MINIMAL SURFACES

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1. Introduction

One of the problem sets this year had a question about a very strange parametrized surface, which turned out to be a conformal map from the plane:

$$\sigma(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right).$$

This surface, known more commoonly as the *Enneper surface*, is also a minimal surfaces.

Recall that the mean curvature is defined as $H = \frac{1}{2}\mathrm{Tr}(\mathcal{W})$, where $\mathcal{W}$ is the Weingarten map.

Definition 1.1. A *minimal surface* is a surface with zero mean curvature.

It is easy to see that since the mean curvature (ie. the trace) is 0, the product of the principal curvatures (eigenvalues) of the Weingarten map matrix, or the Gaussian curvature, must be nonpositive.

Theoretically, the minimal surface is one that has the least area when stretched over a closed contour. These arise naturally in many scenarios, such as the draping of a circus tent, the film between two soap films, or even our cells' endoplasmic reticulum. All surfaces parametrized as $(x, y, f(x, y))$ with 0 mean curvature satisfy the equation

$$\frac{d}{dx}\left(\frac{f_x}{\sqrt{1+f_x^2+f_y^2}}\right) + \frac{d}{dy}\left(\frac{f_y}{\sqrt{1+f_x^2+f_y^2}}\right) = 0.$$

Using the formula

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = 0,$$

one can also obtain the equation

$$(1+f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1+f_x^2)f_{yy} = 0$$

to describe minimal surfaces.

Example. The *helicoid*, parametrized by

$$\sigma(u,v) = (u\cos v, u\sin v, v)$$

, is a minimal surface.

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Proof. For the helicoid, we can check that it is minimal by finding the coefficients of the first fundamental form:

$$E = \sigma_u \cdot \sigma_u = (\cos v, \sin v, 0) \cdot (\cos v, \sin v, 0) = \cos^2 v + \sin^2 v = 1,$$
$$F = \sigma_u \cdot \sigma_v = (\cos v, \sin v, 0) \cdot (-u \sin v, u \cos v, 1) = 0,$$
$$G = \sigma_v \cdot \sigma_v = (-u \sin v, u \cos v, 1) \cdot (-u \sin v, u \cos v, 1) = u^2 \sin^2 v + u^2 \cos^2 v + 1 = u^2 + 1.$$

The unit normal vector can be computed to be $\mathbf{N} = (\sin v, -\cos v, u)/\sqrt{u^2+1}$, so for the second fundamental form we have the following coefficients.

$$L = \mathbf{N} \cdot \sigma_{uu} = 0,$$
$$M = \mathbf{N} \cdot \sigma_{uv} = -\frac{1}{\sqrt{u^2+1}},$$
$$N = \mathbf{N} \cdot \sigma_{vv} = 0.$$

Note that in the numerator for our expression of the mean curvature H, all terms contain either $F = 0, L = 0$ or $N = 0$, so the mean curvature is indeed 0. A helicoid soap film is presented in Figure 1. □

Figure 1. A helicoid in real life.

Notice that the minimal surfaces do not necessarily correspond to having a minimal surface area to volume ratio: spheres have constant positive mean curvature, for example.

In addition to helicoids, catenoids and planes are the most commonly known minimal surfaces. These can be checked in a similar manner to have zero mean curvature.

2. Complex Analysis Crash Course

For the next section, we will need knowledge of some topics from complex analysis.

Definition 2.1. Let $f(x, y)$ be a function of two variables. We say that f is a *harmonic function* if it is a twice-differentiable function that satisfies the equation

$$\nabla^2 f = f_{xx} + f_{yy} = 0.$$

This equation is known as *Laplace's equation*.

Example. Let $f(x,y) = x^2 - y^2$. Then

$$\nabla^2 f = (x^2 - y^2)_{xx} + (x^2 - y^2)_{yy} = 2 + (-2) = 0.$$

We also need the concept of *holomorphic functions*, which are basically the "nice" functions of complex analysis.

Definition 2.2. Let $U \in \mathbb{C}$ be an open set. A function $f(z) : U \to \mathbb{C}$ is *holomorphic* if it is differentiable, ie.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, at every point $z_0 \in U$.

