CALCULUS OF VARIATIONS

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ABSTRACT. In this article we will discuss calculus of variations. We will go over the basic theory of calculus of variations. We will cover and prove the Euler-Lagrange Equations. Finally, we will go over many applications of calculus of variations both in math and physics.

1. INTRODUCTION

First, we must define a functional.

Definition 1.1. A functional is any function that maps a function to a real number.

Next, we will go over examples of functionals.

Example. Take the functional $F(y) = \int_0^1 y(x)dx$.
We have $F(y = x) = \int_0^1 xdx = 1/2$
$F(y = e^x) = \int_0^1 e^x dx = e$

Example. Two important functionals are the *arclength functional*, C, and the *action*, S.
$C(f) = \int_b^a \sqrt{1 + \dot{f}(x)^2}dx$, and $S = \int_{t_1}^{t_2} L(t)dt$, where in physics $L = T - V$ is known as the *Lagrangian*, where T and V are the kinetic and potential energies of the system respectively. The *Principle of Stationary Action* states that a system will always move in such a way between times t_1 and t_2 that has a stationary value of the action. This will be useful later on.

Now we define informally what calculus of variations is.
Calculus of Variations is the branch of mathematics dealing with the optimization of functionals using variational methods. Now, we will define the functional derivative.
First, consider an ordinary function $f(y_1, y_2, \ldots, y_n)$. The total derivative of f is $df = \sum_{i=1}^n \frac{\partial f}{\partial y_i}\partial y_i$. Now we consider a functional $F(L(x))$, and let $L(x)$ be defined on $[a, b]$. While f depends on the set $\{y_i | i \in [n]\}$, F depends on the uncountably-infinite set $\{L(x)|x \in [a, b]\}$. So out total derivative analogy would look something like $\sum \frac{\delta F}{\delta L(x)}\delta L(x)$, with x continuously ranging from a to b. We can turn this into the integral $\int_a^b \frac{\delta F}{\delta L(x)}\delta L(x)dx$. And so our derivative analogy for functionals should be $\frac{\delta F}{\delta L(x)}$, leading us to the following definition.

Definition 1.2. The *functional derivative* of F is $\frac{\delta F}{\delta L(x)}$ with F and $L(x)$ defined in the previous section.

The functional derivative can be thought of as a directional derivative in the following way. If we take $\eta(x)$ to be an arbitrary function, and take the directional derivative of F in

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the direction of $\eta(x)$, we get

$$\lim_{\epsilon\to 0}\frac{F(L(x)+\epsilon\eta(x))-F(L(x))}{\epsilon}=\int\frac{\delta F}{\delta L(x)}dx$$

We also see that $\delta L(x)=\eta(x)$. So at a high level, a functional derivative is the change in a functional when the function argument it takes is perturbed slightly.
Now we will go over a calculation of the functional derivative, and in the next section show a simple way to calculate it using the Euler-Lagrange Equations.

Example. Consider the functional $F(y(x))=\int_0^1 y(x)^2dx$. In order to calculate the functional derivative, let's calculate $F(y+\delta y)$ for a small change in y, δy:

$$F(y+\delta y)=\int_0^1(y(x)+\delta y(x))^2dx$$

$$=\int_0^1 y(x)^2+2y(x)\delta y(x)+\delta y(x)^2dx$$

$$=F(y(x))+\int_0^1 2y(x)\delta y(x)dx+\int_0^1\delta y(x)^2dx$$

The last term goes to 0, and computing $\delta F=F(y(x)+\delta y(x))-F(y)$ gives $\delta F=\int_0^1 2y(x)\delta y(x)dx$ This resembles our integral $\int_a^b\frac{\delta F}{\delta L(x)}\delta L(x)dx$, and we can see that our functional derivative $\frac{\delta F}{\delta y(x)}=2y(x)$.

We can see that if the functional derivative of $F(y(x))$ with respect to y is 0, then a perturbation in y won't produce a first-order change in the functional. So we can think of y as producing an extrema for $F(y(x))$. In calculus of variations, we optimize functionals, and so calculus of variation problems are mostly concerned with our functional derivative being 0.

2. Euler-Lagrange Equations

The Euler-Lagrange equations are immensely important for calculus of variations problems, and make computing certain functional derivatives much easier:
Consider $F(y(x))=\int_a^b L(x,y,\dot{y})dx$. The *Euler-Lagrange Equation* states $\frac{\delta F}{\delta y(x)}=\frac{\partial L}{\partial y}-\frac{d}{dx}\frac{\partial L}{\partial\dot{y}}$.

