Calculus on Manifolds and Exterior Algebra

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Abstract

In the first section we focus on the machinery needed for generalized stokes' in $\mathbb{R}^n$ a powerful higher dimensional analogue of the Fundamental Theorem of Calculus, Green's Theorem, Stokes' Theorem, and Divergence Theorem over a much more general type of space. Among the machinery it introduces are that of tensors and differential forms. We finish off the first section with a generalization of Gauss-Bonnet to all even-dimensional hypersurfaces. The theory of tensors is used to introduce the reader to exterior algebras, and we discuss various aspects of the wedge product. The paper concludes with a reformulation of Maxwell's laws as a single equation.

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1 Generalized Stokes'

A note before we begin:
The proofs will be omitted because for them to be clear and not overly concise would require them to be several pages in length. However, the proofs along with more detail can be found in a few books. Our goal for this section is to cover Generalized Stokes' in $\mathbb{R}^n$, and our main source for the $\mathbb{R}^n$ case is [8] with [10] also being very helpful. [7] and [5] in particular were useful for the definitions associated with manifolds.

1.1 Manifolds and Orientation

We're used to working in euclidean space, however, many things that work on it work on locally euclidean space as well. So we don't actually need euclidean space, we just need something that is locally resembles it. That is essentially what a manifold is. For example the earth is a manifold. Now, maps exist, this should be pretty uncontroversial. So even though the earth isn't actually flat, we can still describe portions of it on a flat surface with negligible distortion to reality. We will soon use this to describe something called charts and atlases. Adding some additional structure can allow us to talk about smooth manifold and do calculus on it. While we won't actually venture beyond Euclidean space here, manifolds provide us a very useful tool that eventually lets us generalize many of the familiar theorems of calculus. The only type of manifold we will consider here is a smooth manifold.

Before we do so, we define what is a topological manifold.

Definition 1 (Hausdorff)**.** *Let M be a topological space. M is Hausdorff if for all distinct $p, q \in M$, there exists disjoint open subsets $U, V \subseteq M$, such that $p \in U$ and $q \in V$.*

Definition 2 (Second-Countable)**.** *Let M be a topological space. M is second-countable if there exists a countable basis of M.*

Definition 3 (Locally Euclidean)**.** *Let M be a topological space. M is locally Euclidean of dimension n if each point of M has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^n$.*

Definition 4 (Topological Manifold)**.** *Let M be a topological space. M is a topological k-manifold if M is Haussdorf, second-countable, and locally Euclidean of dimension k.*

Since we are only considered with the $\mathbb{R}^n$ case, then we only need to ever worry about the Hausdorff condition since every subset of $\mathbb{R}^n$ is second-countable and locally Euclidean of dimension n.

Now, we define a chart. What we are essentially doing here is cutting off pieces of the manifold.

Consider the following problem. If we can only use flat pieces of paper, how do we accurately map the earth? Now that might be kind of hard since the earth is not flat(hopefully you were taught this) and so we can't really make a single flat map of the earth in all of its details without seriously distorting it. However, every part of the earth, if you zoom in close enough, is relatively flat. So what we can do is make a map of each small part of the earth(chart), then collect them together to form an atlas, which can describe each part of the earth pretty well with very little distortion. If you actually took each piece and physically stitched them together, you'd get a rough approximation of the earth. And if we allow for a bit of stretching and bending then we get several perfect maps of the earth. In fact, you could cover a globe with parts of the atlas. So then we take this idea and run with it, although it might seem to get a bit lost in a sea of terminology.

Definition 5 (Chart)**.** *Let M be a topological k-manifold. A chart on M is a pair (U, ϕ) where U is an open subset of M and $\phi : U \to \hat{U} = \phi(U)$ is a homeomorphism. By definition of a topological manifold, each point $p \in M$ is contained in the domain of some chart (U, ϕ). We say the chart is centred at p if $\phi(p) = 0$. Note that we can obtain a chart centred at p by subtracting from ϕ the constant vector $\phi(p)$. We will sometimes write ϕ in terms of its component functions as $(x_1, \cdots, x_n)$ which we will call local coordinates. You may also sometimes see these written in superscript elsewhere instead of subscript.*

There's a specific type of topological manifolds called smooth manifolds which is the kind with which we will be working with. To define them, we will need to define what smoothness means for maps in general. The definition of C^∞ you are probably familiar with is the one that says it is smooth if it has continuous partial derivatives of all orders. However, we encounter a problem when the domain of a mapping is not open, then there are points on which the partial derivative is not defined, and so we extend this definition a bit. We only need the mapping to be able to be extended locally to a smooth map on open sets, which yields the following definition.

Definition 6 (Smooth Maps)**.** *A function $f : X \to \mathbb{R}^m$ where $X \subseteq \mathbb{R}^m$ is called smooth if for each $x \in X$, there exists an open set $x \in U \subseteq \mathbb{R}^n$ and a smooth map $F : U \to \mathbb{R}^m$ such that $F = f$ on $U \cap X$.*

We now define a more convenient homeomorphism called a diffeomorphism.

Definition 7 (Diffeomorphism)**.** *A smooth function $f : U \to V$ is called a diffeomorphism if it has a smooth inverse and is bijective.*

Note: For our purposes, all charts will be diffeomorphisms.

Now that we can slice off parts of the manifold with coordinate patches, we need a way to collect them all, which we will an atlas, now there's actually a specific type of atlas we're looking for called a smooth atlas which is used to define a smooth manifold.

Definition 8 (Transition Map)**.** *Let M be a topological manifold. If $(U, \phi), (V, \phi)$ are two charts such that $U \cap V \neq \emptyset$, $\psi \circ \phi^{-1}$ is called the transition map from ϕ to ψ.*

Definition 9 (Smoothly Compatible)**.** *We call two charts $(U, \phi), (V, \psi)$ smoothly compatible if either $U \cap V = \emptyset$, or the transition map is a diffeomorphism.*

Definition 10 (Smooth Atlas)**.** *Let M be a topological manifold. We define an atlas $\mathcal{A}$ of M to be a collection of charts whose domains cover M. We call $\mathcal{A}$ a smooth atlas if any two charts in $\mathcal{A}$ are smoothly compatible.*

Definition 11 (Maximal Atlas)**.** *Let M be a topological manifold. A smooth atlas $\mathcal{A}$ on M is maximal if is not a proper subset of another smooth atlas. Sometimes this is referred to as a complete atlas instead of a maximal atlas.*

Definition 12 (Smooth Structure)**.** *Let M be a topological manifold. A smooth structure on M is a maximal smooth atlas.*

Definition 13 (Smooth Manifold)**.** *Let M be a topological manifold. A smooth manifold is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.*

We now make a distinction. There is a definition of boundary you are probably familiar with, the one that says any neighbourhood containing a boundary point of a set will contain points from both the set and its complement. This is not to be confused with the definition we'll be using here.

