GROUP ACTIONS AND MÖBIUS SYMMETRIES IN HYPERBOLIC TESSELLATIONS

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Abstract. This expository paper explains tessellations of the hyperbolic plane from a differential–geometric viewpoint. We review the upper half–plane ($\mathbb{H}$) and Poincaré disk ($\mathbb{D}$) models and the Cayley transform linking them. We prove that orientation–preserving isometries are precisely $\mathrm{PSL}(2, \mathbb{R})$ acting by Möbius/Blaschke maps and show these maps are conformal and hyperbolic isometries (hence length– and area–preserving). Using group actions of discrete subgroups (Fuchsian groups), we construct tessellations from fundamental domains; the modular group serves as a guiding example. Via Gauß—Bonnet and reflection/triangle–group constructions we derive the existence criterion for regular $\{p, q\}$ tilings, namely $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, and relate angle defect to area. Throughout the article, we highlight geodesics, curvature $K \equiv -1$, and the role of isometries in organizing hyperbolic tilings.

1. Introduction

Hyperbolic geometry emerged in the nineteenth century from the realization that Euclid's parallel postulate could be replaced, producing a logically consistent geometry of constant negative curvature. This geometry exhibits phenomena entirely unlike those of the Euclidean plane: triangles have angle sum less than π, circles grow exponentially in size with radius, and infinitely many distinct regular tessellations are possible.

The paper begins with a review of the main models of the hyperbolic plane and the algebraic structures underlying their isometry groups. We then introduce group actions, followed by the role of discrete symmetries in generating tessellations. Finally, we present the Gauss–Bonnet theorem and its application to regular tilings, before concluding with open problems and possible extensions.

2. Preliminaries

2.1. Models of the Hyperbolic Plane.

There are many models of the hyperbolic plane. We define two that we use throughout.

Definition 2.1 (Upper Half-Plane)**.** The *upper half-plane* $\mathbb{H}$ is the set of complex numbers $z = x + iy$ such that $\Im(z) > 0$.

Definition 2.2 (Boundary of $\mathbb{H}$)**.** The (ideal) boundary of $\mathbb{H}$ is $\partial\mathbb{H} := \mathbb{R} \cup \{\infty\}$.

In the upper half-plane model, hyperbolic geodesics are precisely the vertical Euclidean lines and the Euclidean semicircles whose centers lie on $\mathbb{R}$. Equivalently, they are exactly those Euclidean lines and circles that intersect $\partial\mathbb{H}$ at right angles; note that not every curve orthogonal to $\partial\mathbb{H}$ in the Euclidean sense is a geodesic.

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Definition 2.3 (Metric on $\mathbb{H}$)**.** The hyperbolic metric (first fundamental form) on $\mathbb{H}$ is

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Remark 2.4. This metric is conformal to the Euclidean metric, so Euclidean and hyperbolic angles agree. Its Gaussian curvature is constant $K \equiv -1$, and the area element is $dA_{\mathbb{H}} = \dfrac{dx\,dy}{y^2}$. For a region $A \subset \mathbb{H}$,

$$\text{Area}_{\mathbb{H}}(A) = \iint_A \frac{dx\,dy}{y^2}.$$

Another common model is the Poincaré disk model of the hyperbolic plane.

Definition 2.5 (Poincaré Disk Model)**.** The Poincaré disk model, denoted $\mathbb{D}$, is

$$\mathbb{D} = \{\, z \in \mathbb{C} : \ |z| < 1 \,\},$$

the open unit disk in the complex plane.

The ideal boundary of $\mathbb{D}$ is the unit circle $\partial\mathbb{D} = \{\, |z| = 1 \,\}$. Hyperbolic geodesics in $\mathbb{D}$ are the Euclidean diameters and circular arcs that meet $\partial\mathbb{D}$ orthogonally; between any two points in $\mathbb{D}$ there is a unique geodesic segment.

Definition 2.6 (Metric on $\mathbb{D}$)**.** The hyperbolic metric on $\mathbb{D}$ is

$$ds^2 = \frac{4\,(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}$$

In complex form, it can be represented as

$$ds^2 = \frac{4\,|dz|^2}{(1 - |z|^2)^2}.$$

This metric is also conformal and has $K \equiv -1$. The hyperbolic distance between $z_1, z_2 \in \mathbb{D}$ is given by

$$\cosh d_{\mathbb{D}}(z_1, z_2) \;=\; 1 + \frac{2\,|z_1 - z_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)}.$$

For polygons in either model, the area equals the angle defect; for a geodesic triangle with interior angles α, β, γ one has Area $= \pi - (\alpha + \beta + \gamma)$ (a special case of Gauß–Bonnet that we use later). We prove this metric in the section on Connections to Differential Geometry.

2.2. Symmetry Groups, Projectivization, and the Cayley Map.

Definition 2.7 (Lie group)**.** A *Lie group* is a smooth manifold G equipped with a group structure such that the multiplication map $G \times G \to G$ and inversion map $G \to G$ are smooth. Lie groups serve as the bridge between algebraic symmetry (via groups) and geometric structure (via manifolds).

Remark 2.8. Classical matrix groups such as $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{R})$ are Lie groups when given the submanifold topology from $\mathbb{R}^{n \times n}$.

Definition 2.9 (General and special linear groups)**.** The *general linear group* $\mathrm{GL}(2, \mathbb{R})$ consists of all 2×2 real matrices with nonzero determinant. The *special linear group* $\mathrm{SL}(2, \mathbb{R})$ is the subgroup with determinant 1.

Remark 2.10. Matrix multiplication makes $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R})$ Lie groups. Elements of $\mathrm{SL}(2,\mathbb{R})$ are linear transformations that preserve oriented area in $\mathbb{R}^2$.

Definition 2.11 (Projective linear groups)**.** The *projective linear group* is

$$\mathrm{PGL}(2,\mathbb{R}) \;:=\; \mathrm{GL}(2,\mathbb{R})/\{\lambda I : \lambda \in \mathbb{R}^\times\},$$

and the *projective special linear group* is

$$\mathrm{PSL}(2,\mathbb{R}) \;:=\; \mathrm{SL}(2,\mathbb{R})/\{\pm I\}.$$

where I is the 2×2 identity matrix.

