# MANIFOLDS, SYMPLECTIC GEOMETRY, AND MECHANICS

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Abstract. In this paper, we introduce coordinate systems, manifolds, holonomic constraints, differential forms, and constructs from symplectic geometry. We also discuss how they apply to classical mechanics and help us understand the motion of objects in our 3-dimensional world.

## 1. A *Symple* Introduction

To be able to describe the motion of objects, we first must create a foundation of vocabulary and notation to avoid confusing language (though I will not deny symplectic geometry can have intimidating symbols). We begin with some definitions:

**Definition 1.1** (Coordinate Map)**.** Take some space $M$ in $d$-dimensions and some open subset of $\mathbb{R}^d$. In addition, consider an open subset of $M$, which we will call $U$. A **coordinate map** $\psi$ is a homeomorphism $\psi$ mapping $U$ to some open subset of $\mathbb{R}^d$.

**Definition 1.2** (Coordinate System)**.** Any pair $(U, \psi)$ is called a **coordinate system** or a **coordinate chart**. [1]

As we will see, defining coordinate systems on manifolds will be very helpful in specifying position, momentum, and paths of objects in classical mechanics.

In addition, it helps easily define what a locally Euclidean space is:

**Definition 1.3** (Locally Euclidean Space)**.** A space $M$ is a **locally Euclidean space** if such a local coordinate system exists around every point in $M$. In other words, consider any point $\mathbf{p} \in M$. Also, consider the set of points in $M$ close to $\mathbf{p}$, denoted by $U$. In other words, let $\varepsilon > 0$ be any sufficiently small real number, $\Gamma$ be the set of all paths from $\mathbf{p}$ to $\mathbf{q}$ along the manifold, and $U = \{\mathbf{q} \in M : \inf_{\gamma \in \Gamma} \int_{\mathbf{p}}^{\mathbf{q}} \|\dot{\gamma}\| dt < \varepsilon\}$. Then, there exists a homeomorphism $\psi$ that maps $U$ to $\mathbb{R}^d$.

Locally Euclidean spaces are nice and easy to work with, which jives well with the purpose symplectic geometry: to focus on simplistic cases.

**Remark 1.4.** Another way to simplify things is to assume that every map or function one works with is infinitely differentiable. This assumption can be seen in numerous definitions of the subject.

We also have *transition functions* that takes us from one coordinate chart to another. More formally,

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[1] Definitions taken from [Tob17].

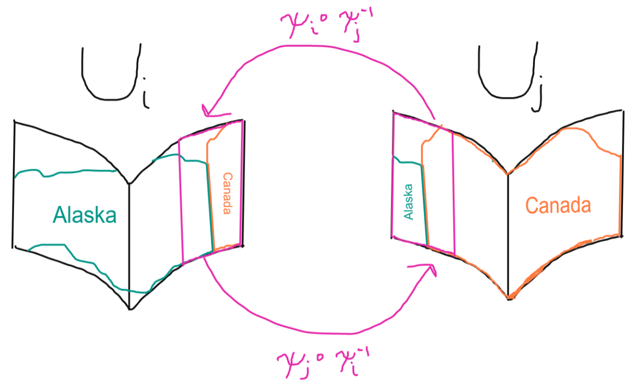


FIGURE 1. A pictorial representation of the idea discussed in the paragraph above.

**Definition 1.5** (Transition Function)**.** Say we have two coordinate systems $(U_1, \psi_1)$ and $(U_2, \psi_2)$. A **transition function** is a function $q : \psi_1(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2)$ [2].

Notice how if $U_1 = U_2$, we could still have two different coordinate systems on the same set, and could give rise to a transition function.