Definition 14 (Upper Half Space)**.** *The subset $\mathbb{H}^k$ of k-tuples $(x_1, \cdots, x_k) \in \mathbb{R}^k$, such that $x_k \geq 0$ is called the upper half plane of $\mathbb{R}^k$. $\mathbb{H}^k_+$ is the set of all k-tuples$(x_1, \cdots, x_k) \in \mathbb{R}^k$, such that $x_k > 0$*

Definition 15 (Topological Manifold with Boundary)**.** *A k-dimensional topological manifold with bounday is a second-countable Hausdorff space M in which every point has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^k$ or a relatively open subset of $\mathbb{H}^k$.*

This is a more general definition than the one we gave before of a manifold, and so we will adjust our existing definitions to be compatible with this one.

Definition 16 (Chart, again)**.** *An open subset $U \subseteq M$ combined with a map $\phi : U \to \mathbb{R}^n$ that is a homeomorphism onto an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ is called a chart for M.*

The rest of the terms needed for manifolds are defined analogously(e.g. Smooth Manifold With Bounday)

Note: We will only be working with Smooth Manifolds with boundary

Definition 17 (Boundary and Interior of Manifold)**.** *Let M be a k-manifold with boundary in $\mathbb{R}^n$, let $\alpha : U \to V$ be a coordinate patch about the point $p \in M$.*

1. *If U is open in $\mathbb{R}^k$, then p is an interior point of M.*
2. *If U is open in $\mathbb{H}^k$ and if $p = \alpha(x_0)$ for $x_o \in H^k_+$, then p is an interior point of M.*
3. *If U is open in $\mathbb{H}^k$ and $p = \alpha(x_0)$ for $x_0 \in \mathbb{R}^{k-1} \times 0$, then p is a boundary point of M.*

The set of all boundary points is called the boundary of a manifold and is denoted ∂M, not to be confused with Bd M

Here's a nice little lemma that becomes useful.

Lemma 1. *Let M be a manifold with boundary in $\mathbb{R}^n$, and let $\alpha : U \to V$ be a coordinate patch on M. If U_0 is a subset of U that is open in U, then the restriction of α to U_0 is also a coordinate patch on M*

The other definitions are analogous as we switch from manifold to manifold with boundary. For example, a smooth structure is still defined as the maximal smooth atlas, and M along with the smooth structure is called a smooth manifold with boundary, and so on.

The next topic to be discussed is orientation, which can be a bit confusing. Orientation of manifolds comes into play later on once we get to integration on manifolds, it's like a generalization of how flipping around the order of integration in an integral will flip the sign.

Definition 18 (Orientation Preserving)**.** *We call a diffeomorphism α orientation preserving if $\det(D\alpha) > 0$. For two diffeomorphisms $\alpha_i : U_i \to V_i$ and $\alpha_j : U_j \to V_j$, if $V_i \cap V_j$ is nonempty, then the transition map is defined as $\alpha_j \circ \alpha_i^{-1}$*

Definition 19 (Orientable Manifold)**.** *We call an atlas an oriented atlas if for any two overlapping charts (U_i, α_i) and (U_j, α_j), the transition map is orientation preserving. If there exists an oriented atlas for a manifold M, then we call call M an orientable manifold.*

Definition 20 (Orientation)**.** *We call a maximal oriented atlas of a manifold its orientation*

Definition 21 (Oriented Manifold)**.** *Let M be an orientable manifold. M, together with an orientation of M is called an oriented manifold.*

On a curve, it's possible to the reverse the direction of travel and thus the orientation, so we define something similar for this version of orientation.

Definition 22 (Reflection Map)**.** *Let $r : \mathbb{R}^k \to \mathbb{R}^k$ be the reflection map $r(x_1, x_2, \cdots, x_k) = (-x_1, x_2, \cdots, x_k)$*

Definition 23 (Reverse/Opposite Orientation)**.** *Let M be an oriented k-manifold in $\mathbb{R}^n$. If $\alpha_i : U_i \to V_i$ is a coordinate patch on M belonging to the orientation of M, let β_i be the coordinate patch*

$$\beta_i = \alpha_i \circ r : r(U_i) \to V_i$$

We can check that β_i is not part of the same orientation as α_i.

$$\begin{aligned}\det(D\alpha_i^{-1} \circ \beta_i) &= \det(D\alpha_i^{-1} \circ \alpha_i \circ r) \\ &= \det \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \\ &= -\det(I_n) \\ &= -1\end{aligned}$$

Furthermore, as it so happens, each β_i forms an orientation as well, which we call the reverse(or opposite) orientation to that given by α_i.

Now, if you look at enough manifolds you might realize something interesting, if the manifold has a nonempty boundary, then the boundary itself is a manifold, just in one less dimension. We state this in the following theorem.

Theorem 1. *Let M be a k-manifold in $\mathbb{R}^n$, of class C^r. If ∂M is non-empty, then ∂M is a k-1 manifold without boundary in $\mathbb{R}^n$ of class C^r*

You may have noticed that we've defined orientation for manifolds, and not manifolds with boundaries. This is not the same for every book, in particular [8] and [10] define the induced orientation to compensate whereas [7] simply extends the definition. Since we are mostly following [8], we will take their approach. Now that we know that the boundary is a manifold, is it also orientable? As it turns out, yes it is. However we first need a method of finding coordinate patches to cover ∂M.

Definition 24 (Restricting Coordinate Patches to ∂M)**.** *Let U_0 be the open set of $\mathbb{R}^{k-1}$ such that $U_0 \times 0 = U \cap \partial\mathbb{H}^k$ If $x \in U_0$, we define $\alpha_0(x) = \alpha(x, 0)$. By our definition of boundary point, $\alpha_0(x)$ must be a boundary point. It then follows from lemma 1 that $\alpha_0(x)$ is a coordinate patch on ∂M.*

Theorem 2. *Let $k > 1$. If M is an orientable k-manifold with non-empty boundary, then ∂M is orientable.*

While we won't explicitly mention it here, there is a choice of a "natural orientation", and as it turns out, every manifold has at last two orientations. The natural orientation and its reverse orientation. Connected manifolds have exactly two orientations.

Definition 25 (Induced Orientation)**.** *Let M be an orientable k-manifold with nonempty boundary. Given an orientation μ of M, the corresponding induced orientation of ∂M is defined as follows. If k is even, it is the orientation obtained by restricting(recall definition 24) coordinate patches belonging to the orientation of M. If k is odd, it is the opposite of the orientation of ∂M obtained in this way*

1.2 Differential Forms

Differential forms involve a lot of abstraction and it's hard to see the point at times. Here we do not present a picture of what they represent, but rather why they exist, and why the abstractions. The definition of a differential form is, as was put here, something that can be integrated. The weird part is that these layers of seemingly endless abstraction give way to a very elegant generalization of all the fundamental theorems of calculus in any number of dimensions, called Generalized Stokes'(GS). It lends itself to many more purposes beyond GS but GS will be the focus of this section. We will stick to the real numbers here as the technicality increases massively in generalizations. We will primarily be using [8] and to a lesser extent [10] here. Check out [7] if you are interested in a much more general discussion.

To start, we first introduce a generalization of a linear map called a tensor, which uses the same idea.

