Minimal Surfaces

A surface that locally minimizes its area.

Shihan Kanungo

Euler Circle, Palo Alto, CA 94306

Abstract

In this expository paper we will introduce the *theory of minimal surfaces.* In the chapter on Geodesics (Week 7), we considered the problem of finding the shortest distance between two points. In this paper we investigate the higher dimensional analogue of this, where we find ways to construct a surface of "minimal" area with a given boundary. Such surfaces can be represented by soap films, where the surface tension of the film ensures that it attains a shape with the minimal surface area. Minimal surfaces can be found in anything from the event horizons of black holes, to biomolecules for drug delivery, to the designs of roofs.

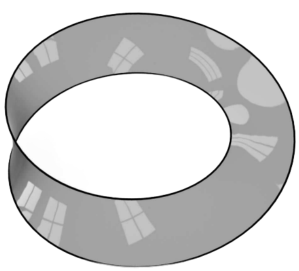
1 Overview

In 1762, Lagrange considered the problem of finding the surface $z = z(x, y)$ with the least area with a given boundary. In doing so, he derived the Euler-Lagrange equation for the solution and showed that a "minimal surface" would satisfy

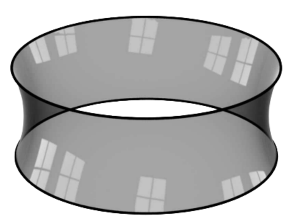
$$\frac{\partial}{\partial x}\left[\frac{Z_x}{\sqrt{1+Z_x^2+Z_y^2}}\right]+\frac{\partial}{\partial y}\left[\frac{Z_y}{\sqrt{1+Z_x^2+Z_y^2}}\right]=0. \tag{1.1}$$

Lagrange pointed out that the plane would be a trivial solution to the equation but made no further investigations to see what other possibilities existed.

In 1744 Euler discovered the catenoid, the first non-planar minimal surface. This surface is readily realised by a soap film, spanning coaxial circular bounding wires. The film shrinks under the action of its surface tension, forming the minimal surface.

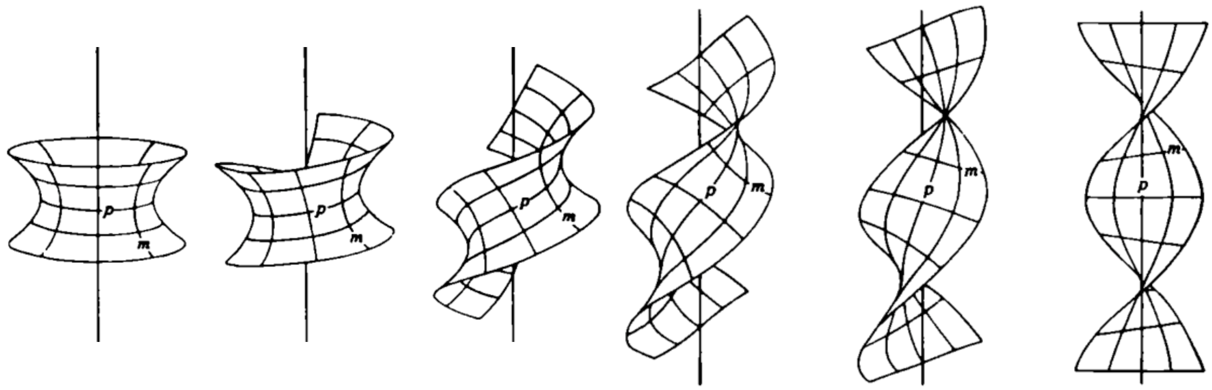


An example of a soap film which looks like a Möbius strip.



Another soap film, which is a piece of the catenoid (the frames are circles).

Jean-Baptiste Meusnier found, in 1776, a further non-trivial solution to (1.1), the helicoid. The helicoid is the surface swept out by a line that always intersects a fixed axis at right angles and that rotates uniformly as its point of intersection moves uniformly along the axis. The helicoid is the only ruled minimal surface other than the plane built up entirely of straight lines, i.e. a ruled surface (Catalan 1842, do Carmo 1986) and the catenoid is the only minimal surface of revolution. These surfaces are related through the Bonnet transformation.



The Bonnet transformation from the catenoid to the helicoid

The Bonnet rotation. The helicoid, $\boldsymbol{\varphi}$, and the catenoid, $\widetilde{\boldsymbol{\varphi}}$ are related through the Bonnet rotation, $\boldsymbol{\sigma}_\theta$, which is the weighted sum of the two minimal surfaces:

$$\boldsymbol{\sigma}_\theta = (\cos\theta)\boldsymbol{\varphi} + (\sin\theta)\widetilde{\boldsymbol{\varphi}}.$$

In a Bonnet rotation every surface element maintains its normal vector but rotates a given angle in its tangent plane. If and only if the surface is a minimal surface, then the surface elements all fit together again. The Bonnet rotation is an isometry of the surface.

Meusnier, in 1776, established a link between curvature and minimal surfaces. He proved that (1.1) implies that the mean curvature is zero everywhere on a minimal surface. In his own words: "*la surface de moindre étendue entre ses limites a cette propriété, que chaque elément a ses deux rayons de Courbure de signe contraire & egaux*".

This is the defining property of a minimal surface: *For a minimal surface, the principal curvatures are equal, but opposite in sign at every point.* The Gaussian curvature is then always non-positive, and the mean curvature is zero.