Definition 2.12 (Real Möbius action on $\widehat{\mathbb{R}}$ and $\mathbb{H}$)**.** A matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts on $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and on $\mathbb{H}$ by

$$A \cdot z \;=\; \frac{az+b}{cz+d},$$

whenever the right-hand side is defined, with the usual conventions at $cz + d = 0$ or $z = \infty$.

Proposition 2.13. *The formula* $z \mapsto \dfrac{az+b}{cz+d}$ *depends only on the class of* A *in* $\mathrm{PGL}(2,\mathbb{R})$*. Its restriction to* $\mathrm{SL}(2,\mathbb{R})$ *descends to an injective homomorphism* $\mathrm{PSL}(2,\mathbb{R}) \hookrightarrow \mathrm{Homeo}^+(\mathbb{H})$*.*

Proof. If A is replaced by λA with $\lambda \neq 0$, the numerator and denominator are both multiplied by λ, leaving the quotient unchanged; hence the action factors through $\mathrm{PGL}(2,\mathbb{R})$. If $A \in \mathrm{SL}(2,\mathbb{R})$ acts trivially, then $az + b = z(cz + d)$ for all z, forcing $c = 0$, $a = d$, and $b = 0$; with $\det A = 1$ this gives $A = \pm I$, which is trivial in $\mathrm{PSL}(2,\mathbb{R})$. ■

Proposition 2.14. *Every orientation-preserving isometry of* $\mathbb{H}$ *is given by*

$$\varphi(z) \;=\; \frac{a\,z+b}{c\,z+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}),$$

unique up to an overall sign.

Sketch. One checks directly that these fractional linear maps preserve the metric $ds^2 = (dx^2 + dy^2)/y^2$ and that any orientation-preserving isometry arises in this way, modulo $\pm I$. ■

Proposition 2.15 (Isometry group of the hyperbolic plane)**.** *The full isometry group of the hyperbolic plane satisfies*

$$\mathrm{Isom}(\mathbb{H}) \;\cong\; \mathrm{PGL}(2,\mathbb{R}),$$

while the subgroup of orientation-preserving isometries is

$$\mathrm{Isom}^+(\mathbb{H}) \;\cong\; \mathrm{PSL}(2,\mathbb{R}) \;=\; \mathrm{SL}(2,\mathbb{R})/\{\pm I\}.$$

Proof. Proof sketch ■

Definition 2.16 (Cayley transform)**.** The *Cayley map* $\Phi : \mathbb{H} \to \mathbb{D}$ and its inverse are

$$\Phi(z) = \frac{z-i}{z+i}, \qquad \Phi^{-1}(w) = i\,\frac{1+w}{1-w}.$$

Remark 2.17. Φ is a biholomorphism sending $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ to $\partial\mathbb{D} = \{\,|w| = 1\,\}$; vertical lines and semicircles orthogonal to $\mathbb{R}$ map to Euclidean diameters and circular arcs orthogonal to $\partial\mathbb{D}$.

Proposition 2.18 (Cayley isometry)**.** *Φ is an isometry between $(\mathbb{H}, ds^2_{\mathbb{H}})$ and $(\mathbb{D}, ds^2_{\mathbb{D}})$, i.e.*

$$\Phi^*\left(\frac{4\,|dw|^2}{(1-|w|^2)^2}\right) = \frac{|dz|^2}{\Im(z)^2}.$$

Proof. Compute $\Phi'(z) = \dfrac{2i}{(z+i)^2}$ and $1 - |\Phi(z)|^2 = \dfrac{2\,\Im(z)}{|z+i|^2}$. Then

$$\frac{4\,|\Phi'(z)|^2}{\left(1-|\Phi(z)|^2\right)^2} = \frac{4\,\dfrac{4}{|z+i|^4}}{\left(\dfrac{2\,\Im(z)}{|z+i|^2}\right)^2} = \frac{1}{\Im(z)^2},$$

which is the density of $ds^2_{\mathbb{H}}$. ■

Corollary 2.19 (Conjugacy of isometry groups)**.** *Conjugation by Φ yields an isomorphism of groups*

$$\mathrm{PSL}(2,\mathbb{R}) \xrightarrow{\cong} \mathrm{Isom}^+(\mathbb{D}), \qquad \gamma \longmapsto \Phi \circ \gamma \circ \Phi^{-1}.$$

3. Groups and Group Actions

3.1. Definition of Group Actions.

Definition 3.1 (Group Action)**.** Let G be a group and X a set. A *group action* of G on X is a map

$$G \times X \longrightarrow X, \qquad (g, x) \mapsto g \cdot x,$$

such that for all $g, h \in G$ and all $x \in X$:

(1) $e \cdot x = x$, where e is the identity of G.
(2) $(gh) \cdot x = g \cdot (h \cdot x)$.

In our context, X will be one of the hyperbolic models $\mathbb{H}$ or $\mathbb{D}$, and G acts by isometries.

3.2. Möbius Transformations.

Definition 3.2 (Möbius transformation)**.** A *Möbius transformation* is a map

$$z \;\mapsto\; \frac{a\,z+b}{c\,z+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$

Let Mob($\mathbb{H}$) denote the set of all Möbius transformations of $\mathbb{H}$.

Theorem 3.3. *Let $\gamma \in \mathrm{Mob}(\mathbb{H})$. Then γ is conformal.*

Proof. Let $\gamma(z) = \dfrac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Its complex derivative is $\gamma'(z) = \dfrac{ad-bc}{(cz+d)^2} = \dfrac{1}{(cz+d)^2}$. If $cz + d = 0$ then $z = -d/c \in \mathbb{R}$, which cannot occur for $z \in \mathbb{H}$. Hence $\gamma'(z) \neq 0$ on $\mathbb{H}$, so γ is holomorphic with nonzero derivative and therefore conformal. ■

We now extend Möbius transformations to $\mathbb{D}$.

