Calculus of Variations

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May 2022

1 Introduction

Calculus of variations is, fundamentally, the study of paths. Calculus of variations can be used to prove mundane facts, such as the shortest distance between two points on a plane is a line. However, it can also solve considerably more complex problems, such as the Brachistochrone problem. Let us start by introducing the basic form of calculus of variations problems.

2 General Form and Euler-Lagrange Differential Equation

We can apply calculus of variations when we wish to maximize or minimize an equation of the following:

$$J = \int f(x, y, y_x)dx$$

We can use the Euler-Lagrange Differential Equation to find stationary values for this equation, where a stationary value is basically a maximum or a minimum.

Theorem 1 *We have that J is stationary when:*

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_x}\right) = 0$$

We will not be proving the Euler-Lagrange Differential Equation, but we will use it to derive our next theorem: the Beltrami Identity.

Theorem 2 *If we have that $\frac{\partial f}{\partial x} = 0$, then:*

$$f - \left(\frac{\partial f}{\partial y_x}\right) = C$$

is true for stationary points of J.

We can prove it as follows:

$$\begin{aligned}
\frac{df}{dx} &= \frac{f(x+dx, y+dy, y_x+dy_x) - f(x,y,y_x)}{dx} \\
&= \frac{f(x+dx, y+y_x dx, y_x+y_{xx}dx) - f(x,y,y_x)}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y_x + \frac{\partial f}{\partial y_x} y_{xx} \\
\frac{\partial f}{\partial y} y_x &= \frac{df}{dx} - \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y_x} y_{xx}
\end{aligned}$$

Now, return to the Euler-Lagrange Differential:

$$\begin{aligned}
= \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_x}\right) &= 0 \\
\frac{\partial f}{\partial y} y_x - \frac{d}{dx}\left(\frac{\partial f}{\partial y_x}\right) y_x &= 0 \\
\left(\frac{df}{dx} - \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y_x} y_{xx}\right) - \frac{d}{dx}\left(\frac{\partial f}{\partial y_x}\right) y_x &= 0 \\
-\frac{\partial f}{\partial y_x}\frac{\partial y_x}{\partial x} + \frac{d}{dx}\left(f - y_x\left(\frac{\partial f}{\partial y_x}\right)\right) &= 0 \\
-\frac{\partial f}{\partial x} + \frac{d}{dx}\left(f - y_x\left(\frac{\partial f}{\partial y_x}\right)\right) &= 0
\end{aligned}$$

Sice $\frac{\partial f}{\partial x} = 0$:

$$\begin{aligned}
\frac{d}{dx}\left(f - y_x\left(\frac{\partial f}{\partial y_x}\right)\right) &= 0 \\
f - y_x\left(\frac{\partial f}{\partial y_x}\right) &= C
\end{aligned}$$

as desired.

We will investigate three problems in this paper: points on a plane, points on a sphere, and the Brachistochrone problem.

3 The Shortest Distance Between Two Points in a Plane

This is perhaps one of the most well-known facts in geometry: the shortest distance between two points on a plane is a line segment. However, we will make this formal using the Beltrami identity. We have:

$$\begin{aligned} J &= \int_{x_0}^{x_1} \sqrt{dx^2 + dy^2} \\ &= \int_{x_0}^{x_1} \sqrt{1 + y_x^2}dx \\ f(x, y, y_x) &= \sqrt{1 + y_x^2}dx \end{aligned}$$

Then, by the Beltrami identity:

$$\begin{aligned} f - y_x \left(\frac{\partial f}{\partial y_x}\right) &= C \\ &= \sqrt{1 + y_x^2} - y_x \left(\frac{d}{dy_x}\left(1 + y_x^2\right)\right)\left(\frac{1}{2\sqrt{1 + y_x^2}}\right) \\ &= \sqrt{1 + y_x^2} - \frac{y_x^2}{\sqrt{1 + y_x^2}} \\ \frac{1}{\sqrt{1 + y_x^2}} &= C \end{aligned}$$

Clearly, this implies that y_x must be constant, the very definition of a line. Therefore, the shortest distance between any two points in a plane is a line.