For the rest of this paper, we will not use the area property of minimal surfaces. In fact, a minimal surface only needs to be a local minimum to the area function. The fact that minimal surfaces have zero mean curvature is what concerns us here.

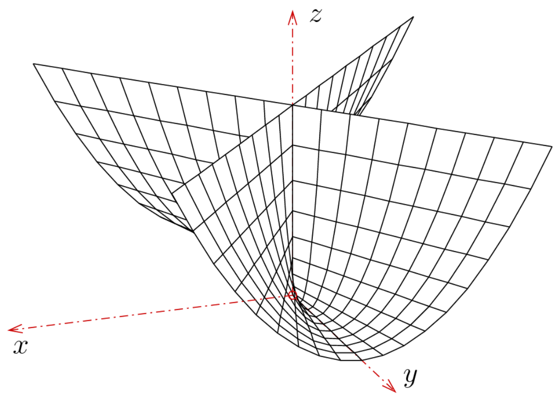
The main point of this paper is the Weierstrass-Enneper representation, which gives a feasible method of constructing minimal surfaces. But before we get to that, we should first formally define minimal surfaces.

2 What are minimal surfaces, really?

The notion of surface we will use in this paper may be slightly different from the one we have used thus far.

Definition 2.1. A *regular parametrized surface* is a differentiable map $\varphi : U \to \mathbb{R}^3$, such that for any $\mathbf{p} \in U$ the vectors $\varphi_u(\mathbf{p})$ and $\varphi_v(\mathbf{p})$ are linearly independent.

So we will consider (images of) local parametrizations which are not necessarily injective. For example, the figure below is the image of a regular parametrized surface.



The Whitney umbrella, $\varphi(u, v) = (uv, u, v^2)$.

Minimal surfaces can be thought of as saddle surfaces. This picture can be described mathematically with the following definition.

Definition 2.2. A *minimal surface* is a surface M with the mean curvature $H = 0$ at all points $\mathbf{p} \in M$.

That is, at each point the bending upward in one direction is matched with the bending downward in the orthogonal direction.

3 Examples of minimal surfaces.

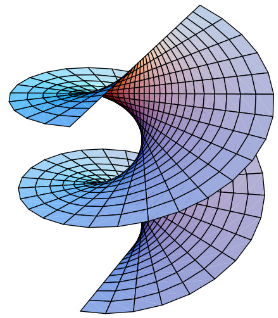
The following surfaces are minimal. We will only justify this later.

1. The *helicoid*, given by
$$\boldsymbol{\varphi}(u, v) = (u \cos v, u \sin v, v)$$
We saw this in [6, *Parametrized Surfaces*, Week 3, Problem 3].

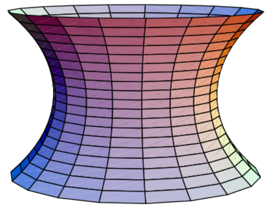
2. The *catenoid*, given by
$$\boldsymbol{\varphi}(u, v) = (\cosh u \cos v, \cosh u \sin v, u)$$
If a heavy flexible cable is suspended between two points at the same height, then it takes the shape of a curve that can be described mathematically by the function $y = \cosh(x)$. Such a curve is called a catenary from the Latin word that means "chain". A catenoid is generated by rotating a catenary on its side about the z-axis. We saw this in [6, *The First Fundamental Form*, Problem 5].

3. *Enneper's minimal surface*, discovered by Alfred Enneper in 1864, and given by
$$\boldsymbol{\varphi}(u, v) = \left(u + uv^2 - \tfrac{1}{3}u^3, v + u^2 v - \tfrac{1}{3}v^3, u^2 - v^2\right)$$

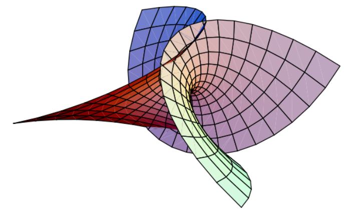
The Enneper surface is a minimal surface formed by bending a disk into a saddle surface. We saw this in [6, *The First Fundamental Form*, Problem 7].



The helicoid.



The catenoid.



Enneper's surface.

4. *Scherk's doubly periodic surface*, given by

$$\boldsymbol{\varphi}(u,v) = \left(u, v, \ln\frac{\cos u}{\cos v}\right)$$

This is an example of a minimal surface that is the graph of a function.

The plane, the catenoid, the helicoid, and Scherk's doubly periodic surface are examples of boundaryless (complete) surfaces that have no self-intersections (embedded). However, the Enneper surface is not embedded, because it has self-intersections as its domain increases.

Example. It is easy to see that the helicoid is a minimal surface: If a straight line on a surface is a symmetry line, then the mean curvature along this line is automatically zero, because rotating about the line changes the direction of the normal, thus the sign of the Weingarten map, thus the sign of the mean curvature. For the helicoid, all its straight lines are symmetry lines, so its mean curvature must vanish everywhere.

The mean curvature can be expressed in terms of the coefficients of the first and second fundamental forms, (cf. [6, *Principal Curvatures*, Corollary 1.3]),

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}, \tag{3.1}$$

We will use (3.1) to show that a surface with a specific parametrization is minimal.

Example. Recall that a catenoid can be parametrized by

$$\boldsymbol{\varphi}(u,v) = (\cosh u \cos v, \cosh u \sin v, u)$$

In [6, Week 4], we compute $E = G = \cosh^2 v$ and $F = 0$. In [6, Week 6, Problem 2], we computed $L = -1$, $M = 0$, and $N = 1$. Substituting these values in (3.1), we get

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{-G + E}{2(EG - F^2)} = 0.$$

And so the catenoid is a minimal surface.