Theorem 3.4. *Every Möbius transformation of the Poincaré disk can be written uniquely as*

$$f(z) \;=\; e^{i\theta}\,\frac{z-a}{1-\overline{a}\,z}, \qquad a \in \mathbb{D},\;\; \theta \in \mathbb{R}.$$

Proof. Use the Cayley transform $\phi : \mathbb{D} \to \mathbb{H}$ given by $\phi(z) = i\,\dfrac{1+z}{1-z}$, with inverse $\phi^{-1}(w) = \dfrac{w-i}{w+i}$. Let $\gamma(w) = \dfrac{aw+b}{cw+d}$ be a real Möbius transformation with $ad - bc = 1$. Define

$$f(z) \;=\; \phi^{-1}\big(\gamma(\phi(z))\big) \;=\; \frac{\alpha z + \beta}{\overline{\beta}\, z + \overline{\alpha}},$$

where

$$\alpha = \frac{a+d}{2} + \frac{i}{2}(c-b), \qquad \beta = \frac{a-d}{2} + \frac{i}{2}(b+c),$$

which satisfy $|\alpha|^2 - |\beta|^2 = 1$. Write $\alpha = re^{i\phi}$ and set $\theta = 2\phi$, so $\alpha/\overline{\alpha} = e^{i\theta}$. Let

$$a := -\frac{\beta}{\alpha} \qquad \Big(\Rightarrow\ |a| = \tfrac{|\beta|}{|\alpha|} < 1\Big).$$

Dividing numerator and denominator by $\overline{\alpha}$ gives

$$f(z) \;=\; \frac{\frac{\alpha}{\overline{\alpha}}\, z + \frac{\beta}{\overline{\alpha}}}{\frac{\overline{\beta}}{\overline{\alpha}}\, z + 1} \;=\; \frac{e^{i\theta} z - e^{i\theta} a}{1 - \overline{a}\, z} \;=\; e^{i\theta}\,\frac{z-a}{1-\overline{a}\, z}.$$

Uniqueness of a, θ follows from the uniqueness of the parameters α, β with $|\alpha|^2 - |\beta|^2 = 1$. ■

The formula above also makes it clear that these disk Möbius maps are conformal. We now record metric invariance (and hence area invariance).

Theorem 3.5. *Let $A \subset \mathbb{D}$ and let $f(z) = e^{i\theta}\dfrac{z-a}{1-\overline{a}\, z}$. Then* $\mathrm{Area}_{\mathbb{D}}\big(f(A)\big) = \mathrm{Area}_{\mathbb{D}}(A)$, *equivalently f is an isometry of $(\mathbb{D}, ds^2_{\mathbb{D}})$ with*

$$ds^2_{\mathbb{D}} \;=\; \frac{4\,|dz|^2}{(1-|z|^2)^2}.$$

Proof. For $w = f(z)$,

$$|f'(z)|^2 = \frac{(1-|a|^2)^2}{|1-\overline{a}z|^4}, \qquad 1 - |f(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}.$$

Therefore

$$\frac{4\,|f'(z)|^2}{(1-|f(z)|^2)^2}\,|dz|^2 \;=\; \frac{4}{(1-|z|^2)^2}\,|dz|^2,$$

and so $f^*(ds^2_{\mathbb{D}}) = ds^2_{\mathbb{D}}$. Thus f is an isometry, and by change of variables $\mathrm{Area}_{\mathbb{D}}(f(A)) = \mathrm{Area}_{\mathbb{D}}(A)$. ■

By a similar method (or via the Cayley map Φ), using the upper half–plane metric, one proves the same result for $\mathbb{H}$. In particular, the identity $f^*(ds^2_{\mathbb{D}}) = ds^2_{\mathbb{D}}$ shows f preserves lengths and areas.

3.3. **Fuchsian Groups.**

Definition 3.6 (Fuchsian Group)**.** A *Fuchsian group* Γ is a discrete subgroup $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ acting properly discontinuously on $\mathbb{H}$: for every compact $K \subset \mathbb{H}$, the set $\{\gamma \in \Gamma :\ \gamma K \cap K \neq \emptyset\}$ is finite.

Definition 3.7 (Geodesic Arcs)**.** Let $a, b \in \mathbb{D} \cup \partial D$. Then there exists a unique geodesic passing through a, b. We call the part of this geodesic that connects a and b as an arc or segment of a geodesic.

Definition 3.8 (Hyperbolic Polygons)**.** A hyperbolic polygon is a region $\in \mathbb{D}$ bounded by a sequence of points $(v_1, v_2, v_3, \ldots, v_n)$, with geodesic arcs connecting each $v_i \to v_{i+1} \pmod n$.

A hyperbolic polygon is considered *regular* if all sides have equal hyperbolic length, all interior angles are equal, and the polygon is cyclic. That is, there exists a hyperbolic isometry which acts as a rotation mapping the polygon to itself.

Furthermore, note that the interior angles of hyperbolic polygons are measured within the hyperbolic metric, not the Euclidean metric. We can now use this definition to define a tessellation.

Definition 3.9 (Tessellation)**.** A tessellation is a collection of hyperbolic polygons $\{P_i\}$ satisfying $\bigcup_i \operatorname{int}(P_i) = \mathbb{D}$, $\operatorname{int}(P_i) \cap \operatorname{int}(P_j) = \emptyset$, for $i \neq j$, and each P_i is a hyperbolic polygon.

Essentially, a tessellation is a collection of polygons that tile the whole plane with none intersecting each other.

Definition 3.10 (Fundamental Domain)**.** A *fundamental domain* for Γ is a closed region $D \subset \mathbb{H}$ such that

$$\bigcup_{\gamma \in \Gamma} \gamma(D) \;=\; \mathbb{H}, \quad \text{and the interiors of } D \text{ and } \gamma(D) \text{ are disjoint for } \gamma \neq e.$$

The images $\{\gamma(D) : \gamma \in \Gamma\}$ then tessellate $\mathbb{H}$.

Remark 3.11. Because Γ acts properly discontinuously, the translates $\{\gamma(D)\}_{\gamma \in \Gamma}$ are pairwise interior–disjoint and cover $\mathbb{H}$, hence form a tessellation. Since elements of $\mathrm{PSL}(2, \mathbb{R})$ are hyperbolic isometries, geodesic edges of D map to geodesics, so tiles meet along geodesic arcs. When D is a regular p–gon with interior angle $2\pi/q$ and side-pairings generate Γ, the orbit $\{\gamma(D)\}$ is called a regular $\{p, q\}$ tessellation.

Example 3.12 (Modular Group)**.** The subgroup $\mathrm{PSL}(2, \mathbb{Z})$ generated by

$$z \mapsto z + 1, \qquad z \mapsto -1/z$$

is a classic Fuchsian group. A standard fundamental domain is

$$D = \left\{ z \in \mathbb{H} : \ |z| \geq 1, \ |\Re z| \leq \tfrac{1}{2} \right\},$$

whose $\mathrm{PSL}(2, \mathbb{Z})$-translates yield the well-known tessellation by ideal triangles.

