Minimal Surfaces Paper

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1 Introduction

In this paper, we are motivated by soap films. We will introduce the topic of differential geometry and then dive into minimal surfaces. We will end by solving two problems in minimal surfaces.

2 Soap Films

Anyone who has played with soap films will know they take on interesting shapes. We are motivated in this paper to use the tools of differential geometry to explain and explore the shapes of soap films. Sometimes, they form flat sections of the plane, like in figure 1. Sometimes they form spirals like in figure 3. Sometimes they form a curve between two rings like in figure 2.

3 Differential Geometry Topics

Before we can discuss minimal surfaces, we must establish some definitions. If the reader is already familiar with differential geometry, feel free to skip to section 4.

3.1 Surfaces

A surface is any subset of $\mathbb{R}^3$ that looks like the plane at every point. Informally, if you imagine the plane as a sheet of rubber, a surface is any shape you can form by bending or stretching the rubber, and a surface can be made up of multiple patches. We can formalize this.

Definition 3.1. A **surface patch** is any function $\boldsymbol{\sigma} : \mathbb{R}^2 \to \mathbb{R}^3$ from an open subset $U \in \mathbb{R}^2$ in $\mathbb{R}^2$ to $\mathbb{R}^3$ that is a **homeomorphism**, i.e. $\boldsymbol{\sigma}$ is continuous and its inverse $\boldsymbol{\sigma}^{-1}$ is continuous.

For example, to see how the surface path $\boldsymbol{\sigma}(u, v) = (u, v, u^2 + v^2)$ acts on the open subset $U = \{(u, v) \in \mathbb{R}^2 | -5 < u < 5, -5 < v < 5\}$, look at figure 4.

Figure 1: A Section of the Plane

Figure 2: A Curve Between Two Rings

Figure 3: A Sprial

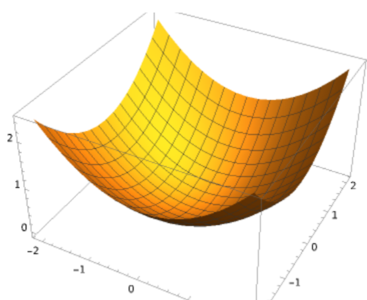

Figure 4: The surface patch $\boldsymbol{\sigma}(u, v) = (u, v, u^2 + v^2)$ acts on an open subset of the plane

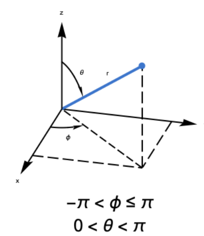

Figure 5: Spherical Coordinates

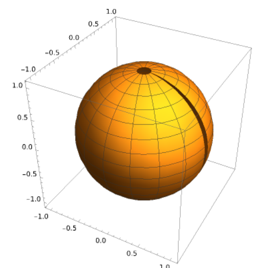

Figure 6: A surface patch of a sphere

Definition 3.2. A subset $\mathcal{S}$ of $\mathbb{R}^3$ is called a **surface** if every point $p \in \mathcal{S}$ is contained in a surface patch.

Example 1. A sphere is a surface because if we take the surface patch

$$\boldsymbol{\sigma}(\theta, \phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta), -\frac{\pi}{2} < \theta < \frac{\pi}{2}, 0 < \phi < 2\pi$$

The reasoning behind this parameterization is to give the sphere in terms of spherical coordinates, see 5. If we do this, we get a sphere with a slice left out. We cannot cover the sphere in just 1 patch because Each subset of $U \in \mathbb{R}^2$ must be open by the definition, but a sphere is closed. In fact we often use multiple surface patches to cover a surface. The second surface patch could be the same as the first, just with the missing slice moved so that it does not intersect and they both cover the entire sphere.

Example 2. A cylinder is a surface because it can be parametrized by the surface patch

$$\boldsymbol{\sigma}(u, v) = (\cos u, \sin u, v), 0 < u < 2\pi$$

just like the sphere, this patch leaves a slice uncovered. See 7

We can find a plane tangent to a surface at a point. We call this plane the **tangent plane**. The tangent plane is spanned by the derivative of $\boldsymbol{\sigma}$ with respect to u, which we denote as $\boldsymbol{\sigma}_u$, and the derivative of $\boldsymbol{\sigma}$ with respect to v, which we call $\boldsymbol{\sigma}_v$. To see an example, look at figure 8.