Definition 26 (Tensor)**.** *Let V be a vector space and $v_i \in V$ for all $i \leq k$. A k-tensor is a function $f : V^k \to \mathbb{R}$ that is linear in each variable. In other words, if c_1 and c_2 are scalars, then for all $i \leq k$, $f(v_1, \cdots, c_1 v_{i_1} + c_2 v_{i_2}, \cdots v_k) = c_1 f(v_1, \cdots, v_{i_1}, \cdots, v_k) + c_2 f(v_1, \cdots, v_{i_2}, \cdots, v_k)$. We denote the space of all k-tensors on V by $\mathcal{L}^k(V)$.*

Tensors are the first thing we generalize. A 0-tensor is a scalar, a 1-tensor is a linear map, a 2-tensor is a bilinear map(e.g. inner product), and so on. Notice that a linear map has a matrix representation of a vector when it has only 1 variable as input. So then a tensor field, depending on the degree of the tensor, could either be a scalar field, a vector field, a field of matrices, and of multilinear maps in general. This idea becomes important later, as you might be able to see. Instead of having to integrate a function over a loop or a vector field over a surface, you only need a tensor field. However, we further divide tensors into symmetric and antisymmetric(aka alternating) tensors, and as it turns out, alternating tensors have the right properties for integration. However, to get to it we first juggle around some terms.

We've established that tensors are a generalization of vectors so you may be wondering if a tensor can be represented in terms of a basis, and the answer is actually yes!

Theorem 3. *Let V be a vector space with basis $a_1, \cdots, a_n$. Let $I = (i_1, \cdots, i_k)$ be a k-tuple of integers from the set $\{1, \cdots n\}$. There is a unique k-tensor ϕ_I on V such that, for every k-tuple $(j_1, \cdots, j_k)$ from the set $\{1, \cdots n\}$,*

$$\phi_I(a_{j_1}, \cdots, a_{j_k}) = \begin{cases} 0 & \text{if } I \neq J \\ 1 & \text{if } I = J \end{cases}$$

The tensors ϕ_I for $\mathcal{L}^k(V)$. We call ϕ_I the elementary k-tensors on V corresponding to the basis $a_1, \cdots, a_n$ for V. We can also define something called the *elementary tensors denoted* ϕ_i *and defined as*

$$\phi_i(a_j) = \begin{cases} 0 & \textit{if } i \neq j \\ 1 & \textit{if } i = j \end{cases}$$

It has the property that if $d_I = f(a_{i_1}, \cdots, a_{i_k})$*, then* $f = \sum_I d_I \phi_I$

Now that we have all of this, we want a way to multiply two tensors and have the result still be a tensor, and in the cases that we are discussing, this comes together as something called the tensor product. Similar to how multiplication adds the exponents of two numbers that have the same base in exponential form, we define something called a tensor product which adds together the orders of two tensors

Definition 27 (Tensor Product)**.** *The tensor product of a* k*-tensor* f *and an* ℓ*-tensor* g *denoted* $f \otimes g$ *is defined by*

$$f(v_1, \cdots, v_k) \otimes g(v_{k+1}, \cdots, v_{k+\ell}) = f(v_1, \cdots, v_k) \cdot g(v_{k+1}, \cdots, v_{k+\ell})$$

The tensor product $f \otimes g$ *is a tensor of order* $k + \ell$

Since the tensor product in this case is just multiplication, you might (correctly) guess that it would share some of the properties of multiplication. We list them in the following theorem

Theorem 4. *Let* f, g, h *be tensors on* V*. Then the following properties hold:*

1. *(Associativity)* $f \otimes (g \otimes h) = (f \otimes g) \otimes h$
2. *(Homogeneity)* $(cf) \otimes g = c(f \otimes g) = f \otimes (cg)$
3. *(Distributivity) Suppose* f *and* g *have the same order. Then:*

$$(f + g) \otimes h = f \otimes g + g \otimes g$$
$$h \otimes (f + g) = h \otimes f + h \otimes g$$

This is a good time to describe what the word alternating means in the context of alternating tensors. While we haven't defined them yet, it does give a bit of a glimpse as to why the next few pieces of terminology are important. Alternating in this case means that it is signed. The funny thing about alternating tensors is that it is actually the determinant, and like the determinant, switching around terms will result in it alternating between positive and negative. It's hard to see the purpose and while there probably exists some long-winded explanation,we can also just say "It just works". Explanations would probably include mention of Theorem 6. For now, we must trudge through.

Definition 28 (Permutation)**.** *A permutation is a bijection from* $\{1, \cdots, n\}$ *to itself. Given* $1 \leq i \leq k$*, an elementary permutation* e_i *is defined by*

$$e_i(v_j) = \begin{cases} v_i & j = i+1 \\ v_{i+1} & j = i \\ v_j & j \neq i, i+1 \end{cases}$$

We denote the symmetry group on n *elements by* S_n

Let's talk about permutations as shuffling cards. Any shuffle of cards can be done as some sequence of card swaps(swapping the position of two cards), and swapping the position of any two cards can be done by a sequence of swapping a card with the card next to it in a specific pattern. It then follows that every card shuffle is just the composition of some number of card swaps between immediate neighbours. Bringing this into the language of math, we can see that this is a specific case of the following lemma

Lemma 2. *If* $\sigma \in S_k$*, then* σ *is a composition of elementary permutations.*

Definition 29 (Sign of Permutation)**.** *Let* σ *be a permutation. We define the sign of* σ *denoted sgn* σ *to be 1 if* σ *is the composition of an even number of elementary permutations and -1 if it is the composition of an odd number.*

Definition 30 (Permutation on Tensor)**.** *If* σ *is a permutation of* $\{1, \cdots, k\}$ *and* f *is a k-tensor, then* $f^\sigma(v_1, \cdots, v_k) = f(v_{\sigma(1)}, \cdots, v_{\sigma(2)})$.

As it turns out, the function sgn has some nice properties, which we list here.

Lemma 3. *Let* $\sigma, \tau \in S_k$

1. *If* σ *equals a composite of* m *elementary permutations, then sgn* $\sigma = (-1)^m$
2. *sgn* $(\sigma \circ \tau) = (sgn\ \sigma) \cdot (sgn\ \tau)$
3. *sgn* $\sigma^{-1} = sgn\ \sigma$
4. *if* $p \neq q$*, and if* τ *is the permutation that exchanges* p *and* q *and leaves all other integers fixed, then sgn* $\tau = -1$

Now we have the tools necessary to define an alternating tensor.

Definition 31 (Alternating Tensor)**.** *If* $f : V^k \to \mathbb{R}$ *is a k-tensor, then* f *is said to be alternating if* $f^{e_i} = -f$ *for all* i*. In other words,*

$$f(v_1, \cdots, v_{i+1}, v_i, \cdots, v_k) = -f(v_1, \cdots, v_i, v_{i+1}, \cdots v_k)$$

for all $1 \leq i \leq k$ *We call the space of all alternating k-tensors on a vector space* V*,* $A^k(V)$

Lemma 4. *Let $\sigma, \tau \in S_k$*

1. *If σ equals a composite of m elementary permutations, then sgn $\sigma = (-1)^m$*
2. *sgn $(\sigma \circ \tau) = (\text{sgn } \sigma) \cdot (\text{sgn } \tau)$*
3. *sgn $\sigma^{-1} = \text{sgn } \sigma$*
4. *if $p \neq q$, and if τ is the permutation that exchanges p and q and leaves all other integers fixed, then sgn $\tau = -1$*

We now introduce a nicer way to represent an alternating tensor, and showing another connection between permutations and alternating tensors

Lemma 5. *The tensor f is alternating if and only if $f^\sigma = (\text{sgn } \sigma) f$ for all σ*

Recall how we had the elementary k-tensor, we now do the same here by defining the elementary alternating k-tensor, which we do right now.