To deduce whether a function is holomorphic, the Cauchy-Riemann equations are of great use.

Theorem 2.3. *(Cauchy-Riemann equations) Let $f(x,y) = u(x,y) + v(x,y)i$ be a function of a complex variable. If the following equations are satisfied,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

f must be a holomorphic function and both u and v must both be harmonic. Additionally, we refer to the functions u and v as harmonic conjugates.

3. Isothermal Surfaces

It is useful to use isothermal surfaces as a means to construct minimal surfaces.

Definition 3.1. A regular (non self-intersecting) parametrized surface $\sigma(u,v)$ is *isothermal* if it satisfies the property $E = \sigma_u \cdot \sigma_u = \sigma_v \cdot \sigma_v = G$ and $F = 0$.

The following proposition shows us another way to prove that a surface is isothermal, which will allow us to find more minimal surfaces.

Proposition 3.2. *For a function of a complex variable f, define $f_z = \frac{1}{2}(f'_u - if'_v)$. Then $\varphi(u,v) = (\varphi^1(u,v), \varphi^2(u,v), \varphi^3(u,v))$ is isothermal if and only if*

$$(\varphi^1_z)^2 + (\varphi^2_z)^2 + (\varphi^3_z)^2 = 0.$$

Proof. Expanding the left-hand side of the equation, we obtain

$$\begin{aligned}
(\varphi^1_z)^2 + (\varphi^2_z)^2 + (\varphi^3_z)^2 &= \frac{1}{4}((\varphi^1)'_u - i(\varphi^1)'_v)^2 + (\varphi^2)'_u - i(\varphi^2)'_v)^2 + (\varphi^3)'_u - i(\varphi^3)'_v)^2 \\
&= \frac{1}{4}\left(\sum_{k=1}^{3}((\varphi^k)'_u)^2 - \sum_{k=1}^{3}((\varphi^k)'_v)^2 - 2i\sum_{k=1}^{3}(\varphi^k)'_u(\varphi^k)'_v\right) \\
&= \frac{1}{4}(E - G + 2iF),
\end{aligned}$$

which is only 0 if $E = G$ and $F = 0$, precisely the case when φ is an isothermal surface. ∎

Now we show how to find minimal isothermal surfaces with an explicit construction.

Theorem 3.3. *Let ψ_1, ψ_2, ψ_3 be holomorphic functions such that*

(1) $(\psi_z^1)^2 + (\psi_z^2)^2 + (\psi_z^3)^2 = 0$,
(2) $|\psi_z^1|^2 + |\psi_z^2|^2 + |\psi_z^3|^2 \neq 0$ *(to ensure that the surface is regular, ie. non-intersecting).*

Then there exists a regular minimal isothermal surface $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ such that

$$\varphi_z^1 = \psi_1, \varphi_z^2 = \psi_2, \varphi_z^3 = \psi_3$$

defined by

$$\varphi^1 = \mathrm{Re} \int \psi_1(z)dz, \varphi^2 = \mathrm{Re} \int \psi_2(z)dz, \varphi^3 = \mathrm{Re} \int \psi_3(z)dz.$$

For the sake of conciseness, we omit the full proof, but the basic idea is that if $h'(z) = g(z)$ is a holomorphic function and $h(z) = a(z) + b(z)i$, then $a_z = g$ with notation as before. Now before we can try an example, we can simplify φ further into a surface based off of just two holomorphic functions using the following theorem.

Theorem 3.4 (Weierstrass Representation Theorem)**.** *Let $g(z)$ and $h(z)$ be two holomorphic functions. Then the isothermal surface $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ defined by*

$$\varphi^1 = \mathrm{Re} \int \frac{1}{2} h(z)(1 - g(z)^2)dz$$

$$\varphi^2 = \mathrm{Re} \int \frac{i}{2} h(z)(1 + g(z)^2)dz$$

$$\varphi^1 = \mathrm{Re} \int h(z)g(z)dz$$

is minimal.