Theorem 3.13 (Proper discontinuity of discrete subgroups)**.** *Let* $\Gamma \leq \mathrm{PSL}(2, \mathbb{R})$ *be discrete, then* Γ *acts properly discontinuously on* $\mathbb{H}$*.*

Proof. If $\gamma K \cap K \neq \emptyset$, pick $z_0 \in K$ and points $p, q \in K$ with $q = \gamma p$. The hyperbolic triangle inequality gives $d_{\mathbb{H}}(\gamma z_0, z_0) \leq d_{\mathbb{H}}(\gamma z_0, \gamma p) + d_{\mathbb{H}}(q, z_0) \leq 2 \max_{r \in K} d_{\mathbb{H}}(r, z_0) =: R$. Thus γ lies in the set $E_R := \{g \in \mathrm{Isom}^+(\mathbb{H}) : d_{\mathbb{H}}(g z_0, z_0) \leq R\}$, which is compact (isometries moving a point by $\leq R$ form a compact set since closed balls in $(\mathbb{H}, d_{\mathbb{H}})$ are compact and the stabilizer of z_0 is compact). A discrete subgroup has only finitely many elements in a compact set, so only finitely many γ satisfy $\gamma K \cap K \neq \emptyset$. ∎

Proposition 3.14 (Area of the quotient equals area of a fundamental domain)**.** *Let* $\Gamma \le \mathrm{PSL}(2,\mathbb{R})$ *be Fuchsian and* $D \subset \mathbb{H}$ *a fundamental domain. Then*

$$\mathrm{Area}(\mathbb{H}/\Gamma) = \mathrm{Area}(D),$$

where area is computed with the hyperbolic area form $dA = \frac{dx\,dy}{y^2}$.

Proof. Up to a boundary set of measure zero, $\mathbb{H}$ is the disjoint union of the translates $\gamma(D)$, $\gamma \in \Gamma$. Hyperbolic area is invariant under Γ, so for any nonnegative compactly supported f,

$$\int_{\mathbb{H}} f\,dA = \sum_{\gamma\in\Gamma} \int_{\gamma(D)} f\,dA = \sum_{\gamma\in\Gamma} \int_D f(\gamma z)\,dA(z).$$

Taking $f \equiv 1$ and passing to the quotient identifies $\int_{\mathbb{H}/\Gamma} 1\,dA = \mathrm{Area}(D)$. ■

4. Tessellations of the Hyperbolic Plane

4.1. **Tessellations and Regular $\{p, q\}$ Tilings.** We now prove some theorems regarding hyperbolic polygons which allow us to examine tessellations more rigorously.

Theorem 4.1. *Let* Δ *be a hyperbolic triangle with internal angles* α, β, γ. *Then,* $\mathrm{Area}_{\mathbb{D}} = \pi - (\alpha + \beta + \gamma)$.

Proof. We use a combination of Möbius transformations and Gauß–Bonnet. Let z_0 be a vertex of Δ. For any $a \in \mathbb{D}$ the disk-preserving Möbius transformation

$$\gamma_a(w) = \frac{w - a}{1 - \overline{a}w}$$

sends $a \mapsto 0$ (indeed $\gamma_a(a) = 0$), and the denominator $1 - \overline{a}w$ never vanishes for $w \in \mathbb{D}$ when $|a| < 1$. Taking $a = z_0$ we obtain $\gamma = \gamma_{z_0} \in \mathrm{Mob}(\mathbb{D})$ with $\gamma(z_0) = 0$. Moreover such γ is an isometry for the hyperbolic metric on $\mathbb{D}$. Let $\Delta' = \gamma(\Delta)$. Since γ is an isometry, $\mathrm{Area}_{\mathbb{D}}(\Delta) = \mathrm{Area}_{\mathbb{D}}(\Delta')$.

Since we sent one of the vertices to 0, the edges of Δ' meeting at that vertex are radial segments. The three sides of Δ' are geodesics, and so the geodesic curvature κ_g of each side is 0. Thus, when we apply Gauß–Bonnet to Δ', $\int_{\partial\Delta'} \kappa_g ds = 0$. Since $K \equiv -1$ in hyperbolic space,

$$\int\int_{\Delta'} -1dA + 0 + \sum_{j=1}^{3} \psi_j = 2\pi,$$

where ψ_j are the exterior turning angles at the three corners. Without loss of generality, $\psi_1 = \pi - \alpha, \psi_2 = \pi - \beta, \psi_3 = \pi - \gamma$. Thus, $-\mathrm{Area}(\Delta') + (3\pi - \alpha - \beta - \gamma) = 2\pi$, and $\mathrm{Area}(\Delta') = \pi - (\alpha + \beta + \gamma) = \mathrm{Area}(\Delta)$. ■

Due to the Cayley Map Φ, this result also applies to $\mathbb{H}$. We now generalize this to polygons.

Corollary 4.2. *Let* P *be an* n*-sided hyperbolic polygon with vertices* $v_1, v_2, \ldots, v_n$ *and internal angles* $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$. *The area of this polygon is*

$$(n - 2)\pi - \sum_{i=1}^{n} \alpha_i.$$

Sketch. This can be done by cutting up the region into triangles and applying the previous theorem. ■

Definition 4.3 ($\{p,q\}$ tilings)**.** A $\{p,q\}$ tessellation means that each face is a regular p-gon and that q faces meet at a vertex.

We now prove a way to easily say whether or not there exists a tessellation given p, q. We split the theorem up into its two directions, as the length of each proof differs.

Theorem 4.4. *There exists a $\{p,q\}$ tessellation for a given p and q if and only if*

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}.$$

Proof. We first prove the forward direction. Assume that there exists a tessellation. For a hyperbolic p-gon, we know that Area $= (p-2)\pi - p\alpha > 0$ in hyperbolic space, where $\alpha = \frac{2\pi}{q}$, so

$$\begin{aligned} (p-2)\pi - \frac{2p\pi}{q} &> 0 \\ p - 2 - \frac{2p\pi}{q} &> 0 \\ \frac{1}{p} + \frac{1}{q} &< \frac{1}{2}. \end{aligned}$$

We now prove the backwards direction. We prove the following lemmas.