Theorem 5. *Let V be a vector space with basis $a_1, \cdots, a_n$. let $I = (i_1, \cdots, i_k)$ be an ascending k-tuple from the set $\{1, \cdots, n\}$. There is a unique alternating k-tensor ψ_I on V such that for every ascending k-tuple $J = (j_1, \cdots, j_k)$ from the set $\{1, \cdots, n\}$,*

$$\psi_I(a_{j_1}, \cdots, a_{j_k}) = \begin{cases} 0 & \text{if } I \neq J \\ 1 & \text{if } I = J \end{cases}$$

The tensors form a basis for $A^k(V)$. The tensor ψ_I satisfies

$$\psi_I = \sum_{\sigma \in S_k} (\text{sgn } \sigma)(\phi_I)^\sigma$$

Lastly, we have this tidbit of information which shows how alternating tensors are a generalization of the determinant.

Theorem 6. *Let ψ_I be an elementary alternating tensor on $\mathbb{R}^n$ corresponding to the usual basis for $\mathbb{R}^n$, where $I = (i_1, \cdots, i_k)$. Given vectors $(x_1, \cdots, x_k)$ of $\mathbb{R}^n$, let X be the matrix $X = [x_1, \cdots, x_k]$. Then*

$$\psi_I(x_1, \cdots, x_k) = \det X_I$$

where X_I denotes the matrix whose successive rows are rows $i_1, \cdots, i_k$

We've shown how we can define a basis for alternating tensors, just as with normal tensors, and so it is only natural that we now define a product for alternating tensors. The problem with the tensor product is that $f \otimes g$ is rarely alternating, even if f and g are.

Definition 32 (Wedge Product)**.** *To define the wedge product, we first define a transformation Alt* $: \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ *by*

$$Alt\ F = \sum_\sigma (sgn\ \sigma) F^\sigma$$

where σ *extends over all permutations on* $\{1, \cdots, k\}$ *The wedge product of an alternating k-tensor* f *and an alternating* ℓ*-tensor* g *on* V *defined by*

$$f \wedge g = \frac{1}{k!\ell!} A(f \otimes g)$$

$f \wedge g$ *is an alternating* $k + \ell$*-tensor*

The coefficient $\frac{1}{k!\ell!}$ may seem strangely out of place here but it is used in this case for associativity although some texts will use other, similar, coefficients.

Theorem 7. *Let* V *be a vector space. For* $f \in A^k(V)$, $g \in A^\ell(V)$, $h \in A^m(V)$, *the following properties hold*

1. *(Associativity)* $f \wedge (g \wedge h) = (f \wedge g) \wedge h$
2. *(Homogeneity)* $(cf) \wedge g = c(f \wedge g) = f \wedge (cg)$
3. *(Distributivity) If* f *and* g *have the same order,*

$$(f + g) \wedge h = f \wedge h + g \wedge h$$
$$h \wedge (f + g) = h \wedge f + h \wedge g$$

4. *(Anticommutativity) If* f *and* g *have orders* k *and* ℓ*, respectively, then*

$$g \wedge f = (-1)^{k\ell} f \wedge g$$

We now reach the differential parts and the first thing we do is define the tangent space.

Definition 33 (Tangent Space)**.** *Given* $x \in \mathbb{R}^n$*, we define a tangent vector to* $\mathbb{R}^n$ *at* x *to be a pair* (x, v)*, where* $v \in \mathbb{R}^n$*. The set of all tangent vectors to* $\mathbb{R}^n$ *at* x *is called the tangent space of* x *at* $\mathbb{R}^n$ *and is denoted* $T_x(\mathbb{R}^n)$*.*

We can extend this definition to manifolds in general, but before we do that we need to define something called the transformation induced by a differential map.

Definition 34 (Transformation Induced by Differentiable Map)**.** *Let* A *be open in* $\mathbb{R}^k$ *or* $\mathbb{H}^k$*, let* $\alpha : A \to \mathbb{R}^n$ *be of class* C^r*. Let* $x \in A$ *and let* $p = \alpha(x)$*. We define a linear transformation* $\alpha_* : T_x(\mathbb{R}^k) \to T_p(\mathbb{R}^n)$ *by the equation*

$$\alpha_*(x, v) = (p, D\alpha(x) \cdot v)$$

This is called the transformation induced by the differentiable map α

Now we can define the tangent space on manifolds.

Definition 35 (Tangent Space on Manifolds with Boundary). *Let M be a k-manifold with boundary of class C^r in $\mathbb{R}^n$. If $p \in M$, choose a coordinate patch $\alpha : U \to V$ about p where U is open in $\mathbb{R}^k$ or $\mathbb{H}^k$. Let x be the point of U such that $\alpha(x) = p$. The set of all vectors of the form $\alpha_*(x, v)$, where v is a vector in $\mathbb{R}^k$, is called the tangent space to M at p, and is denoted $T_p(M)$.*

Since $\mathbb{R}^k$ is spanned by $e_1, \cdots, e_k$, $T_p(M)$ is spanned by the vectors $(p, D\alpha(x) \cdot e_j) = \left(p, \frac{\partial \alpha}{\partial x_j}\right)$ which form a basis for $T_p(M)$

We can now give a generalization of the scalar and vector field here. Instead of a scalar or a vector, we assign a k-tensor to each point(Recall a 0-tensor is a scalar and a 1-tensor is a vector when there is only a single variable input).

Definition 36 (Differential Form). *Let A be an open set in $\mathbb{R}^n$. A k-tensor field in A is a function ω assigning to each $x \in A$ a k-tensor $\omega(x)$ defined on $T_x(\mathbb{R}^n)$. If each $\omega(x)$ is an alternating tensor, then we call ω a differential form of order k or simply k-form. The set of all (C^∞) k-forms on A is denoted $\Omega^k(A)$*

Recall how earlier we defined elementary tensors, now we do the same except for forms.

Definition 37 (elementary forms). *Let $e_1, \cdots, e_n$ be the usual basis for $\mathbb{R}^n$. Then $(x, e_1), \cdots (x, e_n)$ is called the usual basis for $T_x(\mathbb{R}^n)$. We define a 1-form*

$$\tilde{\phi}_i(x)(x, e_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

called an elementary 1-form on $\mathbb{R}^n$. This is often denoted dx_i. Using this notation, given an ascending k-tuple $I = (i_1, \cdots, i_k)$ from the set $\{1, \cdots, n\}$ we define a k-form

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

The k-forms dx_I are called the elementary k-forms on $\mathbb{R}^n$

A property that becomes very useful is $dx_i \wedge dx_i = 0$.

