The Poincaré-Hopf Theorem and the Hairy Ball Consequence

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Abstract

In this paper we introduce the concept of tangent vector fields, and prove a theorem, the Poincaré-Hopf Theorem, about their zeroes/singularities. We will also discuss corollaries, most notably the Hairy Ball Theorem. Many approaches to the Poincaré-Hopf Theorem involve heavy studies of Differential Topology, however we will show that we can prove the theorem using only basic tools of Differential Geometry. We shall restrict our study of the Poincaré-Hopf Theorem to $\mathbb{R}^3$, but Heinz Hopf has shown it can be generalized to higher dimensions.

1 Vector Fields

The Poincaré-Hopf Theorem concerns a structure called a tangent vector field, and more specifically, it's zeros. We define such a vector field as follows:

Definition 1.1. Let S be a surface with patch $\boldsymbol{\sigma} : U \to \mathbb{R}^3$. A tangent vector field on S is a map $V : S \to T_{\mathbf{p}}S$ that takes each point $\mathbf{p}$ on S to a vector in the tangent space.

Example. A tangent vector field on $\mathbb{S}^2$ with patch $\boldsymbol{\sigma}(u, v) = (\cos u \cos v, \cos u \sin v, \sin u)$ is the vector field $V(u, v) = (-\cos u \sin v, \cos u \cos v, 0)$, which is just $\boldsymbol{\sigma}_v$. Since the tangent space of a surface is spanned by the partial derivatives of the surface patch, it is clear that V is a tangent vector field.

Example. A smooth tangent vector field V is given by $V = \alpha(u, v)\boldsymbol{\sigma}_u + \beta(u, v)\boldsymbol{\sigma}_v$, for smooth α, β.

Definition 1.2. Let S be a surface and V a smooth tangent vector field on it. A *singularity* of V, is a point $\mathbf{p} \in S$ such that $V(\mathbf{p}) = \mathbf{0}$. The *index* of a singularity is an integer given by

$$\operatorname{Ind}(\mathbf{p}) = \frac{1}{2\pi} \int_{\boldsymbol{\gamma}} \phi' \, dt$$

where $\boldsymbol{\gamma}$ is any simple closed, unit-speed, positively oriented, regular curve in $\boldsymbol{\sigma}$ enclosing a singularity $\mathbf{p}$ and no other singularities, and $\phi(t)$ is the signed angle between a nowhere vanishing smooth tangent vector field ζ and V along $\boldsymbol{\gamma}$.

For easy computation, and for proving the Poincaré-Hopf Theorem, we would like this ζ to be σ_u, but we must first prove that the index of a singularity remains unchanged for any two different ζ's; i.e. we have to show the index is independent of the choice of ζ.

Proposition 1.3. *Taking notation as in Definition 1.2, the index of a singularity does not depend on our choice of ζ.*

Proof. Let ζ_1 and ζ_2 be two different smooth tangent vector fields on S. Let ϕ_1 and ϕ_2 be their signed angles with V respectively. To prove our assertion, it suffices to prove that the integral over the derivative of the difference function $\theta = \phi_1 - \phi_2$ is equal to zero. In short, we must prove:

$$\int_{\boldsymbol{\gamma}} \dot{\theta}\, dt = 0.$$

To do so, consider the equalities below by noticing that θ is the signed angle between ζ_1, ζ_2:

$$\rho = \frac{\zeta_1 \cdot \zeta_2}{\|\zeta_1 \cdot \zeta_2\|} = \cos\theta, \quad \dot{\theta} = \frac{\dot{\rho}}{\sqrt{1-\rho^2}}.$$

Taking the line integral yields:

$$\int_{\boldsymbol{\gamma}} \frac{\dot{\rho}}{\sqrt{1-\rho^2}}\, dt = \int_{\boldsymbol{\gamma}} \frac{\rho_u \dot{u} + \rho_v \dot{v}}{\sqrt{1-\rho^2}}\, dt = \int_{\boldsymbol{\gamma}} \frac{\rho_u du + \rho_v dv}{\sqrt{1-\rho^2}}$$

which then simplifies to the following by Green's Theorem:

$$\int_{\mathrm{int}(\boldsymbol{\gamma})} \left(\frac{\partial}{\partial u} \frac{\rho_v}{\sqrt{1-\rho^2}} - \frac{\partial}{\partial v} \frac{\rho_u}{\sqrt{1-\rho^2}} \right) = 0.$$