Proof. Using the functions ψ^1, ψ^2, ψ^3 defined in terms of $\varphi^1, \varphi^2, \varphi^3$ in Theorem 3.3, we notice that

$$\begin{aligned}
(\psi_z^1)^2 + (\psi_z^2)^2 + (\psi_z^3)^2 &= \frac{1}{4}h^2(z)(1 - g(z)^2)^2 - \frac{1}{4}h(z)^2(1 + g(z)^2)^2 + h(z)^2 g^2(z) \\
&= \frac{h(z)^2}{4} - \frac{h(z)^2 g(z)^2}{2} + \frac{h(z)^2 g(z)^4}{4} \\
&- \frac{h(z)^2}{4} - \frac{h(z)^2 g(z)^2}{2} - \frac{h(z)^2 g(z)^4}{4} + h(z)^2 g(z)^2 \\
&= 0.
\end{aligned}$$

Additionally, one can check that

$$\begin{aligned}
(\psi_z^1)^2 + (\psi_z^2)^2 + (\psi_z^3)^2 &= \frac{1}{4}|h(z)|^2(|1 - g(z)^2|^2 + |1 + g(z)^2|^2 + 4|g(z)|^2) \\
&= \frac{1}{2}|h(z)|^2(1 + |g(z)|^2)^2 \\
&\neq 0
\end{aligned}$$

using the identity

$$|1-z^2|^2+|1+z^2|^2+4|z|^2=2(1+|z|^2)^2.$$

Because all three conditions are satisfied, the surface described is indeed minimal. $\square$

Now that the proof is complete, let us try out some examples.

Example. Let $h(z)=-e^{-z}$ and $g(z)=-e^{z}$. We now compute $\varphi^1,\varphi^2,\varphi^3$:

$$\begin{aligned}
\varphi^1(u,v) &= -\frac{1}{2}\mathrm{Re}\int e^{-z}(1-e^{2z})dz \\
&= -\frac{1}{2}\mathrm{Re}(-e^{-z}-e^{z}) \\
&= -\frac{1}{2}\mathrm{Re}(-e^{-u}(\cos v - i\sin v) - e^{u}(\cos v + i\sin v)) \\
&= -\frac{-e^{-u}-e^{u}}{2}\cos v \\
&= \cosh u\cos v.
\end{aligned}$$

$$\begin{aligned}
\varphi^2(u,v) &= -\frac{1}{2}\mathrm{Re}\int ie^{-z}(1+e^{2z})dz \\
&= -\frac{1}{2}\mathrm{Re}(-ie^{-z}+ie^{z}) \\
&= -\frac{1}{2}\mathrm{Re}(-ie^{-u}(\cos v - i\sin v) + ie^{u}(\cos v + i\sin v)) \\
&= -\frac{-e^{-u}-e^{u}}{2}\sin v \\
&= \cosh u\sin v.
\end{aligned}$$

$$\varphi^3(u,v) = \mathrm{Re}\int(-e^{-z})(-e^{z})dz = \mathrm{Re}\int 1dz = \mathrm{Re}z = u.$$

This is the parametrization of the catenoid: $(\cosh u\cos v, \cosh u\sin v, u)$, which is indeed a minimal surface.

Example. If we take $-\mathrm{Im}$ instead of Re in the integrals from the previous example, we obtain the new parametrization $(-\sinh u\sin v, \sinh u\cos v, -v)$. Letting $u'=\sinh u$ and $v'=v+\pi/2$, this parametrization can be rewritten as

$$u'\cos v', u'\sin v', -v'+\frac{\pi}{2}.$$

Because a change of variables from the catenoid leads us to the helicoid, and this can be obtained by taking $-\mathrm{Im}$ instead of Re everywhere, the helicoid and catenoid are *conjugate minimal surfaces* (surfaces whose component functions are pairwise harmonic conjugates).

4. More Minimal Surfaces

In this section, we will be exploring more nontrivial minimal surfaces, in a chronological order. Let us first return to the Enneper surface (discovered in 1864), mentioned at the start of this paper. Recall its parametrization

$$\sigma(u,v)=\left(u-\frac{u^3}{3}+uv^2, v-\frac{v^3}{3}+vu^2, u^2-v^2\right).$$

A portion of the surface is shown in Figure 2. This is actually a remarkably simple surface when applied in the context of Theorem 3.4: the holomorphic functions are $h(z)=1$ and $g(z)=z$.

