigma^i$ giving us a $(k-1)$ simplex, the alternating sign keeping track of orientation.

We now have the following important property:-

Proposition 6.4. $\partial \circ \partial = 0$

Proof. This follows from the way ∂ is defined through the alternating sum. ■

We now define integrals as:-

Definition 6.5. Let M be a smooth manifold. Suppose σ is an n-simplex on M and ω is a differential n-form on M. Define the integral of ω over σ as:-
When $n = 0$, $\int_\sigma \omega = \omega(\sigma(0))$
When $n \geq 1$, $\int_\sigma \omega = \int_{\Delta^n} \sigma^*(\omega)$.

We also require the following notion of supports:-

Definition 6.6. Let ω be a smooth $k-$form on X, a k-dimensional manifold with boundary. The support of ω is defined as the closure of the set of points where $\omega(x) \neq 0$; we say that ω is compactly supported if the support is compact.

Now by some additional machinery, one can prove:-

Theorem 6.7. *If $f : X \to Y$ is an orientation-preserving diffeomorphism, then $\int_X \omega = \int_Y f^*\omega$ for every compactly supported , smooth $k - form$ on X $(k = dimX = dimY)$*

Finally, we now present the generalized Stokes' Theorem as:-

Theorem 6.8. *Suppose that X is any compact oriented k-dimensional manifold with boundary , so ∂X is a $k-1$ dimensional manifold with the boundary orientation. If ω is any smooth $k-1$ form on X then, $\int_{\partial X} \omega = \int_X d\omega$*