Lemma 4.5. *If $\alpha, \beta, \gamma \in (0, \pi)$ satisfy $\alpha + \beta + \gamma < \pi$, then there exists a hyperbolic triangle with interior angles of α, β, γ.*

Proof. Let $A, B \in \mathbb{H}$ be two points at hyperbolic distance $d_{\mathbb{H}}(A, B) > 0$. At A and B, construct geodesics on the same side as AB, which make interior angles α and β with segment AB. If these geodesics meet at a point C, then we will have a triangle ABC whose angles at A, B are α, β and whose angle at C will be denoted as $\varphi(c)$ (it depends upon the chosen base length c).

The hyperbolic law of cosines for angles applied to ABC is

$$\cos \varphi(c) = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cosh c. \tag{4.1}$$

Note that the right-hand side is a continuous, strictly increasing function of $\cosh c$, and hence of c. Since arccos is strictly decreasing on $(-1, 1)$, $\varphi(c)$ is a continuous strictly decreasing function of c on the domain where the right hand side lies in $[-1, 1]$ (where a triangle exists). Evaluate the limit as $c \to 0$. Since $\cosh c \to 1$,

$$\cosh \varphi(0) = -\cos\alpha\cos\beta + \sin\alpha\sin\beta = -\cos(\alpha + \beta),$$

so

$$\varphi(0) = \pi - (\alpha + \beta).$$

Because $\alpha + \beta + \gamma < \pi$,

$$\varphi(0) > \gamma.$$

Let $f(c) = \cos(\varphi(c))$. Note that $f(0) = \cos(\pi - (\alpha+\beta)) = \cos(\varphi(0)) < \cos\gamma$, and $\cos\gamma < 1$ (as $\gamma \in (0, \pi)$. Meanwhile,

$$\lim_{c\to\infty} f(c) = +\infty,$$

so there exists a finite $c_{\max} > 0$ such that $f(c) \le 1$ for $0 \le c \le c_{\max}$ and $f(c) > 1$ for $c > c_{\max}$. On the interval $[0, c_{\max}]$, the right-hand side of 4.1 stays in $[-1, 1]$ and so $\varphi(c)$ is well-defined with $\varphi(c) \in (0, \pi]$. Moreover, f is continuous and strictly increasing on that closed interval and takes value from $f(0) < \cos\gamma$ to $f(c_{\max}) = 1 > \cos\gamma$. By the Intermediate Value Theorem, since f is continuous on $[0, c_{\max}]$ and $f(0) < \cos\gamma < 1 = f(c_{\max})$, there exists some $c_0 \in (0, c_{\max})$ such that $f(c_0) = \cos\varphi(c_0) = \cos\gamma$. Since $\varphi(c_0), \gamma \in (0, \pi)$, the invertibility of cosine on that interval implies that $\varphi(c_0) = c$. Thus a triangle with α, β, γ exists. ■

Lemma 4.6. *Let $T \subset \mathbb{H}$ be a geodesic triangle with vertices A, B, C and interior angles $\alpha, \beta, \gamma \in [0, \pi/2]$ satisfying $\alpha + \beta + \gamma < \pi$. Let a, b, c be the sides opposite A, B, C, and let r_a, r_b, r_c denote reflections in these sides. Let $G = \langle r_a, r_b, r_c\rangle$ be the group generated by these reflections. Then the images $\{g(T) : g \in G\}$ are pairwise interior-disjoint, meet only along full edges or vertices, their interiors cover $\mathbb{H}$, and G is a discrete subgroup of* $\mathrm{Isom}(\mathbb{H})$.

Proof. Each reflection r_s is an isometry, so reflecting T across any side produces a triangle congruent to T sharing that side. Repeated reflections generate all triangles adjacent along sides.

Let

$$\mathcal{M} := \{g(a), g(b), g(c) : g \in G\}$$

be the set of all geodesics obtained from the sides by reflections in G, called *mirrors*. The complement $\mathbb{H} \setminus \bigcup \mathcal{M}$ consists of open convex regions, each congruent to T°. Convexity implies that any geodesic segment intersects each mirror at most once.

For points $x, y \in \mathbb{H}$, define the *crossing-number* $d(x, y)$ to be the number of mirrors intersected by a geodesic segment from x to y. This number is independent of the segment chosen: if γ_0, γ_1 are two segments from x to y, then each mirror separating x from y must be crossed exactly once by any path between x and y within the union of regions, so γ_0 and γ_1 cross the same mirrors.

Suppose $g_1(T)^\circ \cap g_2(T)^\circ \neq \emptyset$. Let x be a point in the intersection. Then $g_2^{-1}g_1$ fixes $g_2^{-1}(x) \in T^\circ$. Since each reflection moves points not on its mirror, no nontrivial composition of reflections can fix a point in the interior of T, and thus $g_2^{-1}g_1 = \mathrm{id}$, so $g_1 = g_2$. Hence interiors of distinct triangles are disjoint.

If closures of two triangles intersect along more than a vertex, the intersection must lie in a mirror. Reflections map sides to sides, so intersections along sides are complete edges. Triangles meet only edge-to-edge or at vertices.

For any $y \in \mathbb{H}$, take a geodesic segment γ from a point $x_0 \in T^\circ$ to y, and let $L_1, \dots, L_N$ be the mirrors crossed in order. Reflecting successively across $L_1, \dots, L_N$ produces a triangle containing y. Therefore, all points of $\mathbb{H}$ lie in some triangle $g(T)$.

If a vertex, say A, is ideal (angle $\alpha = 0$), the sides meeting at A are asymptotic to the same point at infinity. Let s_1, s_2 be these sides. Consider the sequence of triangles obtained by alternating reflections across s_1 and s_2. Each triangle is interior-disjoint from the previous and lies strictly closer to the ideal point. The union of these triangles forms a full sector near the ideal vertex. The same argument applies to any other ideal vertices.

Each triangle has positive area. If $K \subset \mathbb{H}$ is compact, only finitely many triangles intersect K, since the sum of their areas cannot exceed the finite area of K. Therefore G acts properly discontinuously and is discrete.

Hence the images of T under G form an edge-to-edge tiling of $\mathbb{H}$ by congruent triangles with disjoint interiors, covering all of $\mathbb{H}$, and G is a discrete subgroup of $\mathrm{Isom}(\mathbb{H})$. ■

Theorem 4.7. *Suppose we have a hyperbolic triangle T whose interior angles are $\pi/p, \pi/q, \pi/2$ by Lemma 4.5, for positive integers p, q such that $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$. Then, the index-2 subgroup*

G^+ of the reflection group G (the subgroup fo the orientation-preserving isometries) acts on $\mathbb{H}$ with fundamental domain a regular p-gon whose interior angle is $2\pi/q$. The orbit of this p-gon under G^+ under G^+ is a regular $\{p, q\}$ tessellation.

