Differential Forms and the Hodge Operator

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1 Introduction

Differential forms provide an elegant way to state results from vector calculus and physics. They are a natural extension of standard 1-dimensional calculus to not only higher dimensions, but to non-Euclidean spaces.

2 m-forms

2.1 Tangent Spaces and 1-forms

Let S be a surface. We will use the familiar notation of T_pS to denote the tangent space of S at the point p.

We begin with a discussion on 1-forms. A 1-form is a linear map from a vector in a tangent space, say T_pS, to a scalar value. For these tangent vectors to S at p, we describe them as vectors independent of the tangent point p.

For now, we shall assume that this surface S is $\mathbb{R}^n$, and so the tangent space T_pS must also be $\mathbb{R}^n$ for all $p \in \mathbb{R}^n$.

Let ω be a 1-form. Because it is a linear map mapping every vector $v = (x_1, x_2, \dots, x_n) \in T_pS$ onto $\mathbb{R}$, it ought to be represented as $\omega(v) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ (where a_i are constants), as in linear algebra. With differential forms, this is written with the fundamental 1-forms.

In T_pS, $dx_1, dx_2, \dots, dx_n$ are the fundamental 1-forms, where for all $v = (x_1, x_2, \dots, x_n)$,

$$dx_i(v) = x_i$$

Then, for every 1-form ω, we may represent ω as

$$d\omega = \sum_{i=1}^{n} a_i dx_i$$

Here, a_i are constant coefficients.

Example. For example, in $T_p\mathbb{R}^3$, $\omega = 2dx + 3dy + 5dz$ would be a 1-form. It is important to remember that differential forms are all maps of some sort, and we can calculate the value of $d\omega$ given a vector, say $v = (1, 3, 2)$:

$$\omega(v) = 2dx(v) + 3dy(v) + 5dz(v) = 2(1) + 3(3) + 5(2) = 21$$

Because 1-forms are linear, any 1-form ω must satisfy $\omega(c_1v_1 + c_2v_2) = c_1\omega(v_1) + c_2\omega(v_2)$ for vectors v_1, v_2 and scalars c_1, c_2. Also, as one may expect, 1-forms may be added together or multiplied by a scalar.

2.2 Wedge Operator

We shall now introduce the wedge operator, written $\wedge$, which generalizes the 1-form to higher dimensions. Generally, the wedge operator takes in m 1-forms and spits out an m-form.

Let ω_1 and ω_2 be two 1-forms, which map T_pS into $\mathbb{R}$. Then, $\omega_1 \wedge \omega_2$ is a 2-form, which is a function that maps *2 vectors* in T_pS into $\mathbb{R}$. Specifically, given $v_1, v_2 \in T_pS$, then

$$\omega_1 \wedge \omega_2(v_1, v_2) = \det \begin{pmatrix} \omega_1(v_1) & \omega_2(v_1) \\ \omega_1(v_2) & \omega_2(v_2) \end{pmatrix}$$

More generally, given m 1-forms $\omega_1, \omega_2, \cdots, \omega_m$, then $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_m$ is an m-form mapping m vectors T_pS into $\mathbb{R}$. To compute this function, we have that

$$(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_m)(v_1, v_2, \cdots v_m) = \det \begin{pmatrix} \omega_1(v_1) & \cdots & \omega_m(v_1) \\ \vdots & \ddots & \vdots \\ \omega_1(v_m) & \cdots & \omega_m(v_m) \end{pmatrix}$$

Note that $\wedge$ is associative. Now, we may define any general m-form as a linear combination of these wedge products. In other words, if ω is an m-form, then

$$\omega = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_m \le n} a_{i_1 i_2 \cdots i_m} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}$$

where a_I are constant coefficients.

Example. In $T_p\mathbb{R}^3$, $\omega = 3dx \wedge dy + 2dy \wedge dz$ is a 2-form. Recalling that ω takes in 2 3-dimensional vectors, let $v_1 = (1, 2, 3)$ and $v_2 = (0, 4, 5)$. Then,

$$\begin{aligned} \omega(v_1, v_2) &= 3dx \wedge dy(v_1, v_2) + 2dy \wedge dz(v_1, v_2) \\ &= 3 \det \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix} + 2 \det \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \\ &= 3(4) + 2(-2) = 8 \end{aligned}$$

The wedge operator need not only operate on 1-forms. Say $\alpha = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}$ and $\beta = dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_q}$. Then,

$$\alpha \wedge \beta = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_q},$$

as one may expect. Furthermore, $\wedge$ is distributive, which allows us to wedge together any two general m-forms.

There are a couple key properties for $\wedge$.