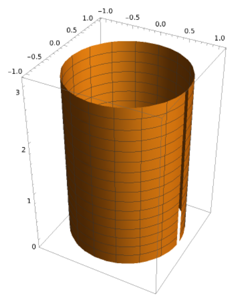

Figure 7: A surface patch of a cylinder

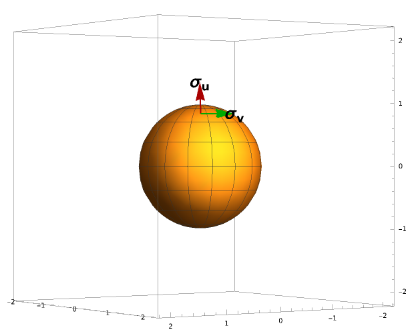

Figure 8: $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ on a sphere

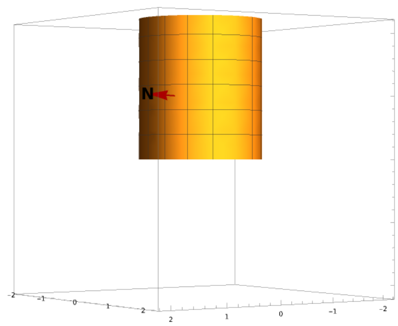

Figure 9: A unit normal vector drawn of a cylinder

3.2 Unit Normal Vector

Definition 3.3. The **unit normal vector** to a surface is defined as the vector at a point that is tangent to the tangent plane at that point. It can be calculated as

$$\mathbf{N} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|}$$

For an example of the unit normal vector, look at figure 9.

Example 3. To see how to find the normal in figure 9, let

$$\boldsymbol{\sigma}(u, v) = (\sin u, \cos v, u)$$

Then we have that

$$\boldsymbol{\sigma}_u = (\cos u, -\sin u, 0), \boldsymbol{\sigma}_v = (0, 0, 1)$$

We can calculate that

$$\mathbf{N} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|} = (\sin u, \cos u, 0).$$

This makes intuitive sense: the normal is contained in a disk around the cylinder.

3.3 Curvature

From intuition, we know that a plane is not curved, but a sphere is curved. To formalize this, we will introduce the notion of mean curvature.

Definition 3.4. The **Gauss Map** is defined as a function $\mathcal{G} : \mathcal{S} \to S^3$ sends every point $\mathbf{p}$ on a surface to it normal vector. A normal vector can be though of as a point on a unit sphere, so the Gauss map sends every point on a surface to a point on the unit sphere

Definition 3.5. The **Weingarten Map** is defined as the derivative of the Gauss map. You can think about it as asking: when a point on a surface moves from $\mathbf{p}_1$ to $\mathbf{p}_2$ how much does its normal change from $\mathbf{N}_1$ to $\mathbf{N}_2$? Look at figure 10.

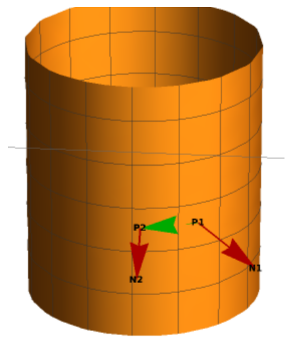

Figure 10: The Weingarten Map

Just like all derivatives, the Weingarten map is a linear operator, so we can represent it as a 2×2 matrix. The Weingarten map is also symmetric, so we can represent it as

$$\mathcal{W} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

3.4 Mean Curvature

Definition 3.6. The **Mean Curvature** of a surface is defined as half the trace of the Weingarten map, or

$$H = \frac{1}{2} tr(\mathcal{W})$$

Definition 3.7. The **Principle Curvatures** are defined as the eigenvectors of the Weingarten map (since the Weingarten map is a matrix), i.e.

$$\mathcal{W}(\mathbf{t}_1) = \kappa_1 \mathbf{t}_1, \qquad \mathcal{W}(\mathbf{t}_2) = \kappa_2 \mathbf{t}_2$$

We call them κ_1 and κ_2

To see the principle curvatures, look at figure 11. The principle curvatures can alternatively be thought of as the directions of most and least curvature. For example, in figure 11, κ_2 is the direction of most curvature and κ_1 is the direction of least curvature.