For the surface $z = f(x,y)$, we computed in [6, *Principal Curvatures*, Problem 1], that

$$H = \frac{(1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy}}{2(1 + f_x^2 + f_y^2)^{3/2}}.$$

Example. Scherk's doubly periodic surface has the parametrization

$$\boldsymbol{\varphi}(u,v)=\left(u,v,\ln\frac{\cos u}{\cos v}\right)$$

Using the formula above, with $f(x,y)=\ln(\cos y)-\ln(\cos x)$, we get

$$H=\frac{\sec^2 x(1+\tan^2 y)-\sec^2 y(1+\tan^2 x)}{2(1+\tan^2 x+\tan^2 y)^{3/2}}=0.$$

And so Scherk's doubly periodic surface is a minimal surface.

4 Constructing Minimal Surfaces

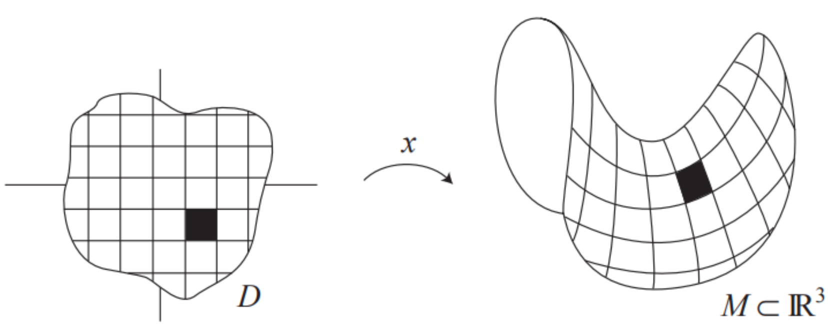
Our goal here is to describe a method of constructing minimal surfaces. As we have seen, determining if a surface is minimal basically involves solving second order differential equations. Since we have yet to set up our framework, we have some degree of freedom to choose certain objects. For example, we have for our surfaces the freedom to choose a parametrization. One way we could use this freedom is to try to find a parametrization that would simplify the underlying differential equations. It turns out we can do so if we choose a specific type of parametrization known as an isothermal parametrization.[1]

4.1. Isothermal parametrization.

We start with the definition.

Definition 4.1. A parametrization $\boldsymbol{\varphi}$ is *isothermal* if $E=\boldsymbol{\varphi}_u\cdot\boldsymbol{\varphi}_u=\boldsymbol{\varphi}_v\cdot\boldsymbol{\varphi}_v=G$ and $F=\boldsymbol{\varphi}_u\cdot\boldsymbol{\varphi}_v=0$.

Indeed, isothermal parameters do exist locally for all minimal surfaces.[2] We refer to [4, Lemma 4.4], for a proof. Isothermal parametrizations preserve angles, i.e. angles in the parameter plane are mapped conformally to angles on the surface. Recall that E, F, and G describe how lengths on a surface are distorted as compared to their usual measurements in $\mathbb{R}^3$. So if $F=\boldsymbol{\varphi}_u\cdot\boldsymbol{\varphi}_v=0$ then the vectors $\boldsymbol{\varphi}_u$ and $\boldsymbol{\varphi}_v$ are orthogonal and if $E=G$, then the amount of distortion is the same in these two orthogonal directions. Thus, we can think of an isothermal parametrization as mapping a small square in the domain to a small square on the surface.



An isothermal parametrization maps small squares to small squares.

[1]Named by Gabriel Lamé in his 1833 study of heat transfer. The reason is, for a thermally isolated surface of heat conduction, the constant coordinate lines are *isotherms* if and only if the coordinates are isothermal.

[2]**Theorem.** *Every minimal surface in* $\mathbb{R}^3$ *has an isothermal parametrization.*

Example. The parametrization

$$\boldsymbol{\sigma}(u,v) = (\cosh u \cos v, \cosh u \sin v, u)$$

for the catenoid is isothermal, because we computed[3] $E = \cosh^2 u = G$ and $F = 0$.

Example. We saw in [6] that the mapping $\boldsymbol{\sigma} \mapsto \widetilde{\boldsymbol{\sigma}}$ from the catenoid to the helicoid

$$\widetilde{\boldsymbol{\sigma}}(u,v) = (u \cos v, u \sin v, v)$$

is a local isometry. Since $\boldsymbol{\sigma}$ is isothermal, it follows that $\widetilde{\boldsymbol{\sigma}}$ is isothermal too.

We will use the property of isothermal parametrization to give us a necessary and sufficient condition for a surface to be minimal. This condition is very important and useful. It will come as a corollary to the following theorem, which gives a simple formula for the mean curvature, H.