There are a few important properties of these elementary forms. The first is that they are C^∞. The second is that if ω is a k-form, then we can write the k-tensor $\omega(x)$ uniquely as

$$\omega(x) = \sum_{[I]} b_I(x) \tilde{\psi}_I(x)$$

for some scalar functions $b_I(x)$, which we call components.

We now define something called the differential of a 0-form.

Convention:
Henceforth, we restirct ourselves to manifolds, maps, and forms of class C^∞

Definition 38 (Exterior Derivative). *Let A be open in $\mathbb{R}^n$. Let $f : A \to \mathbb{R}$ be a function of class C^r. We define a 1-form df on A by $df(x)(x, v)$ by*

$$df(x)(x, v) = f'(x, v) = Df(x) \cdot v$$

Where $Df(x)$ denotes the derivative of f at x. Note that here, we are assigning a 1-tensor $df(x)$ to the point x and v is an input to $df(x)$. The 1-form is called the differential of f and is of class C^{r-1}. Now we define d on k-forms for $k > 0$. If ω is a k-form, we can write it uniquely as

$$\omega = \sum_{[I]} f_I dx_I$$

and define

$$d\omega = \sum_{[I]} df_I \wedge dx_I$$

We denote the set of all C^∞ k-forms by $\Omega^k(A)$

The differential(d) operator has some nice properties. For instance, it has an analaogue of the product rule but it also has the property that the second differential vanishes.

Lemma 6.

1. *The operator d is linear on 0-forms*
2. *The operator d is linear on k-forms for $k > 0$*

Theorem 8. *Let A be an open set in $\mathbb{R}^n$. There exists a unique linear transformation*

$$d : \Omega^k(A) \to \Omega^{k+1}(A)$$

defined for $k \geq 0$ such that

1. *If ω and ν are of orders k and ℓ respectively, then*
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$
2. *For every form ω,*
$$d(d\omega) = 0$$

We now are only missing one more piece of the puzzle before we will have arrived at integration, and that is the pullback.

Definition 39 (Pullback). *Let A be open in $\mathbb{R}^k$, let $\alpha : A \to \mathbb{R}^n$ be of class C^∞, let B be an open set of $\mathbb{R}^n$ containing $\alpha(A)$. We define a dual transformation of forms(Pullback) $\alpha^* : \Omega^\ell(B) \to \Omega^\ell(A)$ as follows: Given an ℓ-form ω on B with $\ell > 0$, we define an ℓ-form $\alpha^*\omega$ on A by the equation*

$$(\alpha^*\omega)(x)((x, v_1), \cdots, (x, v_\ell)) = \omega(\alpha(x))(\alpha_*(x, v_1), \cdots, \alpha_*(x, v_\ell))$$

The following theorem is useful primarily for computational purposes, although it does get used in the proofs as well.

Theorem 9. *Let A be open in $\mathbb{R}^k$, let $\alpha : A \to \mathbb{R}^m$ be a C^∞ map. Let B be open in $\mathbb{R}^m$ and contain $\alpha(A)$, let $\beta : B \to \mathbb{R}^n$ be a C^∞ map. Let ω, η, θ be forms defined in an open set C of $\mathbb{R}^n$ containing $\beta(B)$, assume ω and η have the same order. The transformations α^* and β^* have the following properties:*

1. $\beta^*(a\omega + b\eta) = a(\beta^*\omega) + b(\beta^*\eta)$
2. $\beta^*(\omega \wedge \theta) = \beta^*\omega \wedge \beta^*\theta$
3. $(\beta \circ \alpha)^*\omega = \alpha^*(\beta^*\omega)$

We have all of this theory, but we still don't really have a simple way to compute the pullback for elementary k-forms, which we rectify with the following theorem.

Theorem 10. *Let A be open in $\mathbb{R}^k$, let $\alpha : A \to \mathbb{R}^n$ be a C^∞ map. Let x denote the general point of $\mathbb{R}^k$, let y denote the general point of $\mathbb{R}^n$. Then dx_i and dy_i denote the elementary 10forms in $\mathbb{R}^k$ and $\mathbb{R}^n$ respectively.*

1. $\alpha^*(dy_i) = d\alpha_i$
2. *If $I = (i_1, \cdots, i_k)$ is an ascending k-tuple from the set $\{1, \cdots, n\}$, then*

$$\alpha^*(dy_I) = \left(\det \frac{\partial \alpha_I}{\partial x}\right) dx_1 \wedge \cdots \wedge dx_k,$$

where

$$\frac{\partial \alpha_I}{\partial x} = \frac{\partial(\alpha_{i_1}, \cdots, \alpha_{i_k})}{\partial(x_1, \cdots, x_k)}$$

However, note that even with this theorem, it is still difficult to compute $\alpha^*(dy_I)$ for larger k values, where $I = (i_1, \cdots, i_k)$. We simplify this task using the below theorem.

Theorem 11. *Let A be open in $\mathbb{R}^k$, let $\alpha : A \to \mathbb{R}^n$ be of class C^∞. If ω is an ℓ-form defined in an open set of $\mathbb{R}^n$ containing $\alpha(A)$, then*

$$\alpha^*(d\omega) = d(\alpha^*\omega)$$

1.3 Integrating Forms Over Manifolds

To start off here, we need to first define what the integral of a form over an oriented manifold is. To do so, we first define the integral over a subset of $\mathbb{R}^k$.

Definition 40 (Integral of a Form over a Subset of $\mathbb{R}^n$)**.** *Let $A \subset \mathbb{R}^n$ be open, let η be a k-form defined in A. Then η can be written uniquely as*

$$\eta = \sum_I f_i dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

However, since the integral is linear, we will only need to consider the k-form $\eta = f dx_1 \wedge \cdots \wedge dx_k$ *. We define the integral of such an* η *over* A *by*

$$\int_A \eta = \int_A f$$

if the latter integral exists

This gives new meaning to $\int f \mathrm{d}x$, we now see $f \mathrm{d}x$ as a 1-form, thus giving the dx a use.

Definition 41 (Integral of a Form over a Manifold with Boundary)**.** *Let* M *be a compact oriented k-manifold with boundary in* $\mathbb{R}^n$*. let* ω *be a k-form defined in an open set of* $\mathbb{R}^n$ *containing* M*. Let* $C = M \cap (support\omega)$*, then* C *is compact. Suppose there is a coordinate patch* $\alpha : U \to V$ *on* M *belonging to the orientation of* M *such that* $C \subset V$*. Be replacing* U *with a smaller set if necessary, we can assume that* U *is bounded. We define the integral of* ω *over* M *by*

$$\int_M \omega = \int_{int\ U} \alpha^* \omega$$

Here, int $U = U$ *if* U *is open in* $\mathbb{R}^k$*, and int* $U = U \cap \mathbb{H}^k_+$ *if* U *is open in* $\mathbb{H}^k$ *but not in* $\mathbb{R}^k$

Notice that we can write $\alpha^*(\omega)$ as $h dx_1 \wedge \cdots \wedge dx_k$ for some C^∞ scalar function h. Thus by definition,

$$\int_{\text{int } U} \alpha^* \omega = \int_{\text{int } U} h$$

It can be shown that h is indeed integrable over U and thus also over int U as well as $\int_M \omega$ being well-defined, and independent of choice of coordinate patch.