We can consider multiple different coordinate systems at once:

**Definition 1.6** (Smooth Atlas)**.** A **smooth atlas** $\mathcal{A}$ on a locally Euclidean space $M$ is a collection of coordinate systems $\{(U_i, \psi_i) \mid i \in \mathcal{I}\}$ (where $\mathcal{I}$ is just some indexing set) such that

(1) All points are covered by at least one chart, i.e. $\bigcup_{i\in\mathcal{I}} U_i = M$, and
(2) For any two $U_i \subseteq M$ and $U_j \subseteq M$, $\psi_i \circ \psi_j^{-1}$, a.k.a the transition function between $\psi_j(U_i \cap U_j) \to \psi_i(U_i \cap U_j)$, is smooth for all $i, j \in \mathcal{I}$, and
(3) For any two $\mathbf{p}, \mathbf{q} \in M$ such that $\mathbf{p} \neq \mathbf{q}$, there exist $U_1, U_2 \subseteq M$ such that$\mathbf{p} \in U_1$, $\mathbf{q} \in U_2$, and $U_1 \cap U_2 = \phi$. This last condition is also known as the **Hausdorff condition**.

Sometimes, the word *atlas* is replaced by the word **differentiable structure**, though they signify the exact same construct.

The term *atlas* is actually quite fitting here. You can imagine a set of atlases inside a book with world maps; say there was a map of Alaska on two consecutive pages, and part of the second page also showed part of West Canada. Then, after flipping the page, the next two consecutive pages cover mostly Canada, but on the left page there is a slight part of eastern Alaska. The rightmost part of the previous two pages and the leftmost part of the current two pages overlap, i.e. show the same region. This overlap can be considered the space of the transition functions between $\psi_i(U_i \cap U_j)$ and $\psi_j(U_i \cap U_j)$, where $U_i$ represents the last two pages, and $U_j$ represents the new two pages (or a subset of them which overlap).

**Remark 1.7.** For simplicity, we will refer to smooth atlases simply as atlases.

[2][Wol] plus some modification of language to fit that of the paper

We also have an extension of this concept, which is more commonly used in symplectic geometry:

**Definition 1.8** (Maximal Atlas)**.** A **maximal atlas** $\mathcal{A}_{\max}$ is a smooth atlas that also satisfies the following property: if $(U_o, \psi_o)$ is a coordinate system such that $\psi_o$ satisfies the properties of Definition 1.6 and for all $i \in \mathcal{I}, \psi_i = \psi_o$, then $(U_o, \psi_o) \in \mathcal{A}_{\max.}$ In other words, we cannot add new coordinate systems to this atlas.

**Definition 1.9** (Smooth Manifold)**.** Consider a set $M$ of $d$-dimensions. Then, the set $M$ along with any atlas on that set would define a **smooth manifold** of $d$-dimensions, or simply a **manifold**.

As we will see, manifolds are a useful construction that we can use to describe and analyze motion or objects. In particular, the *tangent space* of the manifold is involved:

**Definition 1.10** (Tangent Space)**.** The **tangent space** to a manifold $M$, denoted as $TM$, is the set of all vectors tangent to $M$ at any point. For example, letting $T_{\mathbf{p}}M$ denote the set of vectors tangent to $M$ at $\mathbf{p}$, then we have

$$TM = \bigcup_{\mathbf{p} \in M} T_{\mathbf{p}}M.$$

We can make this definition more rigorous by using equivalence classes and coordinate systems in the following manner:

Let there be a smooth function $\boldsymbol{\gamma} : (-\varepsilon, \varepsilon) \to M$ such that $\boldsymbol{\gamma}(0) = \mathbf{p}$ and we consider the interval $(-\varepsilon, \varepsilon)$ as the time interval. Furthermore, define the equivalence relation $\sim$ such that $\boldsymbol{\gamma}_1 \sim \boldsymbol{\gamma}_2$ if and only if there exists some coordinate chart $(U, \psi)$ for which $\mathbf{p} \in U$ and

$$\frac{d}{dt}|_{t=0}\psi(\boldsymbol{\gamma}_1(t)) = \frac{d}{dt}\ |_{t=0}\ \psi(\boldsymbol{\gamma}_2(t)).$$

Then, $T_{\mathbf{p}}M$ is the set of *tangent vectors*, which are divided into equivalence classes by the relation $\sim$.

**Remark 1.11.** In the rigorous definition above, to make the picture more intuitive, realize that $\psi(\boldsymbol{\gamma}_1(t))$ and $\psi(\boldsymbol{\gamma}_2(t))$ are the result of taking every point on $\boldsymbol{\gamma}_1$ or $\boldsymbol{\gamma}_2$ and expressing them as points in $U$ in terms of the new coordinate system $(U, \psi)$. Then, the derivative of $\psi(\boldsymbol{\gamma}_1(t))$ at $t = 0$ being equal to the derivative of $\psi_{\boldsymbol{\gamma}_2(t))}$ at $t = 0$ is equivalent to the condition that the speeds of the curves at $t = 0$ are the same.