Proof. Denote the reflection in the sides of T as $r_{s_1}, r_{s_2}, r_{s_3}$. The composition of the two reflections corresponding to the sides meeting at a vertex A is a rotation, $\rho_A^p = r_{s_i} r_{s_j}$ about A through an angle of $2\pi/p$. Since $\rho_A^p = \text{id}$, this rotation has order p. Similarly, the composition of reflections in the sides meeting at vertex B gives a rotation ρ_B about B with order q, $\rho_B^q = \text{id}$.

The group G generated by the three reflections is discrete and acts properly on $\mathbb{H}$ by Lemma 4.6. The orientation preserving subgroup G^+ is generated by even-length words in the reflections, in particular by ρ_A and ρ_B. Consider the union of images of T under $\langle \rho_A \rangle$,

$$P = \bigcup_{k=0}^{p-1} \rho_A^k(T).$$

This set P is bounded by p geodesic edges of equal length meeting at angles equal by symmetry. Around vertex B, since q triangles meet, the interior angle of P at the corresponding vertex is $2\pi/q$. Thus, P is a regular hyperbolic p-gon with interior angle $2\pi/q$. The copies of P under G^+ fit together edge-to-edge, and exactly q such polygons meet at each vertex, so the orbit $G^+(P)$ is a regular $\{p, q\}$ tessellation of $\mathbb{H}$. This proves the backward direction of Theorem 4.4.

■

We below state the Poincare Polygon Theorem. Its proof is out of the scope of this paper, however, we encourage the reader to learn more about it in [Buc11].

Theorem 4.8 (Poincaré Polygon Theorem)**.** *Let $P \subset \mathbb{H}$ be a convex polygon whose sides are paired by isometries $\{g_i\}$ such that:*

(1) Each side of P is paired with exactly one other side via some $g_i \in \text{Isom}(\mathbb{H})$.
(2) For each vertex v of P, the product of the side-pairing isometries around v (in cyclic order) is the identity or a rotation of finite order.
(3) The angles at vertices and the side pairings satisfy the angle condition: the sum of the angles at each vertex divided by the order of the rotation equals 2π.

Then:

(1) The group G generated by the side-pairing isometries $\{g_i\}$ is discrete.
(2) The images $\{g(P) : g \in G\}$ tile $\mathbb{H}$ without overlaps (except along sides or at vertices).

5. Connections to Differential Geometry

5.1. **Curvature in the Hyperbolic Plane.**

Definition 5.1 (Sectional Curvature)**.** Let (M, ds^2) be a Riemannian manifold. The *sectional curvature* $K(\sigma)$ of a two-dimensional tangent plane $\sigma \subset T_pM$ measures how the metric deviates from being flat along σ.

In the upper half–plane model $ds^2 = (dx^2 + dy^2)/y^2$, one computes via the Levi–Civita connection that $K = -1$ everywhere. Geodesic curvature of a curve $\gamma(t)$ with respect to ds^2 likewise reflects this constant negative curvature.

5.2. **Geodesics and Metric Properties.**

Definition 5.2. For (M, g), a Riemannian manifold, the Christoffel symbols of the Levi–Civita connection in local coordinates are defined by

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left(\frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial g_{i\ell}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell} \right),$$

where $g^{k\ell}$ denotes the inverse of the metric tensor $g_{k\ell}$.

Definition 5.3. A smooth curve $\gamma(t)$ in a Riemannian manifold (M, g) is a geodesic if its velocity vector is parallel along itself i.e. $D\dot{\gamma}/dt = 0$.

In local coordinates, this condition is equivalent to the system $\ddot{\gamma}^k + \Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j = 0$, where Γ^k_{ij} are the Christoffel symbols of the Levi–Civita connection.

Proposition 5.4. *The geodesics in the upper half-plane model, when parametrized by constant-speed curves, satisfy the system of second-order differential equations*

$$\ddot{x} - \frac{2\dot{x}\,\dot{y}}{y} = 0, \quad \ddot{y} + \frac{\dot{x}^2 - \dot{y}^2}{y} = 0.$$

Proof. The metric tensor on the upper half-plane model is

$$g = \frac{1}{y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad g^{-1} = y^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The only non-zero derivatives of the metric components are $\partial_y g_{xx} = \partial_y g_{yy} = -\frac{2}{y^3}$. Using the Christoffel formula $\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij} \right)$, we find the non-zero components

$$\Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}, \quad \Gamma^y_{xx} = -\frac{1}{y}, \quad \Gamma^y_{yy} = \frac{1}{y}.$$

Let $\gamma(t) = (x(t), y(t))$. The geodesic equation $\ddot{\gamma}^k + \Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j = 0$ gives, for the x-component:

$$\ddot{x} + \Gamma^x_{xy}\dot{x}\dot{y} + \Gamma^x_{yx}\dot{y}\dot{x} = \ddot{x} - \frac{2}{y}\dot{x}\dot{y}.$$

For the y-component:

$$\ddot{y} + \Gamma^y_{xx}\dot{x}^2 + \Gamma^y_{yy}\dot{y}^2 = \ddot{y} - \frac{1}{y}\dot{x}^2 + \frac{1}{y}\dot{y}^2 = \ddot{y} + \frac{\dot{y}^2 - \dot{x}^2}{y}.$$

Therefore, γ is a geodesic if and only if

$$\ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0, \qquad \ddot{y} + \frac{\dot{x}^2 - \dot{y}^2}{y} = 0.$$

■

Proposition 5.5. *Let* (x_1, y_1) *and* (x_2, y_2) *be two points in the upper half-plane model* $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$*. The hyperbolic distance between these points is*

$$d_{\mathbb{H}}\big((x_1, y_1), (x_2, y_2)\big) = \operatorname{arccosh}\left(1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2y_1y_2}\right).$$