You may have noticed that this definition isn't for the general case. For the general case we use partitions of unity.

Definition 42 (Integration of Forms over Oriented Manifolds with Boundary)**.** *Let* M *be a compact oriented k-manifold with boundary in* $\mathbb{R}^n$*. Let* ω *be a k-form defined in an open set of* $\mathbb{R}^n$ *containing* M*. Cover* M *be coordinate patches belonging to the orientation of* M*, then choose a partition of unity* $\phi_1, \cdots, \phi_\ell$ *on* M *that is dominated by this collection of charts on* M*. We define the integral of* ω *over* M *by*

$$\int_M \omega = \sum_{i=1}^{\ell} \left(\int_M \phi_i \omega \right)$$

Following this definition, we have the usual properties of the integral, given be the following theorem.

Theorem 12. *Let M be a compact oriented k-manifold in $\mathbb{R}^n$. Let ω, η be k-forms defined in an open set of $\mathbb{R}^n$ containing M. Then*

$$\int_M (a\omega + b\eta) = a\int_M \omega + b\int_M \eta$$

If $-M$ denotes M witht he opposite orientation, then

$$\int_{-M} \omega = -\int_M \omega$$

While this suffices for theoretical purposes, for computational purposes we must introduce another theorem.

Theorem 13. *Let M be a compact oriented k-manifold in $\mathbb{R}^n$. Let ω be a k-form defined in an open set of $\mathbb{R}^n$ containing M. Suppose that $\alpha_i : A_i \to M_i$, for $i = 1, \cdots, N$, is a coordinate patch on M belonging to the orientation of M, such that A_i is open in $\mathbb{R}^k$ and M is the disjoint union of the open sets $M_1, \cdots, M_N$ of M and a set K of measure zero in M. Then*

$$\int_M \omega = \sum_{i=1}^{N} \left(\int_{A_i} \alpha_i^* \omega \right)$$

We are now left in a position to understand the statement of Generalized Stokes'.

Theorem 14. *Let $k > 1$. Let M be a compact oriented k-manifold in $\mathbb{R}^n$ give ∂M the induced orientation is ∂M is not empty. Let ω be a k-1 form defined in an open set of $\mathbb{R}^n$ containing M. Then*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega$$

if ∂M is nonempty and $\int_M \mathrm{d}\omega = 0$ if ∂M is empty.

We can use Generalized Stokes' to rederive Green's Theorem, all we need to do in this case is to show that for a compact 2-manifold M oriented naturally, with ∂M being given the induced orientation, that for a 1-form $Pdx + Qdy$ defined in an open set of $\mathbb{R}^2$ about M,

$$\int_{\partial M} Pdx + Qdy = \int_M (D_1Q - D_2P)dx \wedge dy$$

which follows immediately from Generalized Stokes'. Generalized Stokes' Is hidden behind a mountain of definitions and results and we end up with this unassuming result. However, with our rushed approach with a single goal in mind, we have neglected the many other tools we've developed. In the next part, we discuss a use of all this terminology by generalizing Gauss-Bonnet using some of it. However, this application to differential geometry extends far beyond just Gauss-Bonnet. You can in fact extend much of Differential Geometry with it. Using this language of forms brings us into the math of the 20th century, with its most famous application being General Relativity. [9] talks about many of the applications of forms in differential geometry. It uses a phrase to describe this abstract machinery which nicely encapsulates it. They call it "The Devil's Machine", it takes away geometric insight and much visual insight but in return gives an amazingly effective tool.

Something which has not been mentioned here until now is the seemingly bizarre definition given for induced orientation. It is defined the way it does because as it turns out, changing dimensions is like playing hopscotch with the normal vector field on the boundary. This definition was given so that the normal field would stay consistent even as we switched dimensions. A major limitation on this section is that we stay in $\mathbb{R}^n$, and there are certainly more manifolds and much more out there but we stayed in $\mathbb{R}^n$ because it gets vastly more complicated when we leave, [7] covers such a case along with many more topics. Something else that was always bubbling right underneath the surface was the connection of forms to algebra. This will be seen very apparently in the next portion of this section as we cover an application of the terminology with the definition of derivations. Going further with all of this does eventually lead to contact with algebra, and it becomes used extensively. The terminology covered so far in this section allows a reformulation of many classical theorems of Vector Calculus and even such things as Maxwell's Equations, and heading in the opposite direction we see it being used in Exterior Algebra, which is covered in the next section(section, not subsection)

1.4 Gauss-Bonnet Theorem

More connected to differential geometry, we can introduce a vast generalization of the Gauss-Bonnet Theorem. To do so, we will need to generalize surfaces to hypersurfaces and introduce some new terminology. This will be very short with just a brief look into it. The main books used here are [5] and [7], although [8] is also used sometimes.

We begin by defining a hypersurface. To do so, we use another approach to tangent spaces and tangent vectors called derivations. At first glance, it doesn't seem to have any connection to the tangent vector we are familiar with, however, all directional derivatives are secretly a derivation. In fact, the connection runs even deeper than this.

Definition 43 (Derivation)**.** *Let M be a smooth manifold with boundary. A linear map $v : C^\infty(M) \to \mathbb{R}$ is called a derivation at p if $v(fg) = f(p)vg+g(p)vf$ for all $f, g \in C^\infty(M)$.The set of all derivations of $C^\infty(M)$ at p, is the tangent space to M at p, with any derivation being called a tangent vector at p.*

It's a bit confusing but it's useful. Now we define something called the differential.

Definition 44 (Differential)**.** *Let M, N be smooth manifolds with boundary and $F : M \to N$ a smooth map. For each $p \in M$ we define a map $dF_p : T_pM \to T_{F(p)}N$ called the differential of F at p as follows. Given $v \in T_pM$, we let $dF_p(v)$ be the derivation at $F(p)$ that acts on $f \in C^\infty(N)$ by the rule*

$$dF_p(v)(f) = v(f \circ F)$$

Now we can define an immersion, which is essentially asking "What is the most we can ask for from a mapping between manifolds of different dimensions?

Definition 45 (Immersion)**.** *Let X, Y be manifolds with $dim(X) < dim(Y)$. Then, we call f an immersion at x if the differential is injective at x. If this is true for every point, we simply call f an immersion.*

Definition 46 (Hypersurface)**.** *Let M be a manifold with boundary and N a manifold such that $\dim M - \dim N = 1$. Given an injective immersion $f : N \to M$, $f(N)$ is a hypersurace in M.*

We define the curvature as the Jacobian determinant of the Gauss map. We can now state this version of the Gauss-Bonnet Theorem

Theorem 15. *If X is a compact, even-dimensional hypersurface in $\mathbb{R}^{k+1}$, then*

$$\int_X \kappa = \frac{1}{2} y_k \chi(X)$$

where y_k is the surface area of the unit k-sphere $\mathbb{S}^k$ and $\chi(X)$ is the Euler Characteristic of X.