Now, we approach the mechanics side of symplectic geometry.

## 2. Mechanics-related Symplectic Geometry

Up until now, the terms we have defined in the previous section have been seemingly unrelated to the world of classical mechanics. However, now, they provide us with a nice basis for how to express the motion of objects in our 3-dimensional world on manifolds.

However, as many of us know from experience, to understand the motion of objects, we must set constraints on its motion and give it a definite path, which we usually denote as $\boldsymbol{\gamma}$.

**Definition 2.1** (Holonomic Constraints)**. Holonomic constraints** are a set of functions outlining the possible points that $\boldsymbol{\gamma}$ could contain. In other words, at any point in time, if the object is anywhere along the path $\boldsymbol{\gamma}$ and has coordinates $(u_1, u_2 \ldots, u_d)$ on a $d-$dimensional coordinate system, then those coordinates must satisfy the equations $f(u_1, \ldots, u_d, t) = 0$, which known as the **holonomic constraints**.

**Remark 2.2.** There also exist velocity-dependent constraints, which are functions that take in $\dot{u}_i$ as inputs for any $1 \leq i \leq d$. These are known as **non-holonomic constraints**.

**Example 2.3.** Describing the holonomic constraints for simple systems is actually quite easy. A simple example is a car moving on a 2-dimensional coordinate system along the $x$-axis, in which case the holonomic constraint is given by $y = 0$.

**Example 2.4.** Let's look at a slightly more involve example: say you are standing at the edge of a cliff of height $h_o$ above the ground and throw a ball horizontally with initial velocity $v_o$. In this case, the ball follows a parabolic path, so $\boldsymbol{\gamma}$ is a subset of the set of points described by the equation $y = -x^2 + h_o$ (though we could include a degree 1 term involving $x$ as well, we can define the coordinate system in such a way that we can void over-complication).

**Example 2.5.** Finally, let's take a look at a bob on a pendulum of length $L$. For instance, consider a pendulum with a weight located at $(x, y)$ attached to a rigid string of length $L$. Then, we can describe the pendulum bob's motion as $x^2 + y^2 - L^2 = 0$.

From these holonomic constraints, we can define a **configuration space** (also known as a constraint manifold) that describes the change in the coordinates of the object per unit time at each point. For instance, say the directional vector $\mathbf{u_q}$, a $d-$dimensional vector, describes the change in the coordinates at a certain point $\mathbf{q}$ on $\boldsymbol{\gamma}$. Then, the constraint manifold $\mathcal{C} = \bigcup_{\mathbf{q} \in \boldsymbol{\gamma}} \mathbf{u_q}$.

**Example 2.6.** Let's return to the case of the ball being thrown off the edge of a cliff, and derive its configuration space. We know that the $x$-component of velocity remains constant at $\dot{x} = v_o$. For the $y$-component, we realize that at $t = 0$ (the instant the ball is released from your hand), $\frac{dy}{dt}|_{t=0} = 0$. However, $\ddot{y} = \frac{d^2y}{dt^2} = -g$, where $g$ is the acceleration due to gravity ($g = 9.8m/s^2$). We can rearrange to get $d^2y = -gdt^2$. After integrating both sides, we get that at time $t$, $dy = -gt$. Thus, the configuration space is $\mathbf{u} = (v_o, -gt)$.

**Remark 2.7.** Symplectic geometry, besides a way to describe mechanics, is also a form of geometry which focuses on area and volume rather than lengths. This is clearly evident in the repeated use of the **exterior product**, denoted by $\wedge$. It is defined such that $\mathbf{v} \wedge \mathbf{w}$ is equal to the signed area of the region defined by the two vectors, i.e. the area of the region swept by $\mathbf{v}$ as it moves along $\mathbf{w}$ from head to tail (Figure 3). There are some important properties of the exterior product:

(1) Firstly, $\mathbf{v} \wedge \mathbf{v} = 0$ for any vector $\mathbf{v}$.
(2) Secondly, $\mathbf{v} \wedge \mathbf{w} = -\mathbf{w} \wedge \mathbf{v}$ for any vectors $\mathbf{v}, \mathbf{w}$.

