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# MEAN CURVATURE FLOW

AMIT SAHA

Abstract. Mean curvature flow is a fascinating field discussing the evolution of hypersurfaces by the mean curvature at a point. It has multiple applications throughout physics and mathematics. In this paper, we first examine some theorems on the $\mathbb{R}^2$ dimensional analogue of mean curvature flow, known as curve-shortening flow. We then examine some prototypical examples of mean curvature flow in $n$ dimensions and examine theorems on convergence of hypersurfaces evolving by mean curvature flow.

## 1. Preface

In a general setting, *geometric flows* describe the deformation of geometric objects using partial differential equations. Geometric flows have numerous uses throughout geometry and applied mathematics, and generally serve to evolve geometric onjects to "nicer" ones with better properties.

## 2. Curve Shortening Flow

First, we'll restrict our attention to *mean curvature flow* in two dimensions, with curves embedded into $\mathbb{R}^2$.

**Definition 2.1** (Curve Shortening Flow)**.** Let $M \in \mathbb{R}^2$ be a smooth curve with unit normal vector $\mathbf{n}(p,t)$ and scalar curvature $\kappa(p,t)$,Then, the *curve shortening flow* is defined by the family of curves $\{\varphi_t \subset \mathbb{R}^2\}$ that satisfy

$$\frac{\partial}{\partial t}\varphi(p,t) = H(p,t)\mathbf{n}(p,t).$$

We can rephrase this in terms of arclength, so that the family of curves satisfies

$$\frac{\partial}{\partial t}\varphi = \frac{\partial^2}{\partial s^2}\varphi.$$

A prototypical example of curve shortening flow is seen in the sphere.

*Example.* If we have $M = \mathbb{S}^1$, we can write that $\varphi(p,t) = R(t)\varphi_0(p,t)$. Then, (2.1) gives us the differential equation

$$R'(t) = -\frac{1}{R(t)}.$$

*Date*: July 5, 2022.

This admits the solution $R(t) = \sqrt{R^2 - 2t}$, $t \in (-\infty, R^2/2)$.

Notice that the solution $R(t) = \sqrt{R^2 - 2t}$ exists for a finite time. It turns out that there exists a strong condition on the lifetime of a closed curve under curve shortening flow. If a closed curve encloses a region of area $A(t)$, we can show that

$$\frac{\partial}{\partial t} A(t) = -\int_{\varphi} \kappa \, \mathrm{d}s = -2\pi.$$

Integrating this yields

$$A(t) = A(0) - 2\pi t,$$

so any closed curve that converges to a singularity will converge in finite time $\frac{A(0)}{2\pi}$. This raises the natural question – when do we know that curves will converge to a singularity in $\mathbb{R}^2$? The answer is given in the Gage-Hamilton theorem.

**Theorem 2.2** (Gage-Hamilton)**.** *A convex curve $\varphi \subset \mathbb{R}^2$ converges to a single point.*

The proof of the Gage-Hamilton theorem is beyond the scope of this paper. However, the consequences are enormous. Under this setting, all convex curves eventually converge to a finite point. In conjunction with another result, we can learn even more about curve shortening flow.

**Theorem 2.3** (Grayson)**.** *If $\varphi_0$ is a closed curve in $\mathbb{R}^2$, then, there exist $x_0$ and $T < \infty$ such that the mean curvature flow $\varphi$ converges to a round point around $x_0$.*

Grayson's theorem essentially implies that non-convex closed and embedded curves eventually become convex, so that the Gage-Hamilton theorem may be applied. Together, these two results imply that any closed embedded curve shrinks to a unique round point under curve shortening flow. We have now determined a fascinating condition on convergence for curve-shortening flow. We can also make an assertion about the disjunction of curves under curve shortening flow, known as the avoidance principle.

Another fascinating area of interest for curve shortening flows is in travelling solutions. For the curve-shortening flow, we have the traveling solution known as the **grim reaper**.

$$y = -\log\cos(t), t \in (-\pi/2, \pi/2).$$

This curve is essentially translated forwards over time, and never changes or converges in any way.
Finally, we can prove some results on the curvature over time.

