SPHERE EVERSION

EZRA FURTADO-TIWARI

Abstract.
We cover the proof behind sphere eversion, as well as a general description of which surfaces and curves can be turned inside out.

1. Curves and Surfaces

Before we can get into any details about eversions, we need to first rigorously define a curve and a surface.

Definition 1.1. Let a *smooth continuous curve* $\boldsymbol{\gamma}$ be the image of some map $f : [a, b] \to \mathbb{R}^2$ for $a, b \in \mathbb{R}$. We call f a *parametrization* of $\boldsymbol{\gamma}$.

The main curve that we will be looking at in this paper is a circle, which is parametrized by $\boldsymbol{\gamma}(t) = (\cos t, \sin t)$.

Definition 1.2. Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$. X and Y are said to be *homeomorphic* if there exists a function $f : X \to Y$ such that f is a continuous bijection whose inverse is also continuous. We call f a *homeomorphism*.

Definition 1.3. A *surface* is a subset $S \subseteq \mathbb{R}^3$ such that for every point $\mathbf{p} \in S$, there exist open subsets $U \subseteq \mathbb{R}^2$ and $W \subseteq \mathbb{R}^3$ containing $\mathbf{p}$ such that $S \cap W$ and U are homeomorphic. A homeomorphism $\boldsymbol{\sigma} : U \to S \cap W$ is called a *surface patch* or *parametrization* of $S \cap W$.

Definition 1.4. We call a smooth map $f : S_1 \to S_2$ a *diffeomorphism* if it is bijective with a smooth inverse. If a diffeomorphism exists between S_1 and S_2, we say that these are *diffeomorphic*. Similarly, two curves $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are *diffeomorphic* if there exists a bijective function $f : \boldsymbol{\gamma}_1 \to \boldsymbol{\gamma}_2$ with a smooth inverse.

Here we note that diffeomorphic surfaces can be obtained from each other via a continuous deformation, and so can diffeomorphic curves.

Definition 1.5. A curve is a *simple closed curve* if it is diffeomorphic to a circle.

In other words, if we can bend our curve without creating any holes or creases into a circle, then the curve is considered simple and closed.

Theorem 1.6 (Jordan)**.** *All simple closed curves split the plane into two regions R_1 and R_2, such that $R_1 \cup R_2 = \mathbb{R}^2$ and $R_1 \cap R_2 = \emptyset$.*

While we will not provide a proof of this here, a proof is provided in [Hal07]. This theorem allows us to establish the idea of orientability of simple closed curves, or that every simple closed curve has a distinct inside and outside. With this, we can establish the notion of the interior and exterior of a circle.

Date: June 5, 2022.

It is difficult to understand what the inside and outside of a two dimensional curve would look like, so in order to understand what this would look like, we can instead consider a surface which is the vertical extension of the curve. However, it is important to note that the curve is still a two dimensional object.

2. Homotopies

Definition 2.1. A *homotopy* is a map F from $X \times [0,1]$ to Y where X and Y are topological spaces and for which which both parameters are continuous.

Intuitively, we can think of a homotopy as a movie which maps the movement of a curve or surface, so homotopic surfaces and curves can be obtained from each other via some continuous deformations. In the case of the circle, we have $X = \boldsymbol{\gamma}$ and $Y = \mathbb{R}^2$. Similarly, the sphere satisfies $X = S$ and $Y = \mathbb{R}^3$.

Definition 2.2. Define an *immersion* to be a nonsingular map $f : X \to Y$ between manifolds X, Y such that for every point $\mathbf{p} \in X$, the derivative is an injective linear transformation.

This means that at every point in the domain, the map appears to be the inclusion map, up to diffeomorphisms of the tangent space. In other words, we have no creases, holes, or tears.

Definition 2.3. A homotopy is said to be *regular* if it is an immersion at every stage.

Thus we find that any two surfaces for which a regular homotopy exists between them can be obtained from each other without creating a hole, tear, or crease. It remains to find out which surfaces and curves are regularly homotopic.

3. Eversions

First, we consider a smooth continuous curve $\boldsymbol{\gamma}$. We wish to find some invariant such that all curves $\boldsymbol{\gamma}_1$ which $\boldsymbol{\gamma}$ can be mapped to via some regular homotopy.

Definition 3.1. Let $f : X \to Y$ be a map between oriented n-dimensional manifolds X, Y. Let $\mathbf{p} \in X$ be some point such that $D_{\mathbf{p}} : T_{\mathbf{p}}X \to T_{\mathbf{p}}Y$ is a linear isomorphism between oriented vector spaces, where $T_{\mathbf{p}}$ is the tangent space. We let $\alpha(D_{\mathbf{p}}) = 1$ if $D_{\mathbf{p}}$ preserves orientation, and we let $\alpha(D_{\mathbf{p}}) = -1$ if $D_{\mathbf{p}}$ reverses orientation. We define the *degree* of f to be

$$d(f; y) = \sum_{\mathbf{p} \in f^{-1}(y)} \alpha\left(D_{\mathbf{p}}\right).$$

Here we note that the choice of y does not affect the degree, so we denote the degree of a map f as $d(f)$.

In other words, for every point $\mathbf{p}$ which is mapped to y by f and gets turned inside out by the map, we subtract one from the degree, and we add one to the degree otherwise. It turns out that the degree of a map is invariant with respect to manifolds which can be obtained from each other via a regular homotopy. We start by introducing a few lemmas that will help us with our proof.

Lemma 3.2. *If $f : X \to Y$ extends to a smooth map $F : Z \to Y$, then $d(f; y) = 0$ for every y.*

We will take this lemma for granted, although a proof is provided in [Mil65].