As we will see soon, the exterior product plays a key role in the construction of symplectic manifolds.

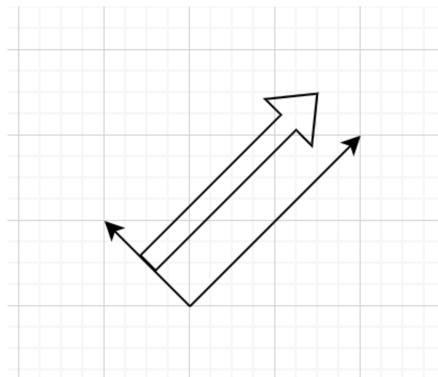

FIGURE 2. Two vectors of which we are about to take the exterior product.

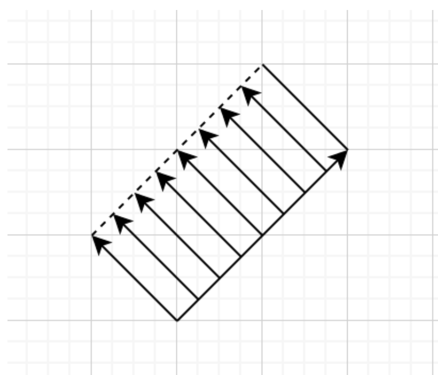

FIGURE 3. The exterior product is the area swept out by one of the vectors as it moves along the other.

## 3. More Symplectic Definitions

On the foundations of the current set of definitions we have created, we continue with more involved definitions, bringing us to why symplectic geometry really matters in mechanics. We continue the introduction of *differential forms*:

**Definition 3.1** (1-form)**.** A **differential 1-form** on a $d$-dimensional manifold $M$ takes the form

$$\sum_{j=1}^{d} F_j(u_1, u_2, \ldots, u_d) d\mathbf{u}_j,$$

where each $F_j$ is a function on an $n$-dimensional manifold $M$ that outputs a real number, and each $\mathbf{u}_j$ for $1 \leq j \leq n$ represents the vector of magnitude $u_j$ along the $j$th-axis.

It turns out we can derive the conservation of energy using a special property of 1-forms, namely:

**Proposition 3.2.** *Let there be a closed 1-form* $\omega = F_1 d\boldsymbol{x} + F_2 d\boldsymbol{y} + F_3 d\boldsymbol{z}$, *so that* $d\omega = 0$ *(where* $d$ *is the* exterior derivative*) and* $F_1, F_2, F_3$ *are all 1-differentiable functions. Then, there exists a 2-differentiable function, namely* $\beta$, *such that* $d\beta = \omega$.

In this situation, the **exterior derivative** of $\omega$ (a.k.a the de Rham derivative of $\omega$), is evaluated in the following manner:

$$\begin{aligned} d\omega &= \frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y} + \frac{\partial \omega}{\partial z} = F_{1,x} d\mathbf{x} \wedge d\mathbf{x} + F_{1,y} d\mathbf{x} \wedge d\mathbf{y} + F_{1,z} d\mathbf{x} \wedge \mathbf{z} \\ &+ F_{2,x} d\mathbf{y} \wedge d\mathbf{x} + \cdots + F_{3,z} d\mathbf{z} \wedge d\mathbf{z} \\ &= 0 + F_{1,y} d\mathbf{x} \wedge d\mathbf{y} + F_{1,z} d\mathbf{x} \wedge \mathbf{z} + F_{2,x} d\mathbf{y} \wedge d\mathbf{x} + \cdots + 0 \\ &= F_{1,y} d\mathbf{x} \wedge d\mathbf{y} + F_{1,z} d\mathbf{x} \wedge d\mathbf{z} + F_{2,x} d\mathbf{y} \wedge d\mathbf{x} + F_{2,z} d\mathbf{y} \wedge d\mathbf{z} + \\ &F_{3,x} d\mathbf{z} \wedge d\mathbf{x} + F_{3,y} d\mathbf{z} \wedge d\mathbf{y} \\ &= (F_{1,y} - F_{2,x}) d\mathbf{x} \wedge d\mathbf{y} + (F_{1,z} - F_{3,x}) d\mathbf{x} \wedge d\mathbf{z} + (F_{2,z} - F_{3,y}) d\mathbf{y} \wedge d\mathbf{z}. \end{aligned}$$

More information regarding the exterior derivatives of higher-dimensional differential forms can be found at [Pen20b].