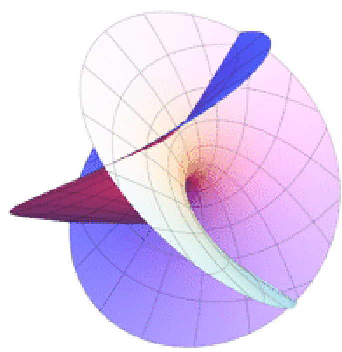

Figure 2. A portion of the Enneper surface. Note the self-intersections.

In the 1880s many triply periodic minimal surfaces were found by Schwarz. Some of them are included in Figure 3.

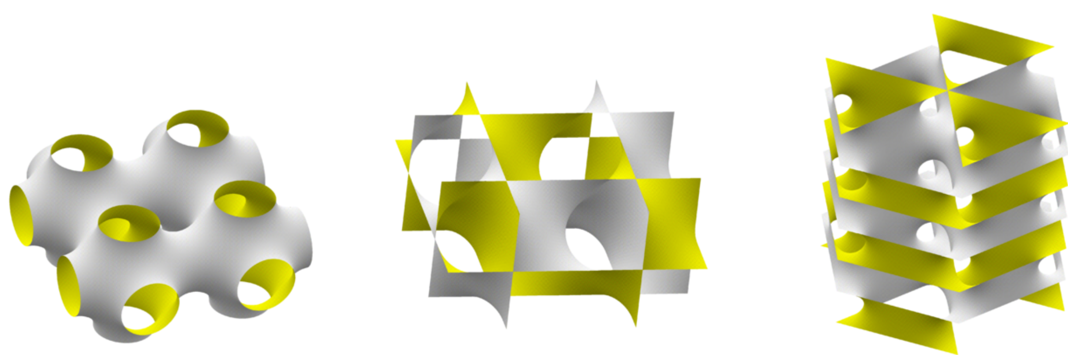

Figure 3. The Schwarz P (primitive), D (diamond) and H (hexagonal) minimal surfaces from left to right.

Finally, we look at *Costa's minimal surface* $C(u,v)$, parametrized by

$$C(u,v) = \left(\frac{1}{2}\mathrm{Re}-\zeta(u+iv) + \pi u + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}\left[\zeta\left(u+iv-\frac{1}{2}\right) - \zeta\left(u+iv-\frac{1}{2}i\right)\right], \right.$$
$$\frac{1}{2}\mathrm{Re}-i\zeta(u+iv) + \pi v + \frac{\pi^2}{4e_1} - \frac{\pi}{2e_1}\left[i\zeta\left(u+iv-\frac{1}{2}\right) - i\zeta\left(u+iv-\frac{1}{2}i\right)\right],$$
$$\left. \frac{1}{4}\sqrt{2\pi}\log\left|\frac{\wp(u+iv)-e_1}{\wp(u+iv)+e_1}\right| \right),$$

where $\zeta(z)$ is the Weierstrass zeta function, $\wp(g_2, g_3; z)$ is the Weierstrass elliptic function with $(g_2, g_3) = (189.07\ldots, 0)$, and $e_1 \approx 6.87519$. The visual representation for Costa's minimal surface can be found in Figure 4. Discovered in 1982 by Celso José da Costa, it disproved the conjecture that the catenoid, helicoid and plane were the only regular minimal surfaces without a boundary.

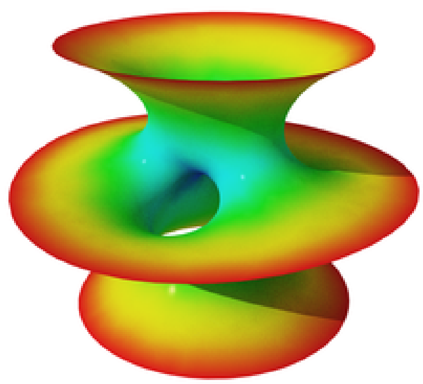

Figure 4. A portion of the Costa minimal surface.

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holomorphic last section