Lemma 4.2. *If $\boldsymbol{\varphi} : U \to \mathbb{R}^3$ is an isothermal surface with $E = G =: \lambda^2$, normal vector $\mathbf{N}$, and mean curvature H then we have*

$$\boldsymbol{\varphi}_{uu} + \boldsymbol{\varphi}_{vv} = 2\lambda^2 H \mathbf{N}.$$

Proof. We have

$$\boldsymbol{\varphi}_u \cdot \boldsymbol{\varphi}_u = \boldsymbol{\varphi}_v \cdot \boldsymbol{\varphi}_v, \quad \text{and} \quad \boldsymbol{\varphi}_u \cdot \boldsymbol{\varphi}_v = 0.$$

Upon differentiating, we get

$$\boldsymbol{\varphi}_{uu} \cdot \boldsymbol{\varphi}_u - \boldsymbol{\varphi}_{vu} \cdot \boldsymbol{\varphi}_v = 0,$$
$$\boldsymbol{\varphi}_{vv} \cdot \boldsymbol{\varphi}_u + \boldsymbol{\varphi}_{vu} \cdot \boldsymbol{\varphi}_v = 0$$

Adding, we get $(\boldsymbol{\varphi}_{uu} + \boldsymbol{\varphi}_{vv}) \cdot \boldsymbol{\varphi}_u = 0$, and by symmetry, $(\boldsymbol{\varphi}_{uu} + \boldsymbol{\varphi}_{vv}) \cdot \boldsymbol{\varphi}_v = 0$. Thus $\boldsymbol{\varphi}_{uu} + \boldsymbol{\varphi}_{vv}$ is perpendicular to both $\boldsymbol{\varphi}_u$ and $\boldsymbol{\varphi}_v$, so it is parallel to $\mathbf{N}$. On the other hand, (3.1) gives

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{L + N}{2\lambda^2}$$

which implies

$$2\lambda^2 H = L + N = \mathbf{N} \cdot (\boldsymbol{\varphi}_{uu} + \boldsymbol{\varphi}_{vv})$$

and finishes the proof. ■

The *Laplacian* of a function $f : U \to \mathbb{R}$ is $\Delta(f) = f_{uu} + f_{vv}$.

Definition 4.3. A function $f : U \to \mathbb{R}$ is *harmonic* if

$$\Delta(f) = f_{uu} + f_{vv} = 0.$$

A straightforward consequence of Lemma 4.2 is the following.

Theorem 4.4. *A surface M with an isothermal parametrization*

$$\boldsymbol{\varphi}(u,v) = (\boldsymbol{\varphi}^1(u,v), \boldsymbol{\varphi}^2(u,v), \boldsymbol{\varphi}^3(u,v)),$$

is minimal if and only if $\boldsymbol{\varphi}^1$, $\boldsymbol{\varphi}^2$, $\boldsymbol{\varphi}^3$ are harmonic.

[3] cf. [6, *First Fundamental Form*, Problem 5]

It is important to understand the significance of this result. Once we have an isothermal parametrization for our surface (remember, every minimal surface in $\mathbb{R}^3$ has an isothermal parametrization), this result tells us we will have a minimal surface if and only if the coordinate functions of that parametrization are harmonic functions. This will provide us another way to create and to prove that a surface is minimal.

Proof. If M is minimal, then $H = 0$ and so by Lemma 4.2, $\boldsymbol{\varphi}_{uu} + \boldsymbol{\varphi}_{vv} = 0$, and hence the coordinate functions are harmonic. Now suppose $\boldsymbol{\varphi}^1$, $\boldsymbol{\varphi}^2$, $\boldsymbol{\varphi}^3$ are harmonic. Then $\boldsymbol{\varphi}_{uu}+\boldsymbol{\varphi}_{vv} = 0$. So by Lemma 4.2, we have that $2(\boldsymbol{\varphi}_u \cdot \boldsymbol{\varphi}_u)H\mathbf{N} = 0$. But $\mathbf{N} \neq 0$ and $E = \boldsymbol{\varphi}_u \cdot \boldsymbol{\varphi}_u \neq 0$. Hence $H = 0$ and so M is minimal. ■

It is not an easy task to produce three functions $\boldsymbol{\varphi}^1(u, v), \boldsymbol{\varphi}^2(u, v), \boldsymbol{\varphi}^3(u, v)$ which are all harmonic and such that $\boldsymbol{\varphi} = (\boldsymbol{\varphi}^1, \boldsymbol{\varphi}^2, \boldsymbol{\varphi}^3)$ is regular. The task will become more handy if we use tools from complex analysis.

4.2. A Review of Complex Analysis.

Let $\mathbb{C}$ be the complex plane. A continuous function $f : \mathbb{C} \to \mathbb{C}$ defined on a domain D is said to be *holomorphic* if the derivative

$$f'(z) := \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists for all $z \in D$. If $f(z) = x(u, v) + iy(u, v)$ is holomorphic, then the Cauchy–Riemann equations hold for f. That is,

$$x_u = y_v, \quad x_v = -y_u.$$

In such a case, y is called the *harmonic conjugate* of x. Also, if f is holomorphic, then

$$f'(z) = x_u + iy_u.$$

If $f : U \to \mathbb{R}$ is differentiable, we denote

$$f_z := \tfrac{1}{2}(f_u - if_v).$$

If $\boldsymbol{\varphi} : U \to \mathbb{R}^3$, with $\boldsymbol{\varphi}(u, v) = (\boldsymbol{\varphi}^1(u, v), \boldsymbol{\varphi}^2(u, v), \boldsymbol{\varphi}^3(u, v))$, we define $\boldsymbol{\varphi}_z : U \to \mathbb{C}^3$ as

$$\boldsymbol{\varphi}_z := (\boldsymbol{\varphi}^1_z, \boldsymbol{\varphi}^2_z, \boldsymbol{\varphi}^3_z).$$