**Example 3.3** (Conservation of Energy)**.** Imagine the force vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ represent the unit vectors in the $x, y, z$ directions, respectively, and $F_1, F_2, F_3$ are 1-differentiable. Suppose that $\nabla \times \mathbf{F} = 0$ (here, $\nabla\times$ is the vector-field-equivalent of derivative), i.e. $\mathbf{F}$ is closed (a.k.a conservative as in classical physics). Then, by the proposition, there exists a 2-differentiable function, which we call the *potential energy*, such that

$$\mathbf{F} = -\nabla P,$$

i.e. $\mathbf{F}$ is equal to the negative gradient of the potential energy of an object.

Now, consider the velocity vector $\mathcal{V} = (\dot{u}_1, \ldots, \dot{u}_d)$. Then, the kinetic energy $K$ is given by $K = \frac{1}{2}m\|\mathcal{V}\|^2$. By Newton's 2nd Law, we also have that $\mathbf{F} = m\dot{\mathcal{V}}$. In the special case that $\mathbf{F}$ is conservative (no energy is lost due to friction, heat, etc., i.e. we have a *nice* system), $\mathbf{F}$ satisfies Proposition 1, and thus we can substitute $\mathbf{F} = -\nabla P$. Now, we can rewrite $K = \frac{1}{2}m\mathcal{V} \cdot \mathcal{V}$, and after some manipulation, we end up with

$$\frac{d}{dt}(K + P) = 0,$$

which is exactly the Law of Conservation of Energy. [3]

We can state this as follows:

**Physical Law** (Conservation of Energy)**.** *In a physical system with no non-conservative forces acting on the system, the mechanical energy, a.k.a the sum of the kinetic and potential energies of the system, remains constant.*

However, we will use the *2-form* more often than the 1-form, which is a higher-dimensional analog of the 1-form, defined as follows:

[3] Example taken from [Ara16].

**Definition 3.4** (2-form)**.** A **differential 2-form**, essentially a higher-dimensional analog of a 1-form, is written as

$$\omega = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} F_j(u_1, u_2, \ldots, u_n) d\mathbf{u}_i \wedge d\mathbf{u}_j,$$

using notation similar to that of Definition 3.1.

Typically, we evaluate the differential 2-form by first considering a specific point $\mathbf{p} \in M$, converting this from a differential 2-form to simply a 2-form, which is a *bilinear form* $\omega_{\mathbf{p}}$ (we will cover this later in the paper) such that $\omega_p : T_{\mathbf{p}}M \times T_{\mathbf{p}}M \to \mathbb{R}$. Then, computing the exterior product is similar to calculating the cross product, since the action of the exterior products on the set of the original vectors, for instance $d\mathbf{u}_i \wedge d\mathbf{u}_j(\mathbf{u}_i, \mathbf{u}_j)$, gives a $2 \times d$ matrix with the $d$ components of $\mathbf{u}_i$ in the first row, and the $d$ components of $\mathbf{u}_j$ in the next row. The exterior product is simply the determinant of this matrix [4].

**Remark 3.5.** Actually, the true interpretation of a differential 2-form also requires another condition, which states that $\omega_{\mathbf{p}}$ varies smoothly (when referring to 2-forms, we will assume this condition is met).