**Theorem 2.4** (Evolution for curvature in curve shortening flow)**.** *If $\varphi$ evolves by curve shortening flow, then*

$$\kappa_t = \kappa_{ss} + \kappa^3.$$

*Proof.* We begin with a unit-speed parameterization. We can first compute that

$$\begin{aligned} \kappa_t &= \partial_t \langle \partial_x^2 \varphi, \mathbf{n} \rangle - 2\langle , t, \partial_x \partial_t \varphi \rangle \langle \partial_x^2 \varphi, \mathbf{n} \rangle \\ &= \langle \partial_x^2 \partial_t \varphi, \mathbf{n} \rangle - 2\kappa \langle t, \partial_x \partial_t \varphi \rangle \end{aligned}$$

We can then continue that

$$\begin{aligned} \partial_t \kappa &= \partial_x^2 \kappa + \kappa \langle \partial_x^2 \mathbf{n}, \mathbf{n} \rangle - 2\kappa^2 \langle t, \partial_x \mathbf{n} \rangle \\ &= \partial_x^2 \kappa - \kappa \langle \partial_x \mathbf{n}, \partial_x \mathbf{n} \rangle + 2\kappa^3 \end{aligned}$$

Observing that $\partial_x^2 \kappa = \kappa_{ss}$ proves the theorem. ∎

## 3. Mean Curvature Flow

We'll now define curve shortening flow in a space of arbitrary dimension. First, we'll cover some preliminaries.

**Definition 3.1** (Mean Curvature)**.** Let $S$ be a hypersurface embedded into $\mathbb{R}^{n+1}$ with second fundamental form $F_2$. Then, the mean curvature is

$$\mathrm{Tr}(F_2) = \sum_{i=1}^{n+1} \kappa_i,$$

where $\kappa_i$ is the $i$-th principal curvature.

We'll denote the unit normal vector to $\varphi$ at a point $p$ at some time $t$ as $\mathbf{n}(p, t)$. As the mean curvature changes at any given point over time, we will use the notation $H(p, t)$ to denote the mean curvature. We can now define the mean curvature flow.

**Definition 3.2** (Mean Curvature Flow)**.** Let $M^n$ be a smooth (i.e. differentiable) manifold. Suppose that there exists some initial embedding of $M$, $\varphi_0 : M^n \to \mathbb{R}^{n+1}$. The *mean curvature flow* of $M^n$ is a family of hypersurfaces $\varphi : M \times [0, t) \to \mathbb{R}^{n+1}$ that satisfy

$$\frac{\partial}{\partial t} \varphi(p, t) = H(p, t)\mathbf{n}(p, t).$$

While hypersurfaces evolving by geometric flows generally do not admit solutions, some simple objects do. We can illustrate the general principle behind mean curvature flow with some prototypical examples, such as the sphere.

*Example.* Consider the $n$-sphere of radius $R$ centered at the origin of $\mathbb{R}^{n+1}$. Intuitively, we can figure out that the sphere simply shrinks over time – the mean curvature is constant everywhere and the unit inwards normal vector also always points towards the origin. This turns out to be true.
If $M = \mathbb{S}^n$, then we find the relation $\varphi(p,t) = R(t)\varphi_0(p)$, where $R(t)$ varies the radius over time and $R(0) = R$. We can then find that

$$R'(t)\varphi_0(p) = \frac{\partial}{\partial t}\varphi(p,t) = -n\frac{\varphi_0(p)}{R(t)},$$

which admits the solution $R(t) = \sqrt{R^2 - 2nt}$ on $t \in (-\infty, R^2/2n)$.

A similar solution exists for the cylinder.

*Example.* Suppose that there exists an $n$-dimensional cylinder $M = \mathbb{S}^j \times \mathbb{R}^{n-j+1}$. The cylinder then has a family of immersions under mean curvature flow described by the radius function

$$R(t) = \sqrt{R^2 - 2(n-j)t}, t \in (-\infty, R^2/2(n-j)).$$

Under a more general setting, there exists no exact formula for the evolution of embedded hypersurfaces. However, we can still prove theorems of existence. We first describe the analogue of the Gage-Hamilton theorem in the $n$-dimensional setting.

**Theorem 3.3** (Huisken)**.** *For $n \geq 2$, a closed convex hypersurface $M \subset \mathbb{R}^{n+1}$ converges to a round point.*

Strikingly, Grayson's theorem does not have a higher dimensional analogue.