Lemma 3.3. *Consider a regular homotopy $F : [0,1] \times X \to Y$ between maps f, g satisfying $f(x) = F(0,x)$ and $g(x) = F(1,x)$. For any common value y, $d(g;y) = d(f;y)$.*

Proof. Orient the manifold $[0,1] \times M^n$ as a cartesian product. Then the boundary will consist of $1 \times M^n$ and $0 \times M^n$. We note that $1 \times M^n$ represents points for which the map is orientation preserving, and $0 \times M^n$ represents points for which F is orientation reversing. Then

$$d(F|\partial([0,1] \times M^n); y) = d(g;y) - d(f;y) = 0$$

by Lemma 3.2, so

$$d(g) = d(f)$$

as desired. ■

Theorem 3.4. *If two maps f and g are regularly homotopic, then $d(f) = d(g)$.*[Mil65]

Definition 3.5. Letting $\boldsymbol{\gamma}(t) = (x(t), y(t))$, we define the turning number $q_{\boldsymbol{\gamma}}$ to be

$$q_{\boldsymbol{\gamma}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}\, dt.\text{[Lev94]}$$

Theorem 3.6. *The turning number of a simple closed curve $\boldsymbol{\gamma}$ is equivalent to that of all simple closed curves $\boldsymbol{\gamma}_1$ which can be obtained via regular homotopies of $\boldsymbol{\gamma}$.*

Proof. This follows from Theorem 3.4. ■

Theorem 3.7. *There is no regular homotopy between the circle $\boldsymbol{\gamma}(t) = (\cos t, \sin t)$ and $\boldsymbol{\gamma}_1(t) = (\cos t, -\sin t)$.*

Proof. Consider the turning numbers of the circle and the eversion of itself. We find that the turning number of the circle is equal to

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin^2 t + \cos^2 t}{\sin^2 t + \cos^2 t}\, dt &= \frac{1}{2\pi} \int_0^{2\pi} 1\, dt \\ &= \frac{1}{2\pi} \cdot 2\pi \\ &= 1, \end{aligned}$$

while the turning number of its eversion is

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \frac{-\sin^2 t - \cos^2 t}{\sin^2 t + \cos^2 t}\, dt &= \frac{1}{2\pi} \int_0^{2\pi} -1\, dt \\ &= -\frac{1}{2\pi} \int_0^{2\pi} 1\, dt \\ &= -1. \end{aligned}$$

Since the two turning numbers are not equal, we find that there is no regular homotopy between $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_1$. ■

Remark 3.8. This proof is equivalent to showing that the circle cannot be turned inside out without creating a hole, tear or crease.

For curves, the direction varies solely based on one parameter t. This allows us to establish the turning number, which in the case of the circle easily allows us to show which curves can be obtained from each other via regular homotopies. This allows us to connect the degree of a map to a much simpler invariant, which we can calculate easily. In the case of surfaces, finding an invariant proves to be a much more difficult task.

Definition 3.9. Consider a surface patch $\boldsymbol{\sigma}(u, v)$, and consider the set U of points $\mathbf{p}$ such that $\boldsymbol{\sigma}(\mathbf{p})_v = \mathbf{0}$. If a neighborhood exists such that the partial derivatives all have the same sign, we let

$$\alpha(\mathbf{p}) = 1,$$

and we let

$$\alpha(\mathbf{p}) = -1$$

otherwise. We then define the *turning number* of the surface patch $\boldsymbol{\sigma}$ to be

$$q = \sum_{\mathbf{p} \in U} \alpha(\mathbf{p}).$$

We let the turning number of a surface S be the sum of the turning numbers of its surface patches.

Theorem 3.10. *Two surfaces S and S_1 are regularly homotopic if and only if their turning numbers are the same.*

Instead of providing a formal proof, we provide some intuition as to why this should be true. First of all, like the turning number for a curve, the turning number for a surface is a measure of the change in direction of a map. As we perform a regular homotopy, it appears that for each point we observe such that $\alpha(\mathbf{p}) = 1$ (visually a dome or a bowl), some other point emerges such that $\alpha(\mathbf{p}) = -1$ (a saddle). The way that these figures interact with each other makes the invariance of the turning number plausible. However, a formal proof of this theorem required enough detail and technicality that mathematicians thought it was incorrect for multiple years after it was published. The proof is available at [Sma59].

Theorem 3.11. *The everted sphere S_2 is regularly homotopic to the sphere S_1.*

Proof. We can simply use the turning number to prove this fact; we notice that there are no saddles in either S_2 or S_1, so both have turning number 2 (as both the north and south poles of the sphere contibute one to this value). ■

4. Conclusion

We explored eversions of curves and surfaces, with a focus on the sphere and circle. We finally proved that the circle is not regularly homotopic to an eversion of itself, but the sphere is. As the proof that the sphere can be everted is extremely complicated, the subject of how to evert the sphere has been much more well studied. Multiple methods exist, including the tobacco-pouch eversion, minimax eversion, and others. More information and visualizations of eversions of the sphere can be found in [Lev94].

References

[Sma59] Stephen Smale. "A Classification of Immersions of the Two-Sphere". In: *"Transactions of the American Mathematical Society"* 90 (1959). URL: https://www.ams.org/journals/tran/1959-090-02/S0002-9947-1959-0104227-9/S0002-9947-1959-0104227-9.pdf.

[Mil65] John W. Milnor. *From the Differentiable Viewpoint*. The University Press of Virginia, 1965.

[Lev94] Sylvio Levy. *Making Waves*. AK Peters, 1994.

[Hal07] Thomas C. Hales. "Jordan's Proof of the Jordan Curve Theorem". In: *"Studies In Logic, Grammar and Rhetoric"* (2007). URL: math.stonybrook.edu/~bishop/classes/math401.F09/HalesDefense.pdf.