**Example 3.6.** For instance, consider the case when $n = 3$ (which most accurately describes our planet, commonly believed to exist in 3 dimensions). Then, the 2-form in 3 dimensions would be

$$\omega = F_1(x, y, z) d\mathbf{x} \wedge d\mathbf{y} + F_2(x, y, z) d\mathbf{y} \wedge d\mathbf{z} + F_3(x, y, z) d\mathbf{x} \wedge d\mathbf{z}.$$

We also have one more concept that is based off of the idea of differential forms:

**Definition 3.7** (Pull-back)**.** Say $F$ is a smooth map from one manifold to another, say $F : M \to N$. Then, the **pull-back** of $F$, denoted $F^*$, is a smooth map from the vector space of 2-forms on $N$ to the vector space of the 2-forms on $M$. This means that wor any $\mathbf{p} \in M$ and $\mathbf{q} \in N$ such that $F(\mathbf{p}) = \mathbf{q}$, then $F^*(\omega_{2,\mathbf{q}}) = \omega_{1,\mathbf{p}}$.

We can imagine the pull-back is a map we get for free between the constructs on $N$ and the constructs on $M$ automatically when we define a map from $M$ to $N$. One could also imagine it as the inverse (though not technically) of $F$ but on a different set of constructs.

Now, we define the basic construct in symplectic geometry: the *symplectic manifold.*

**Definition 3.8** (Symplectic Manifold)**.** A smooth manifold $M$ with an even number of dimensions along with a differential 2-form on that manifold, called the **symplectic form** and denoted by $\omega$, is considered a special kind of manifold, known as a **symplectic manifold**, if

(a) $d\omega = 0$, i.e. $\omega$ is closed, and
(b) For any $v \in TM$, $\omega(v, w) = 0$ implies $w = 0$; i.e. $\omega$ is non-degenerate.

[5]

**Remark 3.9.** The reason to specify that $M$ has an even number of dimensions as above will become evident later during discussion of the *Linear Darboux Theorem.*

[4] [Pen20a]

[5] Definition taken from [Jef22].

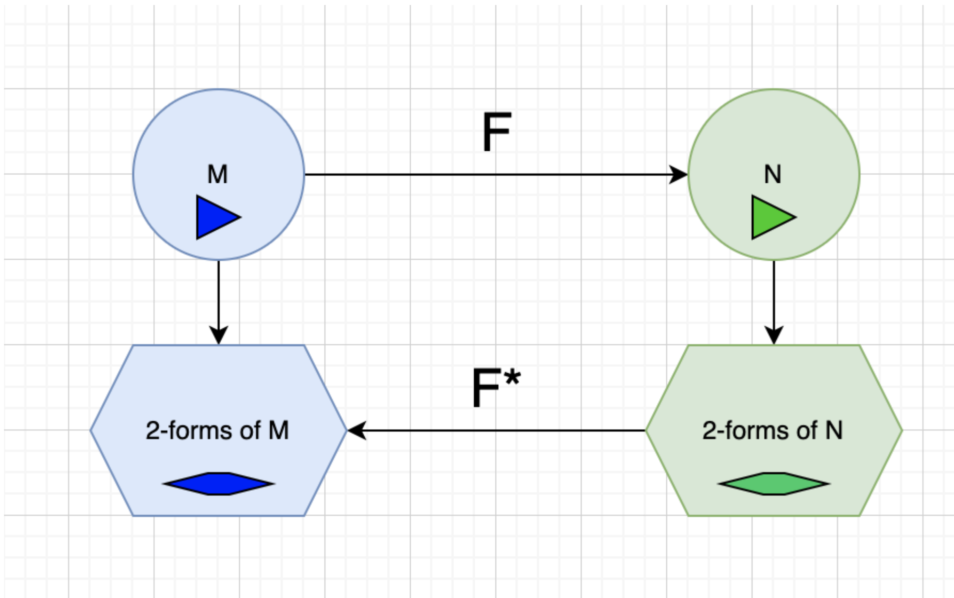


FIGURE 4. A graphical representation of the pull-back of a function. $F$ maps the dark blue triangle to the green triangle. The green diamond represents the space of the differential 2-forms on the green triangle, while the blue diamond represents the space of the differential 2-forms on the blue triangle. The pull-back of $F$, $F^*$, maps the green diamond to the blue diamond.