Theorem 2.1. *Let c, d be scalars, let α be a p-form and let β be a q-form. Then, we have that*

$$c\alpha \wedge d\beta = cd(\alpha \wedge \beta).$$
$$\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha.$$
$$\alpha \wedge \alpha = 0$$

Example. Let us work through an example in $T_p\mathbb{R}^4$ with components x, y, z, w. Let $\omega_1 = dx \wedge dw + 3dy \wedge dw$ and $\omega_2 = 3dx + 2dy$. Then,

$$\begin{aligned}\omega_1 \wedge \omega_2 &= (4dx \wedge dw + 3dy \wedge dw) \wedge (3dx + 2dy)\\ &= 4dx \wedge dw \wedge 3dx + 4dx \wedge dw \wedge 2dy + 3dy \wedge dw \wedge 3dx + 3dy \wedge dw \wedge 2dy\end{aligned}$$

Any term containing two of the same fundamental 1-form becomes 0, as $\omega \wedge \omega = 0$. For exaple, $4dx \wedge dw \wedge 3dx = -12(dx \wedge dx) \wedge dw = -12(0) \wedge dw = 0$. Thus, our above sum becomes

$$\begin{aligned}\omega_1 \wedge \omega_2 &= 4dx \wedge dw \wedge 2dy + 3dy \wedge dw \wedge 3dx\\ &= -8dx \wedge dy \wedge dw + 9dx \wedge dy \wedge dw\\ &= dx \wedge dy \wedge dw\end{aligned}$$

Note our use of the anticommutative property.

We may note that all m-forms make up a vector space themselves, with a basis of the m-forms $dx_{i_1}dx_{i_2}\cdots dx_{i_m}$ for $i_1 \leq i_2 \leq \cdots \leq i_m$. This vector space is denoted $\bigwedge^m$, with dimension $\binom{n}{m}$

2.3 Differential Forms

So far, we have talked about m-forms, but we have not yet touched on the "differential" aspect. A differential form is an m-form, but instead of constant value coefficients, the coefficients are *smooth functions of the tangent point p*. For example, if $S = \mathbb{R}^3$, then $xy^2dx \wedge dy$ is a differential 2-form. Note xy^2 refers to the x and y components of the tangency point p, and not of the tangent vector.

Generally, we may write that

$$\omega = \sum_{I=(i_1,\ldots,i_m)} f_I(p)dx_I$$

We indulge here in some abuse of notation, where we use the multi-index I instead of writing out $i_1, i_2, \ldots, i_m$ all the time.

Evaluating a differential form is exactly the same as our previous forms, except that we must now know the tangency point as well.

Example. In $T_p\mathbb{R}^2$, let us evaluate $\omega = xy^2dx \wedge dy$ at the tangent point $p = (1, 3)$ and with two tangent vectors $v_1 = (2, 3), v_2 = (-1, -2)$. At this p, our differential form becomes $(1)(3)^2dx \wedge dy = 9dx \wedge dy$. Then, since we know the tangent vector, we may evaluate this 2-form like before, which gives $9 \det \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} = 9$.

Finally, note that differential 0-forms exist and are simply scalar functions that take in as an input the point of tangency p.

3 Calculus on Forms

3.1 Integrating Forms

One of the uses of differential forms is to be integrated. To integrate a differential m-form, you must integrate it over some m-dimensional manifold S.

In general, the m-dimensional manifold may be parameterized as some $\sigma : U \to \mathbb{R}^n$, where U is some open subset in $\mathbb{R}^m$. For example, space curves are functions $\boldsymbol{\gamma} : [0,1] \to \mathbb{R}^3$, and surfaces are functions $\boldsymbol{\sigma} : \mathbb{R}^2 \to \mathbb{R}^3$.

To integrate ω over this manifold $\boldsymbol{\sigma}$, we will integrate over U. For every $p \in U$, note that $\boldsymbol{\sigma}(p)$ will be some point on the manifold. This point will be our tangency point.

Then, with the parameterization $\boldsymbol{\sigma}$, we will use the m partial derivatives (which, as we recall, are tangent vectors that form a basis for T_pS) as inputs to our differential m form. In other words, we define

$$\int_S \omega = \int_U \omega_{\boldsymbol{\sigma}(p)}(\boldsymbol{\sigma}_{x_1}(p), \boldsymbol{\sigma}_{x_2}(p), \cdots, \boldsymbol{\sigma}_{x_m}(p))dV$$

Here, dV is $dx_1 dx_2 \cdots dx_m$ (which are standard differentials from vector calculus and not differential forms).

Example. Let us integrate $\omega = ydx$ over the half-circle $\boldsymbol{\gamma} : [0,\pi]$, where $\boldsymbol{\gamma}(t) = (\cos t, \sin t)$.

$$\begin{aligned}
\int_{\boldsymbol{\gamma}} \omega &= \int_0^\pi \omega(\dot{\boldsymbol{\gamma}}(t))dt = \int_0^\pi \omega_{(\cos t, \sin t)}(\langle -\sin t, \cos t\rangle)dt \\
&= \int_0^\pi (\sin t)dx(\langle -\sin t, \cos t\rangle)dt \\
&= \int_0^\pi -\sin^2 t \; dt = -\frac{\pi}{2}
\end{aligned}$$

Note that integrating 1-forms over curves is exactly just a line integral. In other words, if $\omega = f dx$, then $\int_\gamma \omega = \int_\gamma f \; dx$.