This completes our proof. □

Example. Consider the plane given by $\boldsymbol{\sigma}(u,v) = (u,v,0)$, and a tangent vector field $V(u,v) = (u,v,0)$. We have a singularity at $\boldsymbol{\sigma}(0,0)$, and a curve $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t),v(t)) = (\cos t, \sin t, 0)$ enclosing said singularity. Thus:

$$\phi(t) = \arccos\left(\frac{\boldsymbol{\sigma}_u \cdot V}{\|\boldsymbol{\sigma}_u\|\|V\|}\right) = \arccos\left(\frac{u(t)}{\sqrt{u(t)^2 + v(t)^2}}\right) = \arccos\cos t = t$$

which then means:

$$\mathrm{Ind}(\boldsymbol{\sigma}(0,0)) = \frac{1}{2\pi}\int_0^{2\pi} 1\, dt = \frac{1}{2\pi} t\Big|_0^{2\pi} = 1.$$

This type of singularity is known as a source. The same singularity for $\tilde{V} = (-u, -v, 0)$ is known as a sink. Note that we can now choose σ_u as a non-vanishing smooth tangent vector field, due to Proposition 1.3.

Those with a background in Complex Analysis may have seen singularities during their studies. The singularity of vector fields is essentially the same as the singularities in Complex Analysis, due to the very close relation between vectors and complex numbers. The sink and source are characterized by the zeroes of $f(z) = -z$ and $f(z) = z$ respectively. There is also a singularity known as a *centre*, which is a zero where the vector field circles around it. In terms of vector fields, we can have $V(u,v) = (\pm u, \mp v, 0)$, which has the complex-analogue $f(z) = \pm iz$. Centres have index 1. For further reading, refer to Needham's text ([2], Chapter 19.4).

By now, an interesting property may be noticed: the index is an integer. This is the case with vector fields, but there are also other structures called line fields, which can have rational indices.

Proposition 1.4. *Taking notation as in Definition 1.2, the index of a singularity does not depend on our choice of* $\boldsymbol{\gamma}$ *and is also an integer.*

The full, proper proof will make use of the techniques used in proving Hopf's Umlaufsatz, but that would go beyond the realm of Differential Geometry, so we provide a heuristic proof omitting some technicalities.

Proof. Let l be the period of $\boldsymbol{\gamma}$ such that:

$$\mathrm{Ind}(\mathbf{p}) = \frac{1}{2\pi} \int_0^{l(\boldsymbol{\gamma})} \phi' \, dt = \frac{1}{2\pi} \left(\phi(l) - \phi(0) \right).$$

Then it suffices to show that $\phi(l) - \phi(0)$ is some integer multiple of 2π, where ϕ is the signed angle between some ζ and V. We can do so by considering $\zeta / \|\zeta\|$, thanks to Proposition 1.3. Additionally we can consider the orthonormal basis $\{V/\|V\|, \mathbf{N} \times V/\|V\|\} = \{V/\|V\|, U\}$ (which, along $\boldsymbol{\gamma}$, has no singularities by construction) for the tangent space to get:

$$\xi = \frac{\zeta}{\|\zeta\|} = \frac{V}{\|V\|} \cos\phi + U \sin\phi. \tag{1.1}$$

