AN INTRODUCTION TO CONTINUED FRACTION

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ABSTRACT. This paper gives a brief introduction for calculating continued fraction for both the finite and infinite cases through the use of several theorems. It also gives future steps which should involve the calculation of Mobius transformations, which are more detailed expansion of fractions, and represent positive integers **Index Terms.** finite continued fraction, infinite continued fraction,

1. INTRODUCTION

A finite continued fraction is given an expression

$$\frac{p}{q} = a_0 + \cfrac{e_1}{a_1 + \cfrac{e_2}{a_2 + \cfrac{e_3}{\ddots \cfrac{e_n}{a_n}}}}$$

If an algebra student attempts solving the quadratic equation

$$x^2 - 3x - 1 = 0$$

As follows: he starts by diving all through by x, and then writes the equation in the following form:

$$x = 3 + \frac{1}{x}$$

The element x is still in the equation and can hence be replaced by the equation equivalent to it, which is

$$x = 3 + \frac{1}{x} = 3 + \frac{1}{3 + \frac{1}{x}}$$

This result in an equation as follows

This replacement of x can be repeated by

$$3 + 1/x$$

Several times in order to obtain the expression that follows

$$x = 3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{x}}}}}$$

An expression of the form

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \ddots}}}$$

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is called a continued fraction. In general, the numbers all a_1,a_2, a_3, ... ,b_1, b_2, b_3, ... may be any real or complex numbers, and the number of terms may be finite or infinite

For finite continued fractions, they are in the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

If the expression only has finite number of terms $a_1, \cdot a_2, a_3, \ldots, a_n$

In this case, the lower + signs shows that the step-down process forms a continued fraction. A continued fraction can also be conveniently denoted by

$$[a_1, a_2, \cdots, a_n]$$

In this case, the terms $a_1, a_2, \cdots, a_n$

Are known as partial quotients of the continued fraction

1.1. **Expansion of the Rational Fractions.** A rational number is any given fraction in the form p/q in which p and q are integers and $q \neq 0$

It is possible to prove that every rational fraction, or any rational number, can be expressed as a simple continued fraction For instance, given the fraction 67/29, its continued fraction can be expanded to give

$$\frac{67}{29} = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2}}}$$

or

$$\frac{67}{29} = [2, 3, 4, 2]$$

The result is obtained by first dividing 67 by 29, which gives the quotient 2 and the remainder as 9, so that

$$\frac{67}{29} = 2 + \frac{9}{29} = 2 + \frac{1}{\frac{29}{9}}$$

Since on the right side, we have replaced 9/29 by its reciprocal 29/9, the next step involves diving 29 by 9 in order to obtain

$$\frac{29}{9} = 3 + \frac{2}{9} = 3 + \frac{1}{\frac{9}{2}}$$

The final step involves diving 9 by 2 in order to obtain

$$\frac{9}{2} = 4 + \frac{1}{2}$$

This marks the end of the process. When this is substituted in the original expression, we obtain

$$\frac{67}{29} = 2 + \cfrac{1}{\cfrac{29}{9}} = 2 + \cfrac{1}{3 + \frac{1}{\frac{9}{2}}} = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2}}}$$

Or

$$\frac{67}{29} = [2, 3, 4, 2] = [a_1, a_2, a_3, a_4]$$

2. Theorems of Continued fractions

Theorem 2.1. *Every rational number p/q determines a unique finite regular continued fraction.*

Proof. Given p/q, Euclid's algorithm determines $[a_0, a_1, a_2, \dots a_n]$.Note that for $t = [0, a_1, a_2, \dots a_n]$ holds: $0 \leq t < 1$ (with a strict inequality on the right because an > 1), and thus $a_0 \leq [a_0, a_1, a_2, \dots a_n] < a_{0+1}$. Now suppose that also $p/q = [b_0, b_1, \dots b_k]$ for another continued fraction. Then $a_0 = p/q = b_0$ since p/q is the unique integer satisfying p/q ≤ p/q < [p/q] + 1. ∎