Of course, we can also consider maps between symplectic manifolds:

**Definition 3.10** (Symplectomorphism)**.** A **symplectomorphism** $S$ is a diffeomorphism from two symplectic manifolds $(M_1, \omega_1)$ and $(M_2, \omega_2)$ $S : M_1 \to M_2$ which has a pull-back acting on the space of 2-forms on $M_2$ that outputs elements of the space of 2-forms on $M_1$. We consider $M_1$ and $M_2$ as manifolds and $\omega_1, \omega_2$ their differential 2-forms, respectively.

In other words, a symplectomorphism is a set of two maps that essentially came with a construct-one-get-one-free deal.

It turns out that symplectomorphisms are transitive, though we will not be proving that in this paper.

Let's look at a few examples of symplectic manifolds:

**Example 3.11.** Suppose we have an orientable $d$-dimensional manifold $M$, i.e. there exists a volume form $\omega$ (i.e. a differential $d$-form, a higher dimensional analog of the 2-form that involves exterior products of $d$ vectors at a time). Then, since volume forms are nondegenerate on orientable surfaces and are closed (we leave this as an excercise to the reader to show), $(M, \omega)$ is a symplectic manifold.

**Example 3.12.** Say $(M, \omega)$ is a any symplectic manifold. Then, for any open subset $U \subseteq M$, $(U, \omega)$ is also a symplectic manifold.

**Example 3.13.** The $d$-dimensional torus $\mathbb{T}^d = S^1 \times S^1 \times \cdots \times S^1$ (where the right-hand side involves $d$ terms) with the symplectic form

$$\omega = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} d\mathbf{u}_i \wedge d\mathbf{u}_j$$

together constitute a symplectic manifold. [6]

The *Linear Darboux Theorem* gets us closer to some insight on how to understand these abstract constructions. Before introducing the theorem, we cover a few quick definitions:

**Definition 3.14** (Bilinear Forms)**.** Take a vector space $V$. A **bilinear form** $\mathcal{B}$ is a map $\mathcal{B} : V \times V \to \mathbb{R}$ that satisfies the following properties:

(1) For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $\mathcal{B}(\mathbf{u} + \mathbf{v}, \mathbf{w}) = \mathcal{B}(\mathbf{u}, \mathbf{v}) + \mathcal{B}(\mathbf{u}, \mathbf{w})$ and $\mathcal{B}(\mathbf{u} + \mathbf{w}, \mathbf{v}) = \mathcal{B}(\mathbf{u}, \mathbf{v}) + \mathcal{B}(\mathbf{w}, \mathbf{v})$, and
(2) For any scalar $\lambda \in \mathbb{R}$, $\mathcal{B}(\mathbf{u}, \lambda\mathbf{v}) = \mathcal{B}(\lambda\mathbf{u}, \mathbf{v}) = \lambda\mathcal{B}(\mathbf{u}, \mathbf{v})$.

**Remark 3.15.** A differential 2-form can also be viewed a continuous map from $\mathbb{R}^d$ to the set of all alternating bilinear forms on $V \times V$.

**Definition 3.16** (Alternating and Non-degenerate Bilinear Forms)**.** A bilinear form is considered **alternating** if $\mathcal{B}(\mathbf{v}, \mathbf{v}) = 0$ for all $\mathbf{v} \in V$. A **nondegenerate bilinear form** is one such that for any two vectors $\mathbf{v}, \mathbf{w} \in V$, $\mathcal{B}(\mathbf{v}, \mathbf{w}) = 0$ implies that one of $\mathbf{v}$ or $\mathbf{w}$ is $\mathbf{0}$.

We can now state the theorem:

**Theorem 3.17** (Linear Darboux)**.** *Let there be a vector space $V$ of $d$ dimensions. Then, if there exists an alternating non-degenerate bilinear form $\mathcal{B} : V \times V \to \mathbb{R}$, then $d$ must be even.*

*Proof.* The idea of the proof follows along the lines of induction. For the base case, consider any vector $\mathbf{v} \in V$. Then, due to the non-degeneracy property of $\mathcal{B}$, there exists some $\mathbf{v}' \in V$ such that $\mathcal{B}(\mathbf{v}, \mathbf{v}') = 1$. Thus, there does not exist a scalar $\lambda$ for which $\mathcal{B}(\mathbf{v}, \lambda\mathbf{v}') = 0$, which means that $\mathbf{v}'$ is not a scalar multiple of $\mathbf{v}$ (otherwise due to $\mathcal{B}(\mathbf{v}, \mathbf{v}) = 0$, the output of the bilinear form acting on $\mathbf{v}$ and $\mathbf{v}'$ would also be 0). Thus, $V$ must have at least two dimensions.