Example 4. See 12 to see the principle curvatures on a cylinder. κ_1 goes up the cylinder, and it has the least curvature of all directions because it has 0 curvature. κ_2 goes around the circle, and it has the maximum curvature.

Theorem 3.8. *The principle curvatures are always perpendicular.*

Theorem 3.8 is a result of the Spectral Theorem (see [Hal63]) which says that a self-adjoint operator has an ortho-normal basis.

In addition to definition 3.6, we can calculate H using the First Fundamental Form.

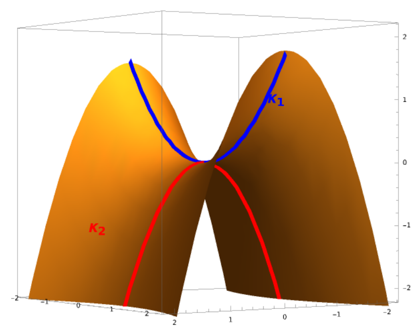

Figure 11: The Principle Curvatures

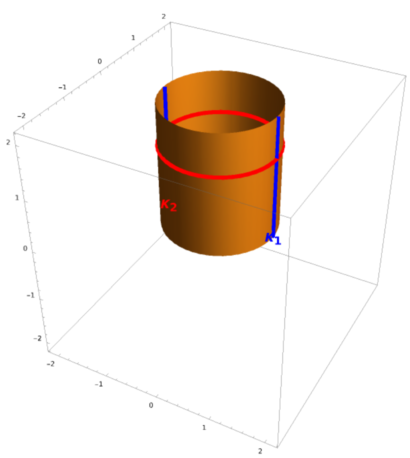

Figure 12: The Principle Curvatures on a Cylinder

Theorem 3.9. *For a surface where the first and second fundamental forms are, respectively,*

$$Edu^2 + Fdudv + Gdv^2, \qquad Ldu^2 + Mdudv + Ndv^2$$

The mean curvature can be calculated as

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}$$

This theorem will be used later.

4 Minimal Surfaces

To mathematically model soap films, we use the definition of minimal surfaces. In soap films, there is potential energy in the inter-molecular attractions between adjacent soap molecules. The potential energy is minimized when the soap molecules are as close together as possible, which is when surface area is minimized. This means that soap films seek to minimize surface area.

To formalize this notion of "minimizing surface area" we have the following definition.

Definition 4.1. A **minimal surface** is defined as a surface where $H = 0$ everywhere.

We use this definition because it is formal, but now let us prove that this definition minimizes surface area.

Theorem 4.2. *A minimal surface has the minimum surface area among all surfaces with the same boundary.*

Proof. Let π be a simple closed curve and let $\text{int}(\pi)$ be the area in the interior of π. Let τ be a small variation in the surface patch $\boldsymbol{\sigma}$. Let $\mathcal{A}(\tau)$ be the surface area. The area is

$$\mathcal{A}(\tau) = \int_{\text{int}(\pi)} d\mathcal{A} = \int_{\text{int}(\pi)} \|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| dudv = \int_{\text{int}(\pi)} \mathbf{N} \cdot (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v) dudv$$

In calculus, a function is minimize when its derivative is 0, so let us find $\dot{\mathcal{A}}$.

$$\dot{\mathcal{A}} = \int_{\text{int}(\pi)} \mathbf{N} \cdot (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v) dudv$$

Through a lot of algebra (if you want to see the full algebra go to [Pre10]) we get

$$\dot{\mathcal{A}}(0) = -2 \int_{\text{int}(\pi)} H(EG - F^2)^{\frac{1}{2}} \alpha dudv$$

in this case when $H = 0$. This means the surface area is minimized when $H = 0$ $\square$

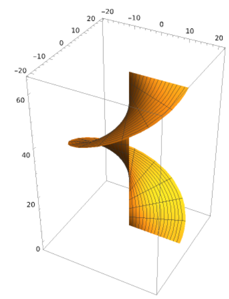

Figure 13: a helicoid

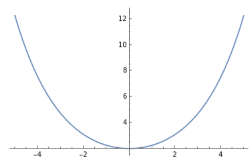

Figure 14: The graph of the catenary $y = 2\cosh\frac{x}{2}$

4.1 Examples of Minimal Surfaces

Let us give some examples of minimal surfaces.