Consider $\frac{1}{[a_1;a_2,\dots,a_n]} = [0; a_1, a_2, \dots, a_n] = \frac{p}{q} - \lfloor p/q \rfloor = [0; b_1, \dots b_k] = \frac{1}{[b_1;b_2,\dots,b_k]}$ then $[a_1; a_2, \dots, a_n] = [b_1; b_2, \dots, b_k]$ so $a_1 = b_1$ as before; etc. then $[a_1; a_2, \dots a_n] = [b_1; b_2, \dots b_k]$ so $a_1 = b_1$ as before; etc. it is possible to consider the regular expansion of p/q without first finding the Euclid's algorithm as follows: determine the integral part of a0, after which, subtract it from the given fraction, take the reciprocal of the given result, then repeat this means that $x_{k+1} = \frac{1}{x_k - \lfloor x_k \rfloor}$ with $x_0 = p/q$ until $x_k - \lfloor x_k \rfloor$ becomes 0. Put $a_k = \lfloor x_k \rfloor$

This results in the convergent as follows By definition $p_0/q_0 = \lfloor a_0; \rfloor = a_0/1$

Then

$$\frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1}$$

and

$$\frac{p_2}{q_2} = a_0 + \frac{1}{a_1 + \frac{1}{a_2}} = a_0 + \frac{a_2}{a_2 a_1 + 1} = \frac{a_2 a_1 a_0 + a_2 + a_0}{a_2 a_1 + 1}$$

Similarly,

$$\frac{p_3}{q_3} = \frac{\left(a_2 + \frac{1}{a_3}\right) a_1 a_0 + a_2 + \frac{1}{a_3} + a_0}{\left(a_2 + \frac{1}{a_3}\right) a_1 + 1} = \frac{a_3 \left(a_2 a_1 a_0 + a_2 + a_0\right) + a_1 a_0 + 1}{a_3 \left(a_2 a_1 + 1\right) + a_1}$$

Theorem 2.2. *The convergents pk/qk of a rational number p/q satisfy* $|p_k| \geq |p_{k-1}|$ *and* $q_k \geq q_{k-1}$ *for* $k \geq 1$*, and even:*

$$|p_k| > |p_{k-1}|, \quad (k \geq 3), \text{ and } \quad q_k > q_{k-1}, \quad (k \geq 2)$$

while

$$p_{k-1} q_k - p_k q_{k-1} = (-1)^k$$

Proof.

$$\frac{p_{k-1} q_k - p_k q_{k-1}}{q_{k-1} q_k} = \frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{p_{k-1}}{q_{k-1}} - \frac{a_k p_{k-1} + p_{k-1}}{a_k q_{k-1} + q_{k-1}}$$
$$= \frac{(-1)\left(p_{k-2} q_{k-1} - p_{k-1} q_{k-2}\right)}{q_{k-1}\left(a_k q_{k-1} + q_{k-2}\right)}$$

which by induction equals

$$\frac{(-1)^k \left(p_{-1} q_0 - p_0 q_{-1}\right)}{q_{k-1}\left(a_k q_{k-1} + q_{k-2}\right)} = \frac{(-1)^k}{q_{k-1}\left(a_k q_{k-1} + q_{k-2}\right)}$$

∎

Theorem 2.3. *The convergents pk/qk of a rational number p/q satisfy for k≥0:*

$$\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_{k-1} q_k}$$

and

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \cdots < \frac{p}{q} = \frac{p_n}{q_n} < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1} < \frac{p_{-1}}{q_{-1}}$$

also

$$\left|\frac{p}{q}-\frac{p_k}{q_k}\right|<\left|\frac{p}{q}-\frac{p_{k-1}}{q_{k-1}}\right|$$

for $0 \leq k \leq n$.