Now, we move on to the inductive step. Consider the set $\mathfrak{V} = \{\mathbf{w} \in V \mid \mathcal{B}(\mathbf{v}, \mathbf{w}) = \mathcal{B}(\mathbf{v}', \mathbf{w}) = 0\}$. In other words, $\mathfrak{V}$ contains the vectors that are scalar multiples of either $\mathbf{v}$ or $\mathbf{v}'$. Due to the non-degeneracy condition, this set must have $d - 2$ dimensions, and also has its own alternating non-degenerate bilinear form. Therefore, we can combine this basis of vectors with $\mathbf{v}$ and $\mathbf{v}'$ and end up with $V$, which must have an even number of dimensions. □

**Corollary 3.18.** *A corollary of the theorem above is that all symplectic manifolds have an even number of dimensions.*

This is why in the definition of a symplectic manifold, we must state that the dimension of the manifold is even (or it can be implied by the corollary above).

There is yet another theorem (by Darboux once more) which presents even more information regarding symplectic manifolds:

[6]Both examples above were taken from [WanAD].

**Theorem 3.19** (Darboux)**.** *Every symplectic d-dimensional manifold* $(M, \omega)$ *is locally symplectomorphic to* $(\mathbb{R}^d, \omega_{\mathbb{R}^d})$*, where* $\omega_{\mathbb{R}^d}$ *is a symplectic form on* $\mathbb{R}^d$ [7]*.*

The idea of the proof is given a $d$-dimensional manifold $M$ and its symplectic form along with the symplectic form of $\mathbb{R}^d$, there is only one diffeomorphism from $M$ to $\mathbb{R}^d$. We will not be proving this theorem in this paper[8].

However, we will be including a corollary that one can derive from this theorem:

**Corollary 3.20.** *Every d-dimensional manifold is symplectomorphic to each other.*

What this is essentially saying is that every $d$-dimensional symplectic manifold is equivalent to every other. This has huge applications, especially in the field of **contact geometry**.

There are numerous more interesting properties of symplectic manifolds that this paper did not cover, and this is still an ongoing area of research. In fact, this field is so new that some believe the foundations for the topic are too weak to properly define abstracts [HM19].

## 4. Conclusion

In this paper, we have introduced coordinate systems and maps; manifolds and atlases; holonomic constraints and configuration spaces; differential forms; and constructs from symplectic geometry, including symplectic manifolds and symplectomorphisms. We also discuss how they apply to classical mechanics and simplify the process of describing the motion of objects. For instance, holonomic constraints and configuration spaces are a very natural way to translate the motion of real-world objects into the language of manifolds. In addition, differential forms allow us to understand force and prove the Law of Conservation of Energy. Finally, constructs from symplectic geometry enable us to investigate some interesting properties in this area.

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## References

[Ara16] Donu Arapura. Introduction to differential forms. May 2016.
[HM19] Kevin Hartnett and Quanta Magazine. A fight to fix geometry's foundations. *Quanta Magazine*, Mar 2019.
[Jef22] Maxim Jeffs. Classical mechanics and symplectic geometry. *Maxim Jeffs*, May 2022.
[Kum15] Nilay Kumar. *Notes on symplectic geometry contents - nu math sites*, 2015.
[Pen20a] Differential forms | 2-forms, May 2020.
[Pen20b] Differential forms | the exterior derivative., Jul 2020.
[Spu13] R Spuzio. Proof of darboux's theorem (symplectic geometry). *Proof of Darboux's theorem (symplectic geometry)*, Mar 2013.

[7] The theorem statement was taken from [Kum15]

[8] A proof can be found at [Spu13] which involves the *Moser trick*, but this is outside the scope of this paper.

[Tob17] Symplectic geometry amp; classical mechanics, lecture 1, Oct 2017.
[WanAD] Wangzuoq Wangzuoq. *Contents symplectic manifolds - USTC*, 0AD.
[Wol] Transition function.

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