Characteristic Classes and Fiber Bundles

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1 Fiber Bundles

A fiber bundle is a bundle of fibers of unimportant length and thickness. The *total space* of a fiber bundle is the *fiber space* bundled along the *base space*. A *fiber bundle* is defined as
I. The base space B, the total space E, and a fiber F.

II. A continuous map, π:E$\rightarrow$B, called the projection map.

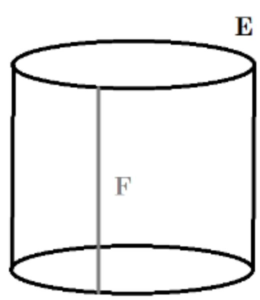
III. An open neighborhood U_x of x for each x$\in$B and a homeomorphism

φ_x:π^{-1}(U$_x$)$\rightarrow U_x\times$F.

E is a fiber bundle over B with fiber F, which can be denoted $\pi : E \rightarrow B$.

2 Example of a Fiber Bundle

Using a cylinder, E, of height 2 with a base disk of radius 1, E=D$^2 \times [-1,1]$, where the unit disk $D^2 = \{(x,y) \in \mathbf{R}^2 | x^2 + y^2 = 1\}$.



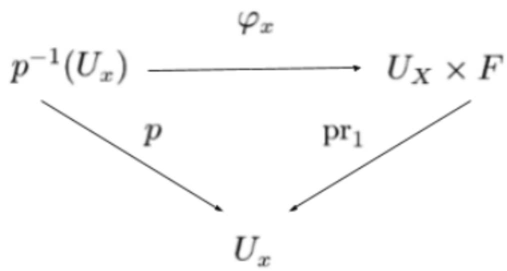
Then, E is copies of [-1,1] on D^2 placed vertically. This means E is a bundle of [-1,1] along D^2. The closed intervals, ([-1,1]), are the fibers.

3 Trivial Bundle

B is a connected topological space. A covering space over B is a continuous map p:E $\to B$ such that each point x $\in$ B and each point $y \in p^-1(x)$ have an open neighborhood U_x and U_y satisfying
1) if $y \neq y', U_y \cap U_y' = 0$ and

2) p(U$_y$) U$_x$ and the restriction insert is a homeomorphism. Here, p is the projection and $p^{-1}(x)$ is a fiber over x.

p$^-1(U_x)$ is a bundle of $p^{-1}(x)$ along U_x.



Each map φ_x is a local trivialization. Given the spaces B,E,and F, a continuous map $p : E$is a fiber bundle with fiber F if and only if B has an open covering $B = \bigcup_{\alpha \in A} U_\alpha$ and there exists a homeomorphism $\varphi_\alpha : p^{-1}(U_\alpha \to U_\alpha \times F$. When E=B$\times F$ and p: E $\to B$ is the projection onto the first factor, it is a trivial bundle.

4 Group Action

A left group action μ of a topological group G on a topological group X is a continuous map μ:G$\times$X$\to$X. A right action of G on X is a continuous map $\mu : X \to X$ satisfying
1) For any $x \in S$, where S is a set, $\mu(x, e) = x$, where e is the unit of G.

2)For g,h in G and $x \in S$, have $\mu(\mu(x, g), h) = \mu(x, gh)$.
A left action G to S switches S and G.

5 Principal Bundle

Let G be a topological group. A fiber bundle $\pi : E \to B$ with structure group G and fiber G. This is a principal bundle if the action of the structure group on the fiber is given by the group multiplication of G. A fiber bundle p:E $\to B$ with fiber G and structure group G is called a principal bundle.

6 Example

The Mobius strip is an example of a non-trivial fiber bundle.
p: M→ S^1 by p(x,y,z)= (cosφ, sin φ) for
(x,y,z)= $((2 + t\cos\frac{\varphi}{2})\cos\varphi, (2 + t\cos\frac{\varphi}{2})\sin\varphi, t\sin\frac{\varphi}{2}) \in$ M.
Using the open subsets,

$U_1\{(\cos\theta,\sin\theta|0 < \theta < 2\pi\}$

$U_2\{(\cos\theta,\sin\theta|\theta \neq 0\}$
$p^{-1}(U_1) = \{((2 + t\cos\frac{\varphi}{2}\cos, (2 + t\cos\frac{\varphi}{2}\sin, t\sin\frac{\varphi}{2})| - \frac{1}{2} \leq t \leq \frac{1}{2}, 0 < \varphi < 2\pi\}$
$p^{-1}(U_2) = \{((2 + t\cos\frac{\varphi}{2}\cos, (2 + t\cos\frac{\varphi}{2}\sin, t\sin\frac{\varphi}{2})| - \frac{1}{2} \leq t \leq \frac{1}{2}, \pi < \varphi < 3\pi\}$
$f_1 : (0, 2\pi) \times [-\frac{1}{2}, \frac{1}{2}] \to p^{-1}(U_1)$ defined by $f_1(\varphi, t) = ((2 + t\cos\frac{\varphi}{2}\cos, (2 + t\cos\frac{\varphi}{2}\sin, t\sin\frac{\varphi}{2})$and
$g_1 : (0, 2\pi) \to U_1$ defined by$g_1(\varphi) = (\cos\varphi,\sin\varphi)$
f_1 and g_1 are homeomorphic. $\varphi_1 = (g_1\varphi_2 : p^{-1}(U_2) \to U_2 \times [-\frac{1}{2}, \frac{1}{2}]$ is a homeomorphism. The map p:M→ $S^{-1}$is a fiber bundle with fiber$[-\frac{1}{2}, \frac{1}{2}]$.
Thus, the Mobius strip is a fiber bundle over S^1 with fiber $[-\frac{1}{2}, \frac{1}{2}]$.

7 Characteristic Classes

When the principle bundle G is a compact Lie group, the cohomology classes of the base space are characteristic classes.

8 Sources

Tamaki, Dai. Fiber Bundles and Homotopy. World Scientific Publishing Co. Pte. Ltd., 2021
Thomas, C.B. Characteristic classes and the cohmology of finite groups. Cambridge University Press, 2008