Example 5. Any open subset of the plane is a minimal surface.

This statement is pretty obvious, so I will not prove it. We saw this before in figure 1. If you have a wire spiral and a wire rod going through the center, the surface formed by a bubble mix is called a **helicoid**. Alternatively, a helicoid is the shape swept out by the propeller of a plane as the plane flies along. We saw a helicoid before in figure 3 The surface patch of a helicoid is

$$\boldsymbol{\sigma}(u, v) = (u\cos(v), u\sin(v), av)$$

for some constant a. To see a helicoid, look at figure 13.

Example 6. The helicoid is a minimal surface. The first and second fundamental forms are, respectively

$$(a^2 + v^2)du^2 + dv^2, \frac{a}{\sqrt{a^2 + v^2}}dudv$$

By theorem 3.9, we get

$$H = \frac{LG - 3MF + NG}{2(EG - F^2)} = 0$$

so the helicoid is a minimal surface.

The graph $y = a\cosh\frac{x}{a}$ is called a catenary (see figure 14). If we rotate $y = a\cosh\frac{x}{a}$ about the x-axis, we get a surface called a **catenoid** (see figure 15).

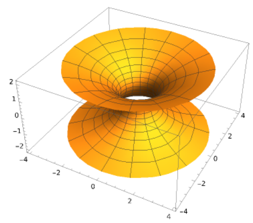

Figure 15: The graph of the catenoid $\boldsymbol{\sigma}(u,v) = (\cos u \cosh v, \sin u \cosh v, v)$

Example 7. If you have 2 rings and dip them in soap, the film will form a catenoid, as we saw in figure 2. The catenoid is a minimal surface. It is defined by

$$\boldsymbol{\sigma} = (\cos u \cosh v, \sin u \cosh v, v)$$

To prove that it is a minimal surface, use theorem 3.9,

$$H = \frac{LG - 2MF + NE}{2(EG - F)^2} = \frac{-\cosh^2 u + \cosh^2 u}{2\cosh^4 4} = 0.$$

4.2 Theorems about Minimal Surfaces

Now that we have seen minimal surfaces, let's prove some theorems about them.

Theorem 4.3. *All surfaces of revolution that are minimal are either a catenoid or an opens subset of the plane*

Proof. The surface patch of a surface of revolution is

$$\boldsymbol{\sigma}(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

where f is the function being rotated and $\dot{f}^2 + \dot{g}^2 = 1$. We can calculate that

$$H = \frac{1}{2}\left(\dot{f}\ddot{g} - \ddot{f}\dot{g} + \frac{\dot{g}}{f}\right)$$

so

$$f\ddot{f} = 1 - \dot{f}^2$$

Let $h = \dot{f}$, then

$$fh\frac{dh}{df} = 1 - h^2$$

and we integrate to get

$$\int \frac{hdh}{1-h^2} = \int \frac{df}{f} \implies h = \frac{\sqrt{a^2 f^2 - 1}}{af}$$

where a is some constant. Let $h = \frac{df}{du}$ and integrate to get

$$f = \frac{1}{a}\sqrt{1 + a^2 u^2}.$$

We computer au as

$$au = \sinh a(g - c))$$

so we finally get

$$f = \frac{1}{a} \cosh a(g - c)$$

This shape is a catenary, so the proof is done. □

Theorem 4.4. *If the Gaussian curvature of a minimal surface is not 0, then the Gauss map is a diffeomorphism.*

Proof. Let $\mathcal{S}$ be a minimal surface where the Gaussian curvature is 0 nowhere (for example, a catenoid). Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a surface patch of a $\mathcal{S}$ containing a point $\mathbf{p}$. Let $\mathbf{p} = \boldsymbol{\sigma}(u_0, v_0)$. We can write

$$\mathbf{N}_u \times \mathbf{N}_v = K\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v.$$