Remark. Note that if $\boldsymbol{\varphi}$ is isothermal

$$\begin{aligned} |\boldsymbol{\varphi}_z|^2 &= \left|\frac{\partial \boldsymbol{\varphi}^1}{\partial z}\right|^2 + \left|\frac{\partial \boldsymbol{\varphi}^2}{\partial z}\right|^2 + \left|\frac{\partial \boldsymbol{\varphi}^3}{\partial z}\right|^2 \\ &= \frac{1}{4}\left(\sum_{k=1}^{3}\left(\frac{\partial \boldsymbol{\varphi}^k}{\partial u}\right)^2 + \sum_{k=1}^{3}\left(\frac{\partial \boldsymbol{\varphi}^k}{\partial v}\right)^2\right) \\ &= \frac{1}{4}(\boldsymbol{\varphi}_u \cdot \boldsymbol{\varphi}_u + \boldsymbol{\varphi}_v \cdot \boldsymbol{\varphi}_v) = \frac{1}{4}(E+G) = \frac{1}{2}E. \end{aligned}$$

We want $|\boldsymbol{\varphi}_z|^2 \neq 0$ because otherwise all the coefficients of the first fundamental form are zero and M degenerates to a point. Similarly, we want $|\boldsymbol{\varphi}_z|^2$ to be finite.

4.3. The Weierstrass–Enneper Representation.

As per the preceeding remarks, we assume, in the following discussion, that all surfaces, $\boldsymbol{\varphi}$, will have $|\boldsymbol{\varphi}_z|^2 \neq 0$ and $|\boldsymbol{\varphi}_z|^2$ be finite.

Proposition 4.5. *Let M be a surface with parametrization $\boldsymbol{\varphi} = (\boldsymbol{\varphi}^1, \boldsymbol{\varphi}^2, \boldsymbol{\varphi}^3)$.*

Then $\boldsymbol{\varphi}$ is isothermal $\iff (\boldsymbol{\varphi}_z)^2 = (\boldsymbol{\varphi}_z^1)^2 + (\boldsymbol{\varphi}_z^2)^2 + (\boldsymbol{\varphi}_z^3)^2 = 0$.

If $\boldsymbol{\varphi}$ is isothermal, then M is minimal $\iff$ each $\boldsymbol{\varphi}_z^k$ is holomorphic (analytic).

Before we prove Proposition 4.4, let's look at applying it to a specific example to help us better understand what the proposition is saying.

Example. Suppose we have the parametrization

$$\boldsymbol{\varphi} = (\boldsymbol{\varphi}^1, \boldsymbol{\varphi}^2, \boldsymbol{\varphi}^3) = (z - \tfrac{1}{3}z^3, -i(z + \tfrac{1}{3}z^3), z^2).$$

Then

$$\begin{aligned} \boldsymbol{\varphi}_z^1 &= \frac{\partial \boldsymbol{\varphi}^1}{\partial z} = 1 - z^2, \\ \boldsymbol{\varphi}_z^2 &= \frac{\partial \boldsymbol{\varphi}^2}{\partial z} = -i(1 + z^2), \text{ and} \\ \boldsymbol{\varphi}_z^3 &= \frac{\partial \boldsymbol{\varphi}^3}{\partial z} = 2z. \end{aligned}$$

Notice that

$$(\boldsymbol{\varphi}_z)^2 = (1 - z^2)^2 + (-i(1 + z^2))^2 + (2z)^2 = 0.$$

Thus, by the proposition, the parametrization $\boldsymbol{\varphi}$ is isothermal. Also, each $\boldsymbol{\varphi}_z^k$ is a polynomial and hence holomorphic. So $\boldsymbol{\varphi}$ is a parametrization of a minimal surface (in fact, it is Enneper's surface).

We are now ready for the following proof.

Proof. Taking partial derivatives and then squaring the terms, we have

$$(\boldsymbol{\varphi}_z^k)^2 = \left(\frac{\partial \boldsymbol{\varphi}^k}{\partial z}\right)^2 = \left[\frac{1}{2}\left(\frac{\partial \boldsymbol{\varphi}^k}{\partial u} - i\frac{\partial \boldsymbol{\varphi}^k}{\partial v}\right)\right]^2 = \frac{1}{4}\left[\left(\frac{\partial \boldsymbol{\varphi}^k}{\partial u}\right)^2 - \left(\frac{\partial \boldsymbol{\varphi}^k}{\partial v}\right)^2 - 2i\frac{\partial \boldsymbol{\varphi}^k}{\partial u}\frac{\partial \boldsymbol{\varphi}^k}{\partial v}\right].$$

Also recall that $\boldsymbol{\varphi}_u \cdot \boldsymbol{\varphi}_u = \sum_{k=1}^3 \left(\frac{\partial \boldsymbol{\varphi}^k}{\partial u}\right)^2$ and similarly $\boldsymbol{\varphi}_v \cdot \boldsymbol{\varphi}_v = \sum_{k=1}^3 \left(\frac{\partial \boldsymbol{\varphi}^k}{\partial v}\right)^2$. Hence,

$$\begin{aligned} (\boldsymbol{\varphi}_z)^2 &= (\boldsymbol{\varphi}_z^1)^2 + (\boldsymbol{\varphi}_z^2)^2 + (\boldsymbol{\varphi}_z^3)^2 \\ &= \frac{1}{4}\left[\left(\frac{\partial \boldsymbol{\varphi}^k}{\partial u}\right)^2 - \left(\frac{\partial \boldsymbol{\varphi}^k}{\partial v}\right)^2 - 2i\frac{\partial \boldsymbol{\varphi}^k}{\partial u}\frac{\partial \boldsymbol{\varphi}^k}{\partial v}\right] \\ &= \frac{1}{4}(E - G - 2iF). \end{aligned}$$