To avoid confusion from the fact that ζ and hence $\zeta/\|\zeta\|$ should be nowhere-vanishing, we emphasize that (1.1) isn't the actual formula for $\zeta/\|\zeta\|$, but is instead a useful representation only along the curve $\boldsymbol{\gamma}$, since we can ensure $V \neq \mathbf{0}$ there. Observe that at the point $\boldsymbol{\gamma}(l)$, the vectors ξ, V, U are equal to their original values at $\boldsymbol{\gamma}(0)$ by the l-periodicity of $\boldsymbol{\gamma}$, so by (1.1) we can deduce:

$$(\cos\phi(l), \sin\phi(l) = (\cos\phi(0), \sin\phi(0)).$$

This then proves the fact that $\phi(l) - \phi(0)$ is an integer multiple of 2π. Next, we consider a new curve $\tilde{\boldsymbol{\gamma}}$, that is also enclosing the singularity $\mathbf{p}$ and has period l as well. Then there is a family of curves $\boldsymbol{\gamma}^\tau$, which are continuous in $\tau \in [0,1]$ and have $\boldsymbol{\gamma}^0 = \boldsymbol{\gamma}$ and $\boldsymbol{\gamma}^1 = \tilde{\boldsymbol{\gamma}}$. The integral

$$\int_{\boldsymbol{\gamma}^\tau} \phi'_\tau \, dt = \int_0^{l(\boldsymbol{\gamma}^\tau)} \phi'_\tau \, dt$$

where ϕ_τ is the signed angle between ζ and V along $\boldsymbol{\gamma}^\tau$, is a continuous function $f(\tau)$ of τ, and we have $f(\tau) = 2\pi n(\tau)$ for $n(\tau) \in \mathbb{Z}$. By the Intermediate Value Theorem, this function/integral must be constant; a continuous function of integer outputs is a constant function, so $f(\tau)/(2\pi)$ must be constant and hence $f(\tau)$ must be constant. This completes the proof that the index is independent of our choice of $\boldsymbol{\gamma}$. □

In the discussions following the Poincaré-Hopf Theorem, the notation of the index being zero will come up implicitly. This can be thought of as a special case when $\mathbf{p}$ isn't a singularity, as that would imply that we can set $\zeta = V$ in Proposition 1.3 (because now V is nowhere-vanishing), and the angle in Definition 1.2 would be zero. Hence the integral over the derivative of said angle will also clearly be zero, making the index be zero as well.

For the rest of this paper, we shall assume tangent vector fields are smooth, surfaces are compact without boundary, and curves are simple closed, unit-speed, regular. Unless stated otherwise of course.

2 The Poincaré-Hopf Theorem

We may now present the heart of this paper: the Poincaré-Hopf Theorem. Much like the celebrated Gauß-Bonnet Theorem, the Poincaré-Hopf relates the topology of a surface to a less global and more local property. In the case of Gauß-Bonnet, that is the curvature. In the case of Poincaré-Hopf, we look at singularities of vector fields; and consequently, their indices. In both cases, we sum the local property in question to relate it to the global topology, but unlike the Gauss-Bonnet Theorem, we do a discrete summation in the Poincaré-Hopf instead of a continuous integration.

Theorem 2.1 (Poincaré-Hopf)**.** *Let V be a smooth tangent vector field on a compact surface S with finitely many **isolated** singularities $\mathbf{p}_1, .., \mathbf{p}_k$. Then we have:*

$$\sum_{i=1}^{k} \text{Ind}(\mathbf{p}_i) = \chi(S)$$

where χ is the Euler Characteristic.

Here, we consider isolated singularities, which just means that there is some ϵ-neighbourhood about one singularity for which no other singularities exist. A simple example of non-isolated singularities would be something like $f(v) = V(u_0, v) = 0$, whereas an isolated singularity must always have both inputs u, v fixed.

Proof. Let V be a tangent vector field on S. Choose disjoint regions $R_1, ..., R_k \subset S$ each containing a singularity $\mathbf{p}_i$, such that it's image is contained in a patch $\boldsymbol{\sigma}_i$, and has positively oriented boundary $\boldsymbol{\gamma}_i$. Let S' denote the compliment of the interiors of these regions:

$$S' = S \setminus \bigcup_{i=1}^{k} R_i.$$

Triangulate S' into curvilinear polygons $P_1, ..., P_n$, each with positively oriented boundary $\tilde{\boldsymbol{\gamma}}_i$. By the Gauß-Bonnet Theorem we have:

$$\int_{S'} K\, dA + \sum_{i=1}^{k} \int_{R_i} K\, dA = 2\pi\chi(S). \tag{2.1}$$

Now, on S, choose an orthonormal basis $\{U_1, U_2\}$ for $T_{\mathbf{p}}S$ such that:

$$U_1 = (\boldsymbol{\sigma}_i)_u / \|(\boldsymbol{\sigma}_u)_i\|, \quad U_2 = U_1 \times \mathbf{N}.$$