Proof. There is a strict increase in the sequence q_i, thus giving the difference between the two consecutive convergents which decrease in the process. This therefore gives the proof for the second part. The final statement is proved by denoting the following

$$\frac{ap_{k-1}+p_{k-2}}{aq_{k-1}+q_{k-2}} \leq \frac{bp_{k-1}+p_{k-2}}{bq_{k-1}+q_{k-2}} \Longleftrightarrow$$
$$0 \leq (a-b)\left(p_{k-2}q_{k-1}-p_{k-1}q_{k-2}\right)=(a-b)(-1)^{k-1}$$
$$k \text{ odd and } b \leq a, \text{ or } k \text{ even and } a \leq b$$

For odd $k<n$ it holds that

$$\frac{p_{k-1}}{q_{k-1}}<\frac{p_{k+1}}{q_{k+1}}<\frac{p}{q}<\frac{p_k}{q_k}$$

and applying the above with $b=a_{k+1} \geq 1$ and a $=1$ we get

$$\frac{p}{q}>\frac{p_{k+1}}{q_{k+1}}=\frac{a_{k+1}p_k+p_{k-1}}{a_{k+1}q_k+q_{k-1}} \geq \frac{p_k+p_{k-1}}{q_k+q_{k-1}}$$

but then

$$\begin{aligned}\frac{p}{q}-\frac{p_{k-1}}{q_{k-1}}>\frac{p_{k+1}}{q_{k+1}}-\frac{p_{k-1}}{q_{k-1}} \geq \frac{p_k+p_{k-1}}{q_k+q_{k-1}}-\frac{p_{k-1}}{q_{k-1}}=\frac{-\left(p_{k-1}q_k-p_kq_{k-1}\right)}{q_{k-1}\left(q_k+q_{k+1}\right)} \\ =\frac{1}{q_{k-1}\left(q_k+q_{k+1}\right)}>\frac{1}{q_k\left(q_k+q_{k+1}\right)}=\frac{-\left(p_{k-1}q_k-p_kq_{k-1}\right)}{q_k\left(q_k+q_{k+1}\right)}= \\ =\frac{p_k}{q_k}-\frac{p_k+p_{k-1}}{q_k+q_{k-1}} \geq \frac{p_k}{q_k}-\frac{a_{k+1}p_k+p_{k-1}}{a_{k+1}q_k+q_{k-1}}=\frac{p_k}{q_k}-\frac{p_{k+1}}{q_{k+1}} \\ >\frac{p_k}{q_k}-\frac{p}{q}\end{aligned}$$

■

This is also similar with the even case

3. Infinite real continued fractions

Taking arbitrary real numbers x for continued fractions, we note that for $x_0=x \in \mathbb{R}$ We obtain

$$x_0=a_0+\frac{1}{x_1}$$
$$x_1=a_1+\frac{1}{x_2}$$

Theorem 3.1. *The convergents pk/qk of any irrational number x satisfy:*

$$\frac{1}{2q_kq_{k+1}}<\frac{1}{q_k\left(q_k+q_{k+1}\right)}<\left|x-\frac{p_k}{q_k}\right|<\frac{1}{q_kq_{k+1}}<\frac{1}{q_k^2}$$

for $k \geq 1$.

Proof.

$$\left|x-\frac{p_k}{q_k}\right|=\left|\frac{x_{k+1}p_k+p_{k-1}}{x_{k+1}q_k+q_{k-1}}-\frac{p_k}{q_k}\right|=\left|\frac{(-1)^k}{q_k\left(q_kx_{k+1}+q_{k-1}\right)}\right|$$

since $a_{k+1} < x_{k+1} < a_{k+1} + 1$ is $q_{k+1} < q_k x_{k+1} + q_{k-1} < q_{k+1} + q_k$

$$\frac{1}{q_k\left(q_k + q_{k+1}\right)} < \left|x - \frac{p_k}{q_k}