There is an open subset W of U containing (u_0, v_0) such that the restriction of the map $\mathbf{N}$ to W is injective. Then $\boldsymbol{\sigma}(W)$ is an open subset of $\mathcal{S}$ containing $\mathbf{p}$ and the restriction of $\mathbb{G}$ to $\boldsymbol{\sigma}(W)$ is injective. This means it is a diffeomorphism. □

5 Parker's Problem

Question 5.1. *In a talk (See [Par19]) , Matt Parker asks at what point a catenoid of soap film becomes two disks when two rings are slowly moved apart.*

Let's assume for symmetry that the radius of the disks is 1, and one of the disks is centered at the origin. See this graph for a 2d image `https://www.desmos.com/calculator/bbc8wbajjt`. Let d be the distance between the 2 rings. If we imagine the 2-dimension case, in order for the catenary to pass through the second ring, we must have

$$1 = a \cosh \frac{d}{2a}$$

We are going to calculate the area of the catenoid and the area of the disks. The catenoid will be preferred by the soap film when it has the smaller area. The area of the 2 disks is 2π. The area of any region R defined by a surface patch $\boldsymbol{\sigma}(u, v)$ is

$$\int_R \|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| du dv$$

The surface patch of a catenoid is

$$\boldsymbol{\sigma} = (a \cos u \cosh \frac{v}{a}, a \sin u \cosh \frac{v}{a}, a)$$

so

$$\|\boldsymbol{\sigma}_v \times \boldsymbol{\sigma}_v\| = a \cosh \frac{v}{a}^2$$

This gives the total area of the catenoid as

$$\int_0^d \int_0^{2\pi} \|\boldsymbol{\sigma}_v \times \boldsymbol{\sigma}_v\| du dv = a\pi(d + a \sinh \frac{d}{a})$$

We set the area of the disks equal to the area of the catenoid to get

$$2\pi = a\pi(d + a \sinh \frac{d}{a})$$

For the catenoid to be favored, we must have that the catenoid has less surface area than the 2 disks, so

$$2 > ad + a^2 \sinh \frac{d}{a}$$

So we solve the system of equations

$$1 = a \cosh \frac{d}{2a}, 2 = ad + a^2 \sinh \frac{d}{a}$$

First, I solved it numerically to get

$$d = 1.0553947939251433, a = 0.8255174536525105$$

We have answered Parker's question: the catenoid will collapse into 2 disks when the distance between the disks is greater than 1.05539. My experimental data backs this up. I had two rings of radius each 4cm that formed a catenoid. I confirmed that the catenoid collapsed when the rings were approximately 4cm apart.

Later I came back to the problem and solved it exactly as

$$a = \operatorname{sech}\left(\frac{1}{2}\left(W\left(\frac{1}{e}\right)+1\right)\right), d = \left(W\left(\frac{1}{e}\right)+1\right)\operatorname{sech}\left(\frac{1}{2}\left(W\left(\frac{1}{e}\right)+1\right)\right)$$

where W is the product log function (also called the Lambert W function). To learn more about the product log, see [Wei02].

Question 5.2. *In response Parker's Problem, I pose a related question. I ask, imagine we have two rings forming a catenoid, but one ring is growing at a constant rate. At what point does the soap film collapse into two disks?*

Let x be the radius of the growing ring, and let the radius of the other ring be 1. The area of the catenoid is

$$\int_0^{2\pi} \int_{-cosh^{-1} 1}^{cosh^{-1} x} \cosh u^2 du dv = \pi(x\sqrt{-1+x^2} + cosh^{-1} x)$$

and the area of the disks is

$$\pi(1 + x^2)$$

so we have

$$1 + x^2 = x\sqrt{-1+x^2} + cosh^{-1} x \implies x = 2.403624734253787165.$$

I'm not sure if this solution is correct.

References

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[Par19] Matthew Thomas Parker. Matt parker: An attempt to visualise minimal surfaces and maximum dimensions. `https://www.youtube.com/watch?v=Iip8VNrHK_8`, 2019.

[Pre10] Andrew N Pressley. *Elementary differential geometry*. Springer Science & Business Media, 2010.

[Wei02] Eric W Weisstein. Lambert w-function. `https://mathworld.wolfram.com/LambertW-Function.html`, 2002.