As a result of the Gauß-Bonnet Theorem for simple closed curves, we have:

$$\int_{R_i} K\, dA = \int_{\boldsymbol{\gamma}_i} U_1 \cdot \dot{U}_2\, dt. \tag{2.2}$$

Repeat a similar process on S', where we construct an orthonormal basis $\{E_1, E_2\}$ for the tangent space such that:

$$E_1 = \frac{V}{\|V\|}, \quad E_2 = E_1 \times \mathbf{N}.$$

We may do so as V has no singularities on S'. With this construction, by the Gauß-Bonnet Theorem for simple closed curves we get:

$$\int_{S'} K\,dA = \sum_{i=1}^{n} \int_{P_i} K\,dA = \sum_{i=1}^{n} \int_{\tilde{\gamma}_i} E_1 \cdot \dot{E}_2\,dt = -\sum_{i=1}^{k} \int_{\gamma_i} E_1 \cdot \dot{E}_2\,dt. \tag{2.3}$$

This last equality follows from the fact that some of the curvilinear polygons will have edges that are segments of $\boldsymbol{\gamma}_i$, and a positive orientation on $\tilde{\boldsymbol{\gamma}}_i$ induces a negative on $\boldsymbol{\gamma}_i$. Two common edges of the curvilinear polygons are traversed twice in each direction, so the sum of their line integrals cancel out. In the end, the only edges that aren't shared by the curvilinear polygons are segments $\boldsymbol{\gamma}_i$, and we have a minus sign from the induced negative orientation of $\boldsymbol{\gamma}_i$ from the positive orientation of $\tilde{\boldsymbol{\gamma}}_i$ as previously mentioned. After combining (2.1), (2.2), and (2.3) we get:

$$2\pi\chi(S) = \sum_{i=1}^{k} \int_{\gamma_i} U_1 \cdot \dot{U}_2\,dt - \sum_{i=1}^{k} \int_{\gamma_i} E_1 \cdot \dot{E}_2\,dt = \sum_{i=1}^{k} \int_{\gamma_i} U_1 \cdot \dot{U}_2 - E_1 \cdot \dot{E}_2\,dt. \tag{2.4}$$

We also have (see [3] pg337, Eq13.4):

$$U_1 \cdot \dot{U}_2 = \dot{\alpha} - \kappa_g, \quad E_1 \cdot \dot{E}_2 = \dot{\beta} - \kappa_g,$$

where α and β are the signed angles between $U_1, \dot{\boldsymbol{\gamma}}_i$ and $E_1, \dot{\boldsymbol{\gamma}}_i$, and κ_g is the geodesic curvature of $\boldsymbol{\gamma}_i$. Thus we can simplify (2.4) as:

$$2\pi\chi(S) = \sum_{i=1}^{k} \int_{\gamma_i} \dot{\alpha} - \dot{\beta}\,dt.$$

As a result of $U_1 = (\boldsymbol{\sigma}_i)_u$ and $E_1 = V/\|V\|$, we get $\alpha - \beta = \phi$, the signed angle between $\boldsymbol{\sigma}_u, V$. Thus by Definition 1.1:

$$2\pi\chi(S) = \sum_{i=1}^{k} \int_{\gamma_i} \dot{\phi}dt = 2\pi \sum_{i=1}^{k} \mathrm{Ind}(\mathbf{p}_i).$$

This completes the proof of the Poincaré-Hopf Theorem. □

3 Corollaries of the Poincaré-Hopf Theorem

With the Poincaré-Hopf Theorem proven, we can prove some very interesting corollaries, most notably the Hairy Ball Theorem.