Example. Let's find the functional derivative of $F(y(x))=\int_0^1=x^3e^{-y}dx$. Taking $L=x^3e^{-y}$, we have

$$\frac{\delta F}{\delta y(x)}=\frac{\partial L}{\partial y}-\frac{d}{dx}\frac{\partial L}{\partial\dot{y}}$$

$$=\frac{\partial x^3e^{-y}}{\partial y}$$

$$=-x^3e^{-y(x)}$$

Now, we move onto the proof:

Proof. Consider $F(L) = \int_b^a L dx$. If we change L to $\tilde{L} = L + \epsilon\eta$, we have

$$\tilde{L}(x, \tilde{y}, \dot{\tilde{y}}) = L(x, y, \dot{y}) + \epsilon\left(\eta\frac{\partial L}{\partial y} + \frac{d\eta}{dx}\frac{\partial L}{\partial \dot{y}}\right) + O(\epsilon^2)$$

If we take $\lim_{\epsilon\to 0}\frac{F(\tilde{L})-F(L)}{\epsilon}$ to first-order in ϵ, we have

$$\lim_{\epsilon\to 0}\frac{F(\tilde{L})-F(L)}{\epsilon} = \int_b^a \eta\frac{\partial L}{\partial y} + \frac{d\eta}{dx}\frac{\partial L}{\partial \dot{y}}dx$$

Now if we integrate the second term using integration by parts, we get

$$u = \frac{\partial L}{\partial \dot{y}}$$

$$dv = \frac{d\eta}{dx}dx$$

And so our second term becomes

$$-\int_b^a \eta\frac{d}{dx}\frac{\partial L}{\partial \dot{y}}dx$$

Now substituting this into our original integral gives

$$\lim_{\epsilon\to 0}\frac{F(\tilde{L})-F(L)}{\epsilon} = \int_b^a \eta\left(\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial \dot{y}}\right)dx$$

But using

$$\lim_{\epsilon\to 0}\frac{F(\tilde{L})-F(L)}{\epsilon}$$

$$= \lim_{\epsilon\to 0}\frac{F(L+\epsilon\eta(x))-F(L)}{\epsilon} = \int\frac{\delta F}{\delta L}\eta dx$$

We see that

$$\frac{\delta F}{\delta L} = \frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial \dot{y}}$$

proving the statement.

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3. Applications

With the Euler-Lagrange Equation, we are ready to start solving important calculus of variations problems.

Example. We first start with a standard mechanics problem:
Consider a point-mass m on a moving in a circle on a frictionless table with a massless string attached to it. The string extends to the center of the table, where it drops down a small hole. Underneath the table, it connects to a second point-mass M. Assume the string remains taut with length l. Find the radius of the circle in which m rotates.

First, we can calculate the lagrangian of the system,

$$\mathcal{L} = T - V = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + Mg(l-r)$$

Using conservation of angular momentum, L, we can see

$$\frac{L^2}{2mr^2} = \frac{1}{2}mr^2\dot{\theta}^2$$

Using the stationary action principle, we see that the functional derivative of our action, S, must be zero, and using the Euler-Lagrange Equation we see

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}$$

We can plug in $T - V$ for $\mathcal{L}$ and see that

$$(M+m)\ddot{r} = \frac{L^2}{mr^3} - Mg$$

During circular motion $\ddot{r} = 0$, and thus our radius of motion is $r_0 = L^2/Mmg$. We can also use our equation to do other things, like calculate the period of oscillation of our system if we perturbed the rotating mass slightly in the radial direction. This method of solving physics problems is known as the *Lagrangian Method*.

Example. Next we prove a classic statement: the shortest path between two points is a line. Consider our functional

$$C = \int_b^a \sqrt{1+\dot{y}^2}dx$$

We use the Euler-Lagrange Equation to see that

$$0 = \frac{d}{dx}\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}$$

This implies $\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = c$ for some constant c. Rearranging our equation gives $\dot{f}^2 = \frac{c^2}{1-c^2}$, and so f is clearly a line, proving the statement.

Example. In our final example, we consider a famous problem. Consider two points, and the paths between them. The problem is to find the path such that a point-mass going through each path under uniform gravitational acceleration starting at rest from one point will take the least time to get to the other point. First, consider our time functional $T(x) = \int_0^{y_0} dy\frac{\sqrt{1+\dot{x}^2}}{\sqrt{2gy}} = \int_0^{y_0} Ldy$. The Euler-Lagrange equations give

$$\frac{\delta T}{\delta x} = 0 - \frac{d}{dy}\frac{\partial L}{\partial \dot{x}} = 0$$

If $\frac{\partial L}{\partial \dot{x}} = C = \frac{1}{2gy}\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}$, we can solve for $x(y) = \int dy\frac{y}{\sqrt{2ay-y^2}}$ where $a = \frac{1}{4C^2g}$. We can evaluate this integral and use boundary conditions to find

$$x(y) = -\sqrt{2ay-y^2} + a\cos^{-1}\frac{a-y}{a}$$

which is the equation of a cycloid, solving the problem.