Thus, $\boldsymbol{\varphi}$ is isothermal $\iff E = G, F = 0 \iff \boldsymbol{\varphi}_z^2 = 0$. Now suppose $\boldsymbol{\varphi}$ is isothermal. Then

$$\frac{\partial^2 \boldsymbol{\varphi}^k}{\partial u \partial u} + \frac{\partial^2 \boldsymbol{\varphi}^k}{\partial v \partial v} = 4\left(\frac{\partial}{\partial \bar{z}}\left(\frac{\partial \boldsymbol{\varphi}^k}{\partial z}\right)\right) = 4\left(\frac{\partial}{\partial \bar{z}}\left(\boldsymbol{\varphi}_z^k\right)\right).$$

Thus $\boldsymbol{\varphi}_z^k$ is holomorphic $\iff \boldsymbol{\varphi}^k$ is harmonic. By Theorem 4.3, we are done. ∎

Conversely, we have

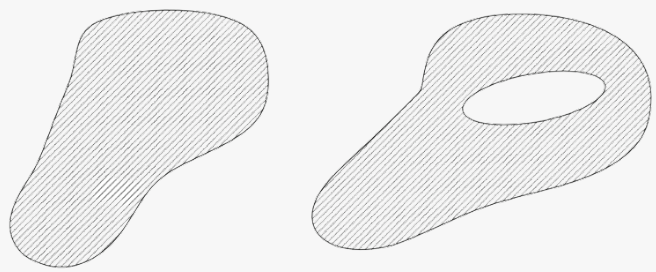
Proposition 4.6. *Let $\psi_1, \psi_2, \psi_3 : U \to \mathbb{C}$ be holomorphic functions such that*

$$(\psi_1)^2 + (\psi_2)^2 + (\psi_3)^2 = 0 \quad \textit{and} \quad |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 \neq 0. \tag{$\dagger$}$$

If U is simply connected then there exists a regular minimal isothermal surface $\boldsymbol{\varphi} = (\boldsymbol{\varphi}^1, \boldsymbol{\varphi}^2, \boldsymbol{\varphi}^3) : U \to \mathbb{R}^3$ such that $\boldsymbol{\varphi}^k_z = \psi_k$ for $k = 1, 2, 3$. More precisely,

$$\boldsymbol{\varphi}^k = \mathrm{Re} \int \psi_k(z)\ dz, \qquad \textit{for } k = 1, 2, 3.$$

Simply connected space. An open subset U of $\mathbb{R}^2$ is said to be *simply-connected* if every simple closed curve in U can be shrunk to a point staying inside U. Intuitively, this means that U has no 'holes'.



Simply–connected Not simply–connected

Proof. We need to solve $\psi_k = \frac{\partial \boldsymbol{\varphi}^k}{\partial z}$ for $\boldsymbol{\varphi}^k$ since the parametrization of the surface is given as $\boldsymbol{\varphi} = (\boldsymbol{\varphi}^1, \boldsymbol{\varphi}^2, \boldsymbol{\varphi}^3)$. Since $\boldsymbol{\varphi}^k$ is a function of two variables u and v, we can write

$$d\boldsymbol{\varphi}^k = \frac{\partial \boldsymbol{\varphi}^k}{\partial u}\, du + \frac{\partial \boldsymbol{\varphi}^k}{\partial v}\, dv. \tag{4.1}$$

Since $dz = du + idv$, we get

$$\begin{aligned}
\psi_k\, dz = \frac{\partial \boldsymbol{\varphi}^k}{\partial z}\, dz &= \frac{1}{2}\left(\frac{\partial \boldsymbol{\varphi}^k}{\partial u} - i \frac{\partial \boldsymbol{\varphi}^k}{\partial v}\right)(du + idv) \\
&= \frac{1}{2}\left[\frac{\partial \boldsymbol{\varphi}^k}{\partial u}\, du + \frac{\partial \boldsymbol{\varphi}^k}{\partial v}\, dv + i\left(\frac{\partial \boldsymbol{\varphi}^k}{\partial u}\, dv - \frac{\partial \boldsymbol{\varphi}^k}{\partial v}\, du\right)\right], \\
\overline{\psi_k\, dz} = \overline{\frac{\partial \boldsymbol{\varphi}^k}{\partial z}}\, \overline{dz} &= \frac{1}{2}\left(\frac{\partial \boldsymbol{\varphi}^k}{\partial u} + i \frac{\partial \boldsymbol{\varphi}^k}{\partial v}\right)(du - idv) \\
&= \frac{1}{2}\left[\frac{\partial \boldsymbol{\varphi}^k}{\partial u}\, du + \frac{\partial \boldsymbol{\varphi}^k}{\partial v}\, dv - i\left(\frac{\partial \boldsymbol{\varphi}^k}{\partial u}\, dv - \frac{\partial \boldsymbol{\varphi}^k}{\partial v}\, du\right)\right].
\end{aligned}$$

Adding these two equations yields

$$\frac{\partial \boldsymbol{\varphi}^k}{\partial u}\, du + \frac{\partial \boldsymbol{\varphi}^k}{\partial v}\, dv = \psi_k\, dz + \overline{\psi_k\, dz} = 2\,\mathrm{Re}(\psi_k\, dz). \tag{4.2}$$