There are however even more general versions of this theorem.

By setting $k = 2$, we get

$$\int_X \kappa = \frac{1}{2} 4\pi\chi(X) = 2\pi\chi(X)$$

which is the familiar Gauss-Bonnet Theorem.

2 Exterior Algebra

2.1 The Wedge Product

We now turn our attention to the use of tensors in exterior algebra, a powerful tool that allows us to relate tensors to the geometry of space. We begin by looking at bilinear mappings, which are basically mappings that take in two vectors and are linear in both inputs.

Note that in all definitions below, vector spaces are assumed to be over an arbitrary field $\mathbb{F}$.

Definition 47. *Suppose E, F, G are vector spaces, and consider a mapping $\phi : E \times F \to G$. We call ϕ bilinear if it satisfies*

1. *For* $x_1, x_2 \in E, y \in F, \lambda, \mu \in \mathbb{F}$,

$$\phi(\lambda x_1 + \mu x_2, y) = \lambda\phi(x_1, y) + \mu\phi(x_2, y)$$

2. *For* $x \in E, y_1, y_2 \in F, \lambda, \mu \in \mathbb{F}$,

$$\phi(x, \lambda y_1 + \mu y_2) = \lambda\phi(x, y_1) + \mu\phi(x, y_2)$$

If $G = \mathbb{F}$*, then* ϕ *is called a bilinear function.*

We can extend the concept of bilinear maps to an arbitrary number of vectors to get the definition of a p-linear mapping.

Definition 48. *Suppose we are given* $p+1$ *vector spaces* E_i *(*$1 \leq i \leq p$*),* G*. A mapping* $\phi : E_1 \times \cdots \times E_p \to G$ *is called* p*-linear if for all* $1 \leq i \leq p$,

$$\phi(x_1, \ldots, \lambda x_i + \mu y_i, \ldots, x_p) = \lambda\phi(x_1, \ldots, x_i, \ldots, x_p) + \mu\phi(x_1, \ldots, y_i, \ldots, x_p),$$

where $x_i, y_i \in E_i, \lambda, \mu \in \mathbb{F}$*. In the case* $G = \mathbb{F}$*, the mapping is called a* p*-linear function.*

We can now redefine the tensor product as in Section 1, but now in terms of bilinear mappings.

Definition 49. *The tensor product of two vector spaces* E *and* F *is a pair* $(T, \otimes)$*, where* $\otimes : E \times F \to T$ *is a bilinear mapping with the universal property that for every mapping* $\phi : E \times F \to H$ *there exists a unique linear map* $T \to H$ *such that Figure 1 commutes.*

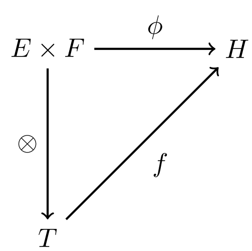

Figure 1

The space T*, which exists and is uniquely determined by* E *and* F *up to an isomorphism, is also called the tensor product of* E *and* F *and is denoted by* $E \otimes F$*.*

We will now prove the uniqueness of the tensor product. The existence is slightly more complicated, but can be seen in [4]. Suppose that we had $\otimes$ and $\tilde{\otimes}$ bilinear mappings onto spaces T and $\tilde{T}$, resp. Then there is linear isomorphism $f : T \tilde{\to} \tilde{T}$ such that $f(x \otimes y) = x \tilde{\otimes} y$ where $x \in E, y \in F$ and linear isomorphism $g : T \tilde{\to} \tilde{T}$ such that $g(x \tilde{\otimes} y) = x \otimes y$ by the diagram. By these, $gf(x \otimes y) = x \otimes y$

and $fg(x \tilde{\otimes} y) = x \tilde{\otimes} y$ and therefore gf and fg are the identity mappings on T and $\tilde{T}$, respectively. Therefore f and g are inverse isomorphisms and so T and $\tilde{T}$ are isomorphic.

Using the new definition of tensor product, we can develop the concept of a tensorial power in terms of p-linear maps. The definition of a tensorial power then gives us the definition of a tensor itself. Now, these definitions are the same as above, but it is useful to view them in this new lens as we will define the exterior algebra similarly.

Definition 50. *Let E be a vector space. For each $p \geq 2$, the pair*

$$(\otimes E^p, \otimes^p) = \underbrace{E \otimes \cdots \otimes E}_{p \ times}$$

is called the p^{th} tensorial power of E, as is the space $\otimes^p E$, whose elements are called tensors of degree p. We can extend this definition to $p \in \{0, 1\}$ by setting $\otimes^1 E = E$ and $\otimes^0 E = \mathbb{F}$.

For more on the tensor algebra, see [4].

Before defining the exterior algebra, we first define skew-symmetric mappings based on the effects that permutations on the input of p-linear mappings have on the outputs of such mappings. Essentially, a mapping is skew-symmetric if any transposition of its inputs flips the sign of its output.

Definition 51. *Let E and F be two vector spaces and let*

$$\phi : \underbrace{E \times \cdots \times E}_{p \ times} \to F$$

be a p-linear mapping. The every permutation σ on p elements determines another p-linear mapping $\sigma\phi$ given by

$$\sigma\phi(x_1, \ldots, x_p) = \phi(x_{\sigma(1)}, \ldots, x_{\sigma(p)}).$$

A p-linear mapping is called skew-symmetric if $\sigma\phi = \varepsilon_\sigma \phi$, where $\varepsilon_\sigma = \pm 1$ depending on whether the permutation is (respectively) even or odd.

The exterior power is one such skew-symmetric mapping.

Definition 52. *The p^{th} exterior power of a vector space E is a pair $(A, \wedge^p)$, where A is a vector space and*

$$\wedge^p : \underbrace{E \times \cdots \times E}_{p \ times} \to A$$

is a skew-symmetric p-linear mapping with the universal property that if

$$\phi : \underbrace{E \times \cdots \times E}_{p \ times} \to H$$

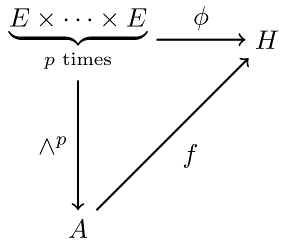

Figure 2

is a skew-symmetric p-linear mapping, there is a unique linear map $f : A \to H$ *such that Figure 2 commutes.*

The space A, which is uniquely determined up to an isomorphism, will also be called the p^{th} *exterior algebra of E and is denoted by* $\wedge^p E$*. The elements of* $\wedge^p E$ *are called p-vectors.*

The uniqueness and existence arguments here are similar as above (see [4]).

Now that we have these definitions in place, we can look at what exterior powers allow us to do in practice. As an aside, we can begin by noting that the entirety of section 1 was due to exterior algebras, as they were used to define differential forms. Note that $\wedge$ is also known as the wedge product (this was also defined in the previous section, but our new definition above provides another lens through which to view them).