Proof. Both sides of the equation are linear in ω, so we may assume ω to have a compact support contained in the image of a local diffeomorphism $h : U \to X$, where U is an open subset of $\mathbb{R}^k$ or H^k. The rest of the proof proceeds y considering the theorem by cases.Our first case will consist of a neighborhood of X locally diffeomorphic to an open subset of $\mathbb{R}^k$, where we expect both sides of the theorem to evaluate to 0 since there is no boundary over which to evaluate the differential form. Our second case will consist of a neighborhood of X locally diffeomorphic to an open subset of H^k. First, we assume U is open in $\mathbb{R}^k$.Then $\int_{\partial X} \omega = 0$ and $\int_X d\omega = \int_U h^*(d\omega) = \int_U d\nu$, where $\nu = h^*\omega$. Since ν is a $(k-1)$-form in $k-$space, it can be expressed as $\nu = \sum_i^k (-1)^{i-1} f_i dx_1..\hat{dx_i}..dx_k$ where $\hat{dx_i}$ means the term dx_i is omitted. Then $d\nu = (\sum_i \frac{\partial f}{\partial x_i})dx_1 \wedge ... \wedge dx_k$ and $\int_{\mathbb{R}^k} d\nu = \sum_i \int_{\mathbb{R}^k} \frac{\partial f}{\partial x_i} dx_1...dx_k$. Integrate the i^{th} term first with respect to x_i:- $\int_{\mathbb{R}^{k-1}}(\int_{-\infty}^{+\infty} \frac{\partial f}{\partial x_i})dx_1...dx_i..dx_k$. Of course, $\int_{-\infty}^{+\infty} \frac{\partial f}{\partial x_i} dx_i$ is the function of $x_1, ...\hat{x_i}, ...x_k$ that maps to any $(k-1)$ tuple $(b_1, ..\hat{b_i}, ..\hat{b_k})$ the number $\int_{-\infty}^{+\infty} g'(t)dt$, where $g(t) = f_i(b_i, ..., t, ...b_k)$. Since ν has a compact support, g vanishes outside any sufficiently large interval $(-a, a)$ in $\mathbb{R}^1$. Therefore, the Fiundamental Theorem of Calculus implies $\int_{-\infty}^{+\infty} g'(t)dt = \int_{-a}^{a} g'(t)dt = g(a) - g(-a) = 0$. Thus $\int_X d\omega = 0$. We now take a look at the second case of the Fundamnetal theorem. When $U \subset H^k$, we repeat the above process for every term except the last term . Since the boundary of H^k is the set where $x_k = 0$, the last integral is $\int_{\mathbb{R}^{k-1}}(\int_0^\infty \frac{\partial f}{\partial x_k} dx_k)dx_1..dx_k$. Now the compact support implies that f_k vanishes if x_k is outside some large interval $(0, a)$, but although $f_k(x_1, ..., x_{k-1}, a) = 0$, $f_k(x_1, ...x_{k-1}, 0) \neq 0$. Thus applying the Fundamental Theorem of Calculus, we obtain $\int_X d\omega = \int_{\mathbb{R}} -f_k(x_1, ..., x_k, 0)dx_1..dx_{k-1}$. On the other hand, $\int_{\partial X} \omega = \int_{\partial H^k} \nu$. Since $x_k = 0$ on ∂H^k, $dx_k = 0$ as well. Consequently, if $i < k$, the form $(-1)^{i-1} f_i dx_1 \wedge .. \wedge \hat{dx_i} \wedge ... \wedge dx^k$ restricts to 0 on ∂H^k. So the restriction of ν to ∂H^k is $(-1)^{k-1} f(x_1, ..., x_k, 0)dx_1 \wedge .. \wedge dx_{k-1}$ whose integral over ∂H^k is therefore $\int_{\partial X} \omega$. Now ∂H^k is diffeomorphic to $\mathbb{R}^{k-1}$ under the map $(x_1, x_2, ..x_k) \to (x_1, ..., x_{k-1}, 0)$ but this diifemorphism does not always carry the usual orientation of $\mathbb{R}^{k-1}$ to the boundary orientation of ∂H^k. Let $e_1, ..e_k$ be the standard ordered basis for $\mathbb{R}^k$, so $e_1, ..., e_k$ is the standard ordered basis for $\mathbb{R}^{k-1}$. Since H^k is the upper half-space, the outward unit-normal to ∂H^k is $e_k = (0, ...0, -1)$. Thus in the boundary orientation, of ∂H^k, the sign of the ordered basis $\{-e_k, e_1, ..e_{k-1}\}$ is the standard orientation of H^k. The latter is easily seen to be $(-1)^k$, so the usual diffeomorphism $\mathbb{R}^k \to \partial H^k$ changes orientation by the factor $(-1)^k$. The result is the following formula $\int_{\partial X} \omega = \int_{\partial H^*} (-1)^{k-1} f_k(x_1, ..., x_{k-1})dx_1...dx_{k-1} = (-1)^k \int_{\mathbb{R}^{k-1}} (-1)^{k-1} f_k(x_1, ..., x_{k-1}, 0)dx_1...dx_{k-1} = -\int_{\mathbb{R}^{k-1}} f_k(x_1, ...x_{k-1}, 0)dx_1..dx_{k-1}$. This is exactly the formula we derived for $\int_X d\omega$. Since both sides of the theorem are evaluated to the same value, they are equivalent, which means the theorem holds for subset of H^k. ■

Stokes theorem then implies that this collection of homomorphisms $\{l_k\}$ commutes with the exterior derivatives and boundary operators, and so it is a co-chain map.

Definition 6.9. Let U be some smooth manifold. We denote the induced homeomorphisms of the k^{th} co homology groups vy $DR_k(U) : H^k_{DR}(U) \to H^k(U)$. We call the collection $\{DR_k(U)\}$ the DeRham homeomorphism on U or simply $DR(U)$ and if each $Dr_k(U)$ is an isomorphism then we say $DR(U)$ is an isomorphism.

In order to make use of the Mayer-Vietrois sequences we need the following lemmaa.

Lemma 6.10. *Let $\psi : M \to N$ be a C^∞ function between smooth manifold M and N. Then this induces two pullbacks on the de rham and singular cohomology groups: $\psi^*_k : H^*_{DR}(N) \to H^*_{DR}(M)$ and $\psi^*_k : H^k(N;\mathbb{R}) \to H^k(M;\mathbb{R})$ which commutes with the kth De rham homomorphism on N and M.*

Proof. This is a consequence of the calculation for $k - form$ ω on N and a $k-$simplex σ in M.$\int_\sigma \psi^*\omega = \int_{\Delta^k} \sigma^*\psi^*\omega = \int_{\Delta^k}(\psi\sigma)^*\omega = \int_{\psi\sigma}\omega$ ■

7. DeRham's Theorem

We now turn to a central theorem of this paper namely, **DeRham's theorem**. In order to do so, we shall first look at 3 important lemmas.