Corollary 3.1 (Hairy Ball Theorem)**.** *The sphere does not admit a smooth nowhere-vanishing tangent vector field.*

Proof. From Topology, it is known that $\chi = 2$ for a sphere, which we shall take as granted without proof. With this in mind, the Poincaré-Hopf Theorem shows that the sum of indices must be 2 for any smooth tangent vector field, which means there must exist some singularities. □

As the name suggests, a nice way to interpret this corollary is by thinking of a hairy ball, with the hairs combed flat as tangent vectors in the smooth tangent vector field. After trying to comb the hairy ball flat for quite some time, it would become apparent that there are at minimum two points that cannot be combed; the two cowlicks at the poles. Trying to comb over these cowlicks will just create more cowlicks, and we will never be able remove any instance of cowlick on the "hair field"; i.e. our smooth tangent vector field cannot be constructed without singularities.

Another nice interpretation of the hairy Ball Theorem is given by Needham ([2] Chapter 19.5). If we pour honey down a sphere, the honey must converge to one point on the bottom of the sphere and drip down. The place we pour the honey is a source, and the place where it drips down is a sink. As we have shown in Section 1, a sink and source each has index 1, so the sum of all indices is simply 1+1 for this honey field; i.e. 2, which is also the Euler Characteristic of the sphere.

Yet another interpretation is by thinking of the Earth, and thinking of wind currents as the tangent vector field; with the generous assumption of smoothness and even "tangent-ness". The Hairy Ball Theorem suggests that there must be at least one singularity of this wind-field on our planet, which would be something like the eye of the storm in a cyclone, or most likely a less extreme place where there is no wind.

A very nice generalization of the Hairy Ball Theorem is:

Corollary 3.2. *If a surface S has nonzero Euler Characteristic, it does not admit a nowhere-vanishing smooth tangent vector field.*

Another corollary of the Poincaré-Hopf Theorem is the following:

Corollary 3.3. *Any two smooth tangent vector fields with isolated singularities on a surface S will have the same sum of indices of their respective singularities. The sum of indices of isolated singularities of any smooth tangent vector field on any surface homeomorphic to S will also be the same.*

Proof. By the Poincaré-Hopf Theorem, for any two vector fields V_1, V_2 with singularities $\mathbf{p}_i^1, \mathbf{p}_j^2$ (say, for $i = 1, .., n$ and $j = 1, ..., k$), we have:

$$\sum_{i=1}^{n} \text{Ind}(\mathbf{p}_i^1) = \chi(S) = \sum_{j=1}^{k} \text{Ind}(\mathbf{p}_j^2).$$

Next, we can borrow another fact of Topology without proof. Namely, if two surfaces are homeomorphic, they have the same Euler Characteristic. Thus if $\tilde{S}$ is a surface with any two smooth tangent vector fields $\tilde{V}_2, \tilde{V}_2$, we can just apply the same argument as before and complete the proof. □

Corollary 3.4. *If a surface S can admit one smooth tangent vector field with a single singularity, it and any other surface homeomorphic to S cannot admit a nowhere-vanishing smooth tangent vector field.*

Proof. If there exists one smooth tangent vector field V on S with a single singularity $\mathbf{p}$, then by the Poincaré-Hopf Theorem we have $\chi(S) = \text{Ind}(\mathbf{p}) \neq 0$, so by Corollary 3.3, any other smooth tangent vector field on S must not be nowhere-vanishing, as the sum of indices must nonzero: $\chi(S) = \text{Ind}(\mathbf{p}) \neq 0$. Since the Euler Characteristic is invariant under homeomorphisms, our above argument applies to any surface $\tilde{S}$ homeomorphic to S. □

References

[1] Butt, Karen. *The Gauß-Bonnet Theorem.* University of Chicago. 2015.

[2] Needham, Tristan. *Visual Differential Geometry and Forms: A Mathematical Drama in Five Acts.* Princeton Press. 2021

[3] Pressley, Andrew. *Elementary Differential Geometry, Second Edition.* Springer. 2012.