Proof. The orientation-preserving isometries of $(\mathbb{H}, ds^2)$ are the Möbius maps

$$z \mapsto \frac{az+b}{cz+d}, \quad ad-bc=1,$$

so distance is invariant under these maps. Geodesics in $\mathbb{H}$ are vertical lines and Euclidean semicircles orthogonal to the boundary at $y=0$. By invariance, we may send (x_1, y_1) and (x_2, y_2) to points on a common vertical geodesic. Along a vertical geodesic, the length between heights y_1 and y_2 is

$$L = \int_{y_1}^{y_2} \frac{dy}{y} = \left|\ln \frac{y_2}{y_1}\right|,$$

so

$$\cosh L = \frac{y_1^2 + y_2^2}{2y_1y_2}.$$

For general points, an isometry straightens their geodesic to the vertical case; expressing $\cosh d$ in the original coordinates gives

$$\cosh d_{\mathbb{H}}\big((x_1, y_1), (x_2, y_2)\big) = 1 + \frac{(x_2-x_1)^2 + (y_2-y_1)^2}{2y_1y_2},$$

hence

$$d_{\mathbb{H}}\big((x_1, y_1), (x_2, y_2)\big) = \operatorname{arcosh}\left(1 + \frac{(x_2-x_1)^2 + (y_2-y_1)^2}{2y_1y_2}\right).$$

■

5.3. **Comparison with Euclidean and Spherical Geometry.** Hyperbolic geometry is one of the three classical geometries of constant Gaussian curvature alongside Euclidean and Spherical geometry. This section aims to provide a comparison for those and develop certain connections to tessellations.

Curvature and Models. Euclidean geometry has $K=0$. The canonical model is the flat plane $\mathbb{R}^2$ with metric $ds^2 = dx^2 + dy^2$. Spherical geometry has constant curvature $K = 1/R^2 > 0$. The canonical model is the sphere of radius R in $\mathbb{R}^3$ with the induced metric. Hyperbolic geometry has constant curvature $K = -1/R^2 < 0$. The upper half-plane model

$$\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}, \quad ds^2 = \frac{dx^2 + dy^2}{y^2}$$

is one of several equivalent models.

Geodesics. In Euclidean geometry, geodesics are straight lines. On the sphere, geodesics are great circles (intersections of the sphere with planes through its center). In the upper half-plane model of hyperbolic geometry, geodesics are vertical lines and semicircles orthogonal to the boundary line $y=0$.

Triangles and Angle Sums. In Euclidean geometry, the sum of the interior angles of a triangle is exactly π. In spherical geometry, the sum of the angles exceeds π; the excess is proportional to the area of the triangle. In hyperbolic geometry, the sum of the angles is strictly less than π; the *angle deficit* $\pi - (\alpha + \beta + \gamma)$ is proportional to the triangle's area.

Parallel Postulate. In Euclidean geometry, through a point not on a given line, there is exactly one line parallel to the given line. In spherical geometry, there are no parallels: all geodesics (great circles) eventually intersect. In hyperbolic geometry, there are infinitely many geodesics through a point not on a given geodesic that do not intersect it.

Growth of Circles. If $C(r)$ and $A(r)$ denote circumference and area of a circle of radius r, then:

$$\begin{aligned} \text{Euclidean:} \quad & C(r) = 2\pi r, \quad A(r) = \pi r^2, \\ \text{Spherical:} \quad & C(r) = 2\pi R \sin(r/R), \quad A(r) = 2\pi R^2\big(1 - \cos(r/R)\big), \\ \text{Hyperbolic:} \quad & C(r) = 2\pi R \sinh(r/R), \quad A(r) = 2\pi R^2\big(\cosh(r/R) - 1\big). \end{aligned}$$

In hyperbolic geometry, both circumference and area grow exponentially with r, a fact directly tied to the richness of possible tessellations.

These metric differences have direct combinatorial implications: in the Euclidean plane, regular tessellations exist only for $\{3,6\}$, $\{4,4\}$, and $\{6,3\}$, while in the hyperbolic plane, infinitely many $\{p,q\}$-tessellations exist whenever $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$.

5.4. **Gauss–Bonnet Theorem and Area Formulas.** The *Gauss–Bonnet Theorem* is a cornerstone result connecting curvature, topology, and geometry.

Theorem 5.6 (Gauss–Bonnet)**.** *Let S be a compact oriented surface with piecewise smooth boundary ∂S. Let K be the Gaussian curvature of S, and κ_g the geodesic curvature of ∂S (with respect to the inward-pointing normal). If the boundary has vertices with interior angles $\theta_1, \dots, \theta_n$, then*

$$\int_S K\, dA + \int_{\partial S} \kappa_g\, ds + \sum_{i=1}^{n} (\pi - \theta_i) = 2\pi\chi(S),$$

where $\chi(S)$ is the Euler characteristic of S.

Geodesic Polygons in Constant Curvature. If ∂S consists entirely of geodesic segments, then $\kappa_g \equiv 0$, and the Gauss–Bonnet formula simplifies to:

$$\int_S K\, dA + \sum_{i=1}^{n} (\pi - \theta_i) = 2\pi\chi(S).$$

Spherical case ($K > 0$): For a geodesic triangle on a sphere of radius R,

$$K = \frac{1}{R^2}, \quad \chi(S) = 1, \quad \Rightarrow \quad \text{Area} = R^2\big(\alpha + \beta + \gamma - \pi\big).$$

The *angle excess* determines the area.
Euclidean case ($K = 0$): We obtain $\alpha + \beta + \gamma = \pi$ for triangles, recovering the classical fact that Euclidean triangles have constant angle sum.
Hyperbolic case ($K = -1$): For a geodesic triangle,

$$-\text{Area} + \sum_{i=1}^{3} (\pi - \alpha_i) = 2\pi,$$

which rearranges to

$$\text{Area} = \pi - (\alpha + \beta + \gamma).$$

The *angle deficit* directly measures the area.

Implications for Tessellations. Consider a regular $\{p,q\}$-tessellation of the hyperbolic plane: each face is a regular p-gon with q meeting at each vertex. The interior angle of each polygon is $\alpha = \frac{2\pi}{q}$. Applying the hyperbolic area formula for a regular p-gon with all sides geodesic:

$$\text{Area}_{\mathbb{H}}(p,q) = (p-2)\pi - p\alpha = \pi\left(p - 2 - \frac{2p}{q}\right),$$

which is positive when $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, the classical condition for hyperbolic tessellations.