Lemma 7.1. *If $U \subset \mathbb{R}^n$ and U is convex, then $DR(U)$ is an isomorphism.*

Proof. Because the de Rham and singular cohomolgy are homotopic invariants and U is homotopy equivalent to $\{x\}$, from the theorem that $H^q_{DR}(\{x\})$ and $H^q_{DR}(\{x\})$ are isomorphic to $\mathbb{R}$ when $q = 0$ and vanish when $q \neq 0$, we have that the qthde rham and qth singular cohomolgy groups vanish for $q \neq 0$ and are isomorphic to $\mathbb{R}$ when $q = 0$. Now consider the case when $q = 0$, in this case $H^0_{DR}(U)$ is a one-dimensional vector space of constant functions on U since the only $0 - forms$ ω such that $d\omega = 0$ are the constant functions. Similarly, the only $0-$simplexes are the maps from $\{0\} \to U$. Since by definition the integral of a $0-$form over a $0-$simplex is just a value of the form at that point the simplex maps 0 into. We find that the de Rham homomorphism can't be the zero map and hence must be an isomorphism. ■

Before moving on to the second lemma, we shall briefly study partitions of unity:-

Definition 7.2. A partition of unity on M is a collection $\{\psi_i | i \in I\}$ of C^∞ functions on M such that

- for each $i \in I$, $\psi_i \geq 0$ on M.
- the collection of supports $\{supp\psi_i | i \in I\}$is locally finite
- $\sum_{i \in I} \psi_i(p) = 1 \forall p \in M$.

Theorem 7.3. *For any smooth manifold M and any open cover $\{U_\alpha\}$ on M. There exists a countable partition of unity $\{\psi_i | i \in I\}$ subordinate to the open cover and for each $i \in I$, $supp(\psi_i)$ is compact.*

We now proceed with the second lemma we wished to establish:-

Lemma 7.4. *Let M be a smooth manifold. Given a basis $\boldsymbol{B}$ on M, there exists a countable open cover $\{U_i\}$ of M such that each U_i can be written as the finite union of basis elements and if $U_i \cap U_j = \emptyset$ then $i \neq j \pm 1$.*

Proof. Let $\{V_i\}$ be an open cover of M and let ψ_i be a partition of unity subordinate to this cover. Let us define a new C^∞function called α on M by setting $\alpha = \sum_{i=1}^{\infty} i\psi_i$. Now suppose $p \in M$ but $p \notin \cup_{i=1}^{N} supp(\psi_i)$. Then $\alpha(p)\sum_{i=1}^{\infty} i\psi_i(p) = \sum_{i=N+1}^{\infty} i\psi_i(p) > \sum_{i=N+1}^{\infty} N\psi_i(p) \geq N\sum_{i=1}^{\infty}\psi_i(p) = N$. Therefore, $\alpha^{-1}([0, N])$ is compact for each $N \in \mathbb{N}$. Because α is C^∞, it is continuous and so $\alpha^{-1}(a, b)$ is open and must have compact closure for any open interval (a, b). Define the sets $A_i = \alpha^{-1}(i+1/4, i+7/4)$ and $A'_i = \alpha^{-1}(i, i+2)$ for $i = -1, 0, 1, 2, ...$Now for each point$x \in \overline{A_i}$, take a basis element $B_x \in \mathbf{B}$ that contains x but is contained in A'_i and form the open cover $\{B_x\}$ of $\overline{A_i}$. Since $\overline{A_i}$ is compact,there is a finite sub collection of $\{B_x\}$ that covers $\overline{A_i}$. Take U_i as the union of this finite sub collection and because we choose each B_x to be contained within A'_i we find that $A_i \subseteq U_i \subseteq A'_i$. Therefore, if $U_i \cap U_j = \emptyset$ ■

Lemma 7.5. *Let M be a smooth manifold. Suppose $M = \cup_{i=1}^{k} U_i$ where $k \in \mathbb{N}$ and U_i are open. If DR is an isomorphismon each of the sets $\{U_i\}$ and each finite intersection of theses sets, then $DR(M)$ is an isomorphism*