Combining (4.1) and (4.2), we have $d\boldsymbol{\varphi}^k = 2\,\mathrm{Re}(\psi_k\, dz)$. Therefore, upto a translation and scaling factor, neither of which affects the geometric shape of the surface, we have

$$\boldsymbol{\varphi}^k = \mathrm{Re} \int \psi_k\, dz.$$

∎

We are especially interested in the following aspect, described in the above proposition: if U is simply connected, we can construct minimal surfaces from U to $\mathbb{R}^3$ by picking a triple of holomorphic functions ψ_1, ψ_2, ψ_3 satisfying (†), integrate each of them and take each time the real part: the three resulting functions are components of a minimal surface, call it $\boldsymbol{\varphi} : U \to \mathbb{R}^3$.

Bonnet rotation (reprise). It is interesting to note that if ψ_1, ψ_2, ψ_3 satisfy (†), then for any $\theta \in \mathbb{R}$, the triple $e^{i\theta}\psi_1$, $e^{i\theta}\psi_2$, $e^{i\theta}\psi_3$ satisfies (†) as well; if we integrate the new triple and take the real parts, we obtain a new minimal surface, call it $\boldsymbol{\varphi}_\theta : U \to \mathbb{R}^3$. We obviously have $\boldsymbol{\varphi}_0 = \boldsymbol{\varphi}$. The family $\{\boldsymbol{\varphi}_\theta\}_\theta$ is called the *associated family* of $\boldsymbol{\varphi}$. The surface $\widetilde{\boldsymbol{\varphi}} := \boldsymbol{\varphi}_{\frac{\pi}{2}}$ is the *conjugate* of $\boldsymbol{\varphi}$. Note that in fact the latter is given by

$$\widetilde{\boldsymbol{\varphi}}^k = -\mathrm{Im} \int \psi_k(z)\, dz.$$

We can easily check that the coefficients of the first fundamental form of $\boldsymbol{\varphi}_\theta$ are independent of θ: we say that $\{\boldsymbol{\varphi}_\theta\}_\theta$ is an isometric deformation of $\boldsymbol{\varphi}$.
It is, in fact, the Bonnet rotation that we saw earlier.

The following theorem is the main result of the paper.

Theorem 4.7 (**The Weierstrass Representation Theorem**). *Let U be simply connected and $h, g : U \to \mathbb{C}$ two holomorphic functions with $h(z) \neq 0$ everywhere on U. Then $\boldsymbol{\varphi} = (\boldsymbol{\varphi}^1, \boldsymbol{\varphi}^2, \boldsymbol{\varphi}^3) : U \to \mathbb{R}^3$ given by*

$$\boldsymbol{\varphi}^1 := \mathrm{Re} \int \frac{1}{2} h(z)(1 - g(z)^2)\ dz$$

$$\boldsymbol{\varphi}^2 := \mathrm{Re} \int \frac{i}{2} h(z)(1 + g(z)^2)\ dz$$

$$\boldsymbol{\varphi}^3 := \mathrm{Re} \int h(z) g(z)\ dz$$

is a isothermal minimal surface.

Proof. The functions

$$\psi_1(z) = \tfrac{1}{2} h(z)(1 - g(z)^2),$$
$$\psi_2(z) = \tfrac{i}{2} h(z)(1 + g(z)^2),$$
$$\psi_3(z) = h(z)g(z)$$

satisfy equations (†): the first equation is obvious; for the second one, we note that

$$\begin{aligned} &|\psi_1(z)|^2 + |\psi_2(z)|^2 + |\psi_3(z)|^2 \\ &= \tfrac{1}{4}|h(z)|^2 \Big(|1 - g(z)^2|^2 + |1 + g(z)^2|^2 + 4|g(z)|^2 \Big) \\ &= \tfrac{1}{2}|h(z)|^2 (1 + |g(z)|^2)^2. \end{aligned}$$

Here we have used the identity

$$|1 - w^2|^2 + |1 + w^2|^2 + 4|w|^2 = 2(1 + |w|^2)^2,$$

where w is any complex number. We now use Proposition 4.5. ∎

The theorem is telling us how to produce minimal surfaces out of two holomorphic functions h and g. We will do a few examples.

5 Examples of Weierstrass Representation

Example 1. (The catenoid) Take

$$h(z) = -e^{-z} \text{ and } g(z) = -e^{z}.$$

The corresponding minimal surface $\boldsymbol{\varphi}$ has

$$\begin{aligned}
\boldsymbol{\varphi}^1(u,v) &= \tfrac{1}{2}\operatorname{Re}\int (e^{-z} + e^{z})\,dz \\
&= \tfrac{1}{2}\operatorname{Re}(e^{-z} + e^{z}) \\
&= \tfrac{1}{2}\operatorname{Re}\big(e^{-u}(\cos v - i\sin v) + e^{u}(\cos v + i\sin v)\big) \\
&= \cosh u \cos v.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\boldsymbol{\varphi}^2(u,v) &= \tfrac{1}{2}\operatorname{Re}\int i(-e^{-z} - e^{z})\,dz \\
&= \tfrac{1}{2}\operatorname{Re}\big(i(e^{-z} - e^{z})\big) \\
&= -\tfrac{1}{2}\operatorname{Im}\big(e^{-u}(\cos v - i\sin v) - e^{u}(\cos v + i\sin v)\big) \\
&= \cosh u \sin v \\
\boldsymbol{\varphi}^3(u,v) &= \operatorname{Re}\int 1\,dz = \operatorname{Re}(z) = u.
\end{aligned}$$

So

$$\boldsymbol{\varphi}(u,v) = (\cosh u \cos v, \cosh u \sin v, u)$$

which describes the catenoid.