6. Future Directions

Despite the extensive development of hyperbolic geometry and tessellation theory, many intriguing open questions remain. Below are several directions for further research:

6.1. Classification of Hyperbolic Tilings.

While regular $\{p,q\}$ tessellations are well understood, the classification of *semi-regular*, *aperiodic*, or more general tilings in hyperbolic space is far less complete. What broader families of tilings exist beyond the regular and semi-regular cases?

Definition 6.1 (Uniform/semi-regular tiling and vertex type)**.** A tiling of $\mathbb{H}$ (or $\mathbb{D}$) is *uniform* if its symmetry group acts transitively on vertices. If k faces meet at a vertex v, record their regular face sizes cyclically as $(p_1,\ldots,p_k)$; this cyclic k–tuple is the *vertex type* at v. A tiling is *semi-regular* if it is uniform and all tiles are regular polygons, possibly of different sizes.

Proposition 6.2 (Necessary angle inequality at a hyperbolic vertex)**.** *If a uniform hyperbolic tiling has vertex type* $(p_1,\ldots,p_k)$*, then*

$$\sum_{i=1}^{k} \frac{1}{p_i} \;<\; \frac{k-2}{2}.$$

Sketch. Let α_i be the interior angle of the regular p_i–gon in the tiling. In $\mathbb{H}$, one has $\alpha_i < \frac{(p_i-2)\pi}{p_i}$. Since the angles around a vertex sum to 2π, we get $2\pi = \sum_i \alpha_i < \sum_i \frac{(p_i-2)\pi}{p_i} = \pi\Big(k - 2\sum_i \frac{1}{p_i}\Big)$, which rearranges to the claim. ■

Corollary 6.3. *The borderline* $\sum_i \frac{1}{p_i} = \frac{k-2}{2}$ *corresponds to Euclidean uniform tilings, while* $\sum_i \frac{1}{p_i} > \frac{k-2}{2}$ *to spherical ones.*

6.2. Growth Rates and Combinatorics.

How does the number of tiles grow as a function of hyperbolic distance from a fixed point? What combinatorial invariants distinguish different tessellations?

Definition 6.4 (Tile-count growth)**.** Fix a regular $\{p,q\}$ tiling and a base point $o \in \mathbb{H}$. Let $N(R)$ be the number of tiles that intersect the hyperbolic ball $B_{\mathbb{H}}(o,R)$.

Proposition 6.5 (Exponential growth)**.** *There exist constants* $C_1, C_2, \lambda_1, \lambda_2 > 0$ *depending only on* $\{p,q\}$ *such that*

$$C_1 e^{\lambda_1 R} \;\le\; N(R) \;\le\; C_2 e^{\lambda_2 R} \qquad (R \ge 1).$$

In particular, $N(R)$ *grows exponentially in* R.

Sketch. $\operatorname{Area}(B_{\mathbb{H}}(o,R)) = 2\pi(\cosh R - 1) \asymp e^R$. In a $\{p,q\}$ tiling, each tile has constant hyperbolic area

$$\operatorname{Area}(T) = (p-2)\pi - p\cdot\frac{2\pi}{q} = \pi\left(p-2-\frac{2p}{q}\right) > 0.$$

Packing/covering of $B_{\mathbb{H}}(o,R)$ by tiles yields two-sided bounds comparing $N(R)$ to $\operatorname{Area}(B_{\mathbb{H}}(o,R))$. ■

Corollary 6.6. *The exponential growth rate of tiles mirrors the exponential growth of hyperbolic area; finer asymptotics depend on adjacency combinatorics of the $\{p,q\}$ tiling.*

6.3. **Tessellations and Topology.** What kinds of topological surfaces (e.g., compact Riemann surfaces or surfaces with boundary) can be realized as quotients of the hyperbolic plane by tessellations induced from Fuchsian groups?

Definition 6.7 (Hyperbolic quotient surface)**.** Let $\Gamma \le \mathrm{PSL}(2,\mathbb{R})$ be torsion-free and discrete (Fuchsian). Then $S = \mathbb{H}/\Gamma$ is a closed hyperbolic surface when Γ is cocompact; its Euler characteristic satisfies Gauß—Bonnet $\operatorname{Area}(S) = -2\pi\,\chi(S) = 2\pi(2g-2)$, where g is the genus of S.

Proposition 6.8 (Fundamental polygons and genus)**.** *If a fundamental polygon for Γ has $4g$ sides with the standard opposite-side identifications producing a single vertex in the quotient, then $\mathbb{H}/\Gamma$ is a closed surface of genus g and $\operatorname{Area}(\mathbb{H}/\Gamma) = 2\pi(2g-2)$.*

Sketch. The side-pairings give a CW structure with one 2–cell, $2g$ 1–cells, and one 0–cell, hence $\chi = 1 - 2g + 1 = 2 - 2g$. Apply Gauß—Bonnet. ■

6.4. **Connections to Group Theory.** How do properties of tessellations reflect properties of the corresponding Fuchsian or triangle groups? Can certain geometric features of a tiling (e.g., curvature distributions or symmetry) be characterized algebraically?

Definition 6.9 (Triangle and reflection groups)**.** For $p,q,r \in \mathbb{N}\cup\{\infty\}$ with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$, the *hyperbolic triangle group*

$$\Delta(p,q,r) = \langle x,y,z \mid x^p = y^q = z^r = xyz = 1\rangle$$

acts on $\mathbb{H}$ with fundamental triangle of angles $(\pi/p, \pi/q, \pi/r)$. The Coxeter reflection group $[p,q,r]$ is generated by reflections in the sides of that triangle; $\Delta(p,q,r)$ is its index-2 orientation-preserving subgroup.

Proposition 6.10 (Tessellations from triangle groups)**.** *The group $[p,q,r]$ tessellates $\mathbb{H}$ by reflected copies of the generating triangle; $\Delta(p,q,r)$ acts with the same triangular fundamental domain.*

Sketch. Apply the Poincaré polygon theorem to the triangle with prescribed angles and side-pairing reflections; the angle conditions encode the relations and ensure discreteness and a global tiling. ■

Example 6.11 (Modular tessellation)**.** $\mathrm{PSL}(2,\mathbb{Z}) \cong \Delta(2,3,\infty)$ with fundamental domain the ideal triangle bounded by $\{\,|z| = 1,\ |\Re z| \le \frac{1}{2}\,\}$ in $\mathbb{H}$; its translates tessellate $\mathbb{H}$ by ideal triangles.

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