Finally, we now have the **DeRham Theorem** :-

Theorem 7.6. *Let M be any smooth manifold. Then $DR(M)$ is an isomorphism*

Proof. First we show that if $DR(U)$ is an isomorphism for each U in some countable collection, it is an isomorphism for the disjoint union. Let $\{M_j\}$ be a countable collection of manifolds where $DR(M_j)$ is an isomorphism for each j. Let $M = \Pi_j M_j$ be the disjoint union of these manifolds. Denote the inclusion maps by $i_J : M_j \to M$. Then the map $i = (i_1, i_2, ...)$ includes isomorphisms between $\oplus_j H^k_{DR}$ and $H^k_{DR}(M)$ as well as $\oplus_j H^k(M_j; \mathbb{R})$ and $H^k(M; \mathbb{R})$. For each k,$\oplus_j DR_k(M_j)$ is an isomorphism between the direct product of the de rham and singular co homology groups and so, $DR_k(M)$ must be an isomorphism. Now, given $\{U_i\}$ be an open cover, let $U_{odd} = \cup U_{2k+1}$, $U_{even} = \cup U_{2k}$ and $U_{int} = \cup(U_k \cap U_{k-1})$ for $k \in \mathbb{N}$. Notice that $U_{odd} \cap U_{even} = U_{int}$. So, if $DR(U_{even}), DR(U_{odd})$ and $DR(U_{int})$ were isomorphic then so would $DR(M)$.Therefore we need only show that DR is isomorphic on each $U_k$and $U_k \cap U_{k+1}$ since $U_{odd}, U_{even}, U_{int}$ are disjoint unions of these sets. It is sufficient to simply show that M has a basis with the property that DR is isomorphic on each basis element and on each finite intersections of basis elements. This is because each $U_k$can be written as the union of finitely many intersections of these basis elements and $U_k \cap U_{k+1}$ can be written as the union of finitely many intersections of these basis elements. This would imply that DR is an isomorphism on these sets. If M is an open subset of $\mathbb{R}^n$ for some integer n. Then M does have a basis with this property . Simply note that M would then have a basis of $n - balls$ and since the intersection of balls is still convex, DR is isomorphic on these intersections. When M is a general smooth manifold of dimension n, we can simply take a basis of domain charts. Each of these domains are diffeomorphic to an open subset of $\mathbb{R}^n$ and hence DR is isomorphic on these intersections. When M is a general smooth manifold of dimension n, we can simply take a basis of domain charts. Each of these domains are diffeomorphic to an open subset of $\mathbb{R}^n$ and hence DR is isomorphic on this basis. ■

8. Acknowledgments

We would like to like to acknowledge Simon Rubinstein-Salzedo and Sarah Bauer for guiding and supporting us to write this paper. We also appreciate and acknowledge the discussions with our peers.

9. Conclusion

To conclude, in this expository paper builds tools to understand 3 critical mathematical results and tools namely; the generalized Stokes' theorem, the De-Rham theorem and the Mayer-Veitoris sequence.

References

[1] Ben Andrews. Lecture 15: de rham cohomology. Lecture notes, Australian National University. Construction of de Rham cohomology via integration of differential forms and Stokes' Theorem.
[2] Chad Berkich. Homology, cohomology, and the de rham theorem. REU paper, University of Chicago, 2021. Includes Poincaré's Lemma, Stokes' Theorem for chains, and proof of de Rham theorem.
[3] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology*. Springer-Verlag, 1982.
[4] Georges de Rham. *Variétés différentiables: Formes, courants, formes harmoniques*. Hermann, 1955.
[5] John M. Lee. *Introduction to Smooth Manifolds*. Springer, 2nd edition, 2013.
[6] Rick Presman. The generalized stokes' theorem. REU paper, University of Chicago, 2012. Proof of the generalized Stokes' Theorem via differential forms and integration on manifolds with boundary.
[7] Benjamin Szczesny. De rham's theorem. AMSI report, 2013. Review of differential forms, singular cohomology, Mayer–Vietoris, and proof of de Rham theorem.
[8] Loring W. Tu. *An Introduction to Manifolds*. Springer, 2011.
[9] Unknown. Chapter 6: De rham cohomology groups. Lecture notes, University of Science and Technology of China. Definition and interpretation of de Rham cohomology groups as.

Euler Circle, Mountain View, CA 94040
Email address: shahaarav103@gmail.com