Example 2. (Enneper's Surface) Take

$$h(z) = 1, \quad \text{and} \quad g(z) = z.$$

The corresponding minimal surface has

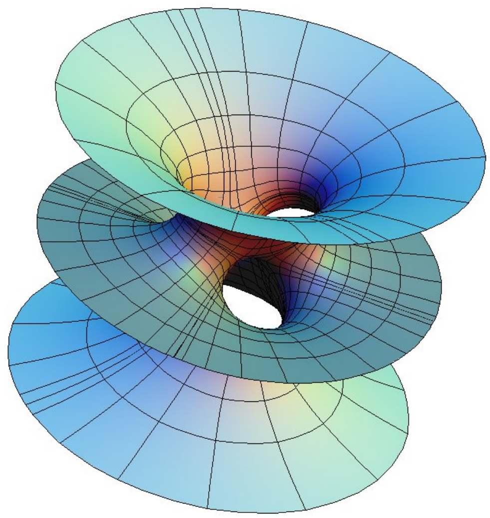
$$\begin{aligned}
\boldsymbol{\varphi}^1(u,v) &= \tfrac{1}{2}\operatorname{Re}\int (1 - z^2)\,dz \\
&= \tfrac{1}{2}\operatorname{Re}\big(z - \tfrac{1}{3}z^3\big) \\
&= \tfrac{1}{2}(u - \tfrac{1}{3}u^3 + uv^2), \\
\boldsymbol{\varphi}^2(u,v) &= \tfrac{1}{2}\operatorname{Re}\left(i\int (1 + z^2)\,dz\right) \\
&= \tfrac{1}{2}\operatorname{Re}\big(i(z + \tfrac{1}{3}z^3)\big) \\
&= \tfrac{1}{2}(-v + \tfrac{1}{3}v^3 - u^2 v), \\
\boldsymbol{\varphi}^3(u,v) &= \operatorname{Re}\int z\,dz = \tfrac{1}{2}\operatorname{Re}(z^2) = \tfrac{1}{2}(u^2 - v^2).
\end{aligned}$$

This is Enneper's surface, up to the factor $\frac{1}{2}$. We saw in [6], *The First Fundamental Form*, Problem 7, that the parametrization

$$\boldsymbol{\varphi}(u,v) = \left(u + uv^2 - \tfrac{1}{3}u^3, v + u^2 v - \tfrac{1}{3}v^3, u^2 - v^2\right)$$

is conformal, hence isothermal.

Example 3. (Costa's Surface)



Costa's minimal surface

We choose

$$h(z) = \wp(z), \quad \text{and} \quad g(z) = \frac{2\sqrt{2\pi}e}{\wp'(z)}$$

where $\wp(z)$ is (a certain choice of) the Weierstrass elliptic function. We will not display the parametrization of the surface resulting via Theorem 4.6. The figure above is intended to give you a (vague) idea of what the surface looks like. In particular, it has no self-intersections.

Remark. It is in general hard to decide if for given h and g the resulting minimal surface is with or without self-intersections (that is, $\boldsymbol{\varphi}$ is injective or not). The catenoid and the helicoid are without self-intersections, but Enneper's surface is not. In fact, minimal surfaces with no self-intersections are very rare. That's why the surface discovered by Costa in 1984 was a surprise for the specialists.

6 Appendix

Parametrizations for some minimal surfaces

Surface	Parametrization $\boldsymbol{\sigma}(u,v)$
The plane:	$(u, v, 0)$
The Enneper surface:	$\left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + u^2v, u^2 - v^2\right)$
The catenoid:	$(\cosh u \cos v, \cosh u \sin v, u)$
The helicoid:	$(\sinh u \cos v, \sinh u \sin v, u)$
Scherk's doubly periodic surface:	$\left(u, v, \ln \frac{\cos u}{\cos v}\right)$
Scherk's singly periodic surface:	$\big(\operatorname{arcsinh}(u), \operatorname{arcsinh}(v), \arcsin(uv)\big)$
Catalan surface:	$\big(1 - \cos(u)\cosh(v), 4\sin(\frac{u}{2})\sinh(\frac{v}{2}), u - \sin(u)\cosh(v)\big)$

References

[1] do Carmo, Manfredo Perdigão. 1976. *Differential geometry of curves and surfaces.* Prentice Hall.

[2] Dorff, Michael et al. 2012. *Explorations in Complex Analysis.* Mathematical Association of America.

[3] Klarreich, Erica. 2022. *Math Duo Maps the Infinite Terrain of Minimal Surfaces.* Quanta Magazine.

[4] Osserman, Robert. 1986. *A Survey of Minimal Surfaces.* Dover, New York.

[5] Pressley, Andrew. 2001. *Elementary Differential Geometry.* London: Springer.

[6] Rubinstein-Salzedo, Simon. 2022. *Differential Geometry.* Lecture notes for Euler Circle, Spring 2022.

[7] Spivak, Michael. 1999. *A Comprehensive Introduction to Differential Geometry.* Publish or Perish, Inc.