4 The Shortest Distance Between Two Points on a Sphere

Using differential geometry in past weeks, we have determined that the shortest distance between points on a sphere is along the minor arc of a great circle. Let us prove this using the calculus of variations. Consider the following parameterization of a sphere:

$$\boldsymbol{\sigma}(\theta, \phi) = (cos(\theta)cos(\phi), sin(\theta)cos(\phi), sin(\phi))$$

Then:

$$
\begin{aligned}
J &= \int \sqrt{d\phi^2 + (d\theta \cdot cos(\phi))^2} \\
&= \int \sqrt{\phi_\theta^2 + cos(\phi)^2} d\theta \\
f(\theta, \phi, \phi_\theta) &= \sqrt{\phi_\theta^2 + cos(\phi)^2} \\
f - \phi_\theta \left(\frac{\partial f}{\partial \phi_\theta} \right) &= C \\
\sqrt{\phi_\theta^2 + cos(\phi)^2} - \left(2\phi_\theta^2 \cdot \left(\frac{1}{2} \cdot \frac{1}{\sqrt{\phi_\theta^2 + cos(\phi)^2}} \right) \right) &= C \\
\frac{cos(\phi)^2}{\sqrt{\phi_\theta^2 + cos(\phi)^2}} &= C
\end{aligned}
$$

With a small extra amount of algebra we won't go over in this paper, you can show that this holds true for great circles on a sphere. Thus, as we've discovered before, great circles are geodesics/distance minimizing paths on spheres.

5 The Brachistochrone Problem

We will now move on to the most famous result due to the calculus of variations: the solution to the Brachistochrone Problem. You are given to points in a plane, say (x_0, y_0) and (x_1, y_1) with $y_1 < y_0$. You wish to find the curve connecting the two points such that the time it would take an object to slide from one to the other is minimized. First, we will introduce the following theorem from physics:

Theorem 3

$$v = \sqrt{2gy}$$

where g is the gravitational constant at Earth's surface, v is the velocity, and y is the height the object has descended from.

Then, we can write the following equation:

$$
\begin{aligned}
t &= \int_{x_0}^{x_1} \frac{ds}{v} \\
&= \int_{x_0}^{x_1} \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gy}} \\
&= \int_{x_0}^{x_1} \frac{\sqrt{1 + y_x^2}}{\sqrt{2gy}} dx \\
f(x, y, y_x) &= \frac{\sqrt{1 + y_x^2}}{\sqrt{2gy}}
\end{aligned}
$$

Since this doesn't depend on x, we can apply the Beltrami identity:

$$\begin{aligned} f - y_x \left(\frac{\partial f}{\partial y_x} \right) &= C \\ \frac{\sqrt{1 + y_x^2}}{\sqrt{2gy}} - 2y_x^2 \frac{1}{2\sqrt{2gy\left(1 + y_x^2\right)}} &= C \\ \frac{1}{\sqrt{2gy}} \left(\frac{1}{\sqrt{1 + y_x^2}} \right) &= C \\ y\left(1 + y_x^2\right) &= k^2 \end{aligned}$$

for some new constant k. Again, skipping the algebra, we find that this matches the parameterization of a cycloid:

$$\begin{aligned} x &= \frac{1}{2}k^2 \left(\theta - \sin\theta\right) \\ y &= \frac{1}{2}k^2 \left(1 - \cos\theta\right) \end{aligned}$$

Therefore, the time-minimizing path between two points is a cycloid.

6 Conclusion

Calculus of variations is a critical tool in differential geometry. It allows us to easily compute the form of geodesics/distance minimizing paths on surfaces, and allows us to solve more general problems relating to time as well.