3.2 The Exterior Derivative

Now that we have the integral, it is only natural that we must have a derivative. The *exterior derivative* d is a linear operator that operates on a differential form. For every 0-form (recall that 0-forms are just functions) f,

$$df = f_{x_1}x_1 + f_{x_2}x_2 + \cdots + f_{x_n}x_n$$

Note that this is a differential 1-form that closely resembles the gradient of f. Then, for any differential m-form $f dx_I$,

$$d(f dx_I) = (df) \wedge dx_I$$

Example. Let us compute $d\omega$ for $\omega = xydx + zdy$ in $T_p\mathbb{R}^3$.

$$d\omega = (ydx + xdy) \wedge dx + (dz) \wedge dy = -xdx \wedge dy - dy \wedge dz$$

An important property of the exterior derivative is that, for any differential form ω,

$$d^2\omega = 0$$

The proof is due to the symmetry of second derivatives and the anticommutativity of $\wedge$.

Moreover, d satisfies some version of the product rule. Given that α is a differential m-form and β is a differential k-form, then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^m \alpha \wedge d\beta.$$

4 The Hodge Operator

An interesting thing to note is that in $\mathbb{R}^n$, $\bigwedge^m$ and $\bigwedge^{n-m}$ have the same dimension. Motivated by this, the Hodge operator is a linear operator denoted $\star$ that maps differential m forms to differential $n-m$ forms.

For any form dx_I, $\star dx_I$ is the unique form such that

$$dx_I \wedge \star dx_I = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

Example. In $\mathbb{R}^3$, $\star dx = dy \wedge dz$ because $dx \wedge \star dx = dx \wedge dy \wedge dz$. Meanwhile, $\star dy = -dx \wedge dz$, since $dy \wedge \star dy = dy \wedge -dx \wedge dz = dx \wedge dy \wedge dz$.

Note that applying the Hodge operator twice on ω returns ω, up to a sign. If ω is a differential m-form, then

$$\star \star \omega = (-1)^{m(n-m)}\omega$$

With the Hodge operator, it is possible to concisely represent the gradient, divergence and curl.

Given a function f, we have already seen that if we treat f as a differential 0-form then df becomes analogous to the gradient.

$$\text{grad} f = df$$

Now, let f be a vector field in $\mathbb{R}^3$ with $f = P\hat{i} + Q\hat{j} + R\hat{k}$. Instead of writing it this way, let us write it as a differential 1-form, as $f = Pdx + Qdy + Rdz$. We claim that

$$\text{curl} f = \star df$$

This is because

$$\begin{aligned}\star df &= \star(-P_y dx \wedge dy - P_z dx \wedge dz + Q_x dx \wedge dy - Q_z dy \wedge dz + R_x dx \wedge dz + R_y dy \wedge dz)\\ &= -P_y dz + P_z dy + Q_x dz - Q_z dx - R_x dy + R_y dx\\ &= (R_y - Q_z)dx + (P_z - R_x)dy + (Q_x - P_y)dz\\ &= \text{curl} f\end{aligned}$$

Furthermore, we claim that

$$\text{div} f = \star d \star f$$

$$
\begin{aligned}
\star d \star f &= \star d(Pdy \wedge dz - Qdx \wedge dz + Rdx \wedge dy) \\
&= \star(P_x dx \wedge dy \wedge dz + Q_y dx \wedge dy \wedge dz + R_z dx \wedge dy \wedge dz) \\
&= P_x + Q_y + R_z = \text{div} f
\end{aligned}
$$

4.1 Stoke's Theorem, Generalized

A generalization of Stoke's theorem may be found with differential forms. In $\mathbb{R}^n$, let C be an orientable manifold with boundary ∂C. Given an $(m-1)$-form ω, we have that

$$\int_{\partial C} \omega = \int_C d\omega$$

When C is a 2-dimensional surface embedded in $\mathbb{R}^3$, then ω is some 1-form $\omega = Pdx + Qdy + Rdz$. Note that $d\omega = \star \star d\omega = \star\text{curl}(\omega)$. Then, we have that

$$\int_{\partial C} Pdx + Qdy + Rdz = \int_C \star\text{curl}(\omega)$$

This is exactly the statement of the Stoke's theorem from vector calculus. Moreover, the formula generalizes not only Stoke's Theorem but also the fundamental theorem of line integrals (when ω is a 0-form) as well as the divergence theorem (when using $\star\omega$ instead of ω).

References

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[3] Rick Presman. *The Generalized Stokes' Theorem*