We can begin by thinking of wedge products geometrically. In real spaces, we can imagine p-vectors as oriented p-parallelepipeds sitting in $\mathbb{R}^n$. This geometric intuition provides a basis and a high-level conceptual understanding of the properties of the wedge product that follow.

One key feature of wedge products in $\mathbb{R}^n$ is that they provide a unified approach to expressing geometric objects that encapsulates such notions as determinants as well as cross-products and scalar triple products (in $\mathbb{R}^3$). Let's look at cross-products first. Consider $\mathbb{R}^3$ under the standard basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, and choose $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}, \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \in \mathbb{R}^3$ arbitrarily. Then

$$\mathbf{u} \wedge \mathbf{v} = (u_1v_2 - u_2v_1)(\mathbf{i} \wedge \mathbf{j}) + (u_2v_3 - u_3v_2)(\mathbf{j} \wedge \mathbf{k}) + (u_3v_1 - u_1v_3)(\mathbf{i} \wedge \mathbf{k}).$$

Notice that the coefficients here match those of the cross-product, except that we have a 2-vector as a result of this wedge product. As a result, we have an oriented parallelogram with the same orientation as the cross product. Thus we can generalize the cross-product through the wedge product, as we can find coefficients for 2-vectors in $\wedge^2(\mathbb{R}^n)$ obtained as a result of the wedge product of two vectors in $\mathbb{R}^n$. Intuitively, this generalized cross product provides us oriented n-parallelepipeds that will have edges parallel to the vectors that we wedge, rather than the n-dimensional vectors that we would have expected to get as a result of the cross-product. The relationship with the cross product will be made more explicit in the next section.

We can also look at determinants. Although the calculations are omitted here, the magnitude of the 2-vector of the wedge of two 2-D vectors $\mathbf{v} \wedge \mathbf{w} \in \wedge^2(\mathbb{R}^2)$ in terms of the standard basis matches up with the determinant of the transformation associated with those vectors. Similarly, the magnitude of the 3-vector of the wedge of three 3-D vectors in $\wedge^3(\mathbb{R}^3)$ is the same as the determinant of the transformation associated with those vectors. In general, we have that for $v_1, \ldots, v_n \in \mathbb{R}^n$ the magnitude of the n-vector $v_1 \wedge \cdots \wedge v_n$ in terms of the standard basis is equal to

$$\det \begin{pmatrix} \mathbf{v_1} \\ \vdots \\ \mathbf{v_n} \end{pmatrix}.$$

In the language of differential forms, we refer to nonzero elements of the space $\wedge^n(V)$ as volume forms, for a vector space V with dimension n.

As an aside, note that for a finite-dimensional vector space V, where $\dim V = n$, the vector space $\wedge^p(V)$ has $\dim \wedge^p(V) = \binom{n}{p}$. This is because we have n basis vectors for V, and can choose p vectors to wedge together in $\binom{n}{p}$ ways.

Ultimately, the wedge product provides us a way to encapsulate and generalize various geometric notions by using p-vectors, unifying various geometric ”languages” within mathematics. A plethora of visual aids relevant to this section can be found in [2].

The wedge product also provides a basis for a coordinate-free approach to linear algebra (see [11]).

2.2 Hodge Duality

We proceed by looking at the duality of the exterior algebra under the Hodge star operator. More information about this duality can be found in [3]. Formally, we restrict ourselves to the case where V is a finite-dimensional inner-product space and define the Hodge star on V as follows:

Definition 53. *The Hodge Star operator is a linear operator on the exterior algebra $\wedge^p(V)$ that maps p-vectors to $(n-p)$-vectors such that*

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle (\omega) \quad \forall \alpha, \beta \in \wedge^p(V),$$

where ω is the volume form obtained by wedging vectors in an oriented orthonormal basis of V.

A dual definition of the Hodge star is in terms of determinants: for any vectors $\alpha, \beta \in \wedge^p(V)$, we have that $\det(\alpha \wedge \star\beta) = \langle \alpha, \beta \rangle$.

The Hodge Star operator is similar to the orthogonal complement in linear algebra. Essentially, this is because we can think of exterior products as being analogous to the span of all linear combinations of vectors from two subspaces (although these two concepts have differences, as the span is itself a subspace while the wedge product is a p-vector). Since the wedge of a p-vector α with the Hodge star of another p-vector β is an n-vector with determinant $\langle \alpha, \beta \rangle$, we can think of $\alpha \wedge \star\beta$ as an n-parallelepiped and α as a p-parallelepiped with edge vectors parallel to $\alpha \wedge \star\beta$. Thus $\star\beta$ is an $(n-p)$-parallelepiped whose edges are parallel to the vectors that are the edges of $\alpha \wedge \star\beta$ but not edges of α.

The relationship between the Hodge star, wedge product, and cross product is expressed in the formulas below:

$$\star(\mathbf{u} \wedge \mathbf{v}) = \mathbf{u} \times \mathbf{v}, \qquad \star(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \wedge \mathbf{v}.$$

2.3 The Geometric Algebra

One algebra intimately linked to the exterior algebra is the geometric algebra. This algebra utilizes both the wedge product and inner product to define a geometric product by the formula $\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$. It is more standard to formulate the geometric product axiomatically and define the dot and wedge in terms of the geometric product, but we will not do so here for the sake of brevity.

For example, if we find the geometric product of two parallel vectors, we reason by skew-symmetry of the wedge product that the geometric product is equal to the inner product in this case. Similarly, for orthogonal vectors the inner product is 0 so the geometric product is equal to the wedge product.

Note that the geometric product inherently adds quantities that may not be of the same "type:" the inner product of two vectors is a scalar, while their wedge is a 2-vector. Thus the principal objects of geometric algebras are not p-vectors, but instead "multivectors" that are the sum of various p-vectors that we can write informally as something in the general form

$$\sum_{p=0}^{n} x_p \qquad x_p \in \wedge^p E.$$

Now, we can write Maxwell's equations in terms of the geometric product. Recall that Maxwell's equations are

$$\nabla \cdot \mathbf{D} = \rho,$$
$$\nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

With geometric algebra, we can combine these four equations into a single equation (see [6] for a full derivation in an arbitrary number of dimensions). In 3-D, we note that the pseudoscalar (the basis element resulting from taking the geometric product of all 3 basis vectors) squares to -1 and therefore behaves like the imaginary unit i. Thus we will refer to this pseudoscalar as i. Set $F = \mathbf{E} + ic\mathbf{B}$, and we call $\mathbf{F}$ the field multivector; we can also set $J = \rho - \mathbf{J}$ to be the current multivector. We find that, after defining ∇ analogously to how it is defined in vector calculus, $\nabla F = J$.

Intuitively, this reformulation makes sense because we have that

$$\nabla F = \nabla \cdot F + \nabla \wedge F = \nabla \cdot F + i\nabla \times F,$$

and therefore it's easy to see that the scalar component is Gauss' law, the vector component is Ampere's law, the psuedovector component (the component with basis elements resulting from the geometric product of two basis vectors) is Faraday's law, and lastly Gauss' law for magnetism is the pseudoscalar component. Thus the use of geometric algebra allows us to generalize Maxwell's laws (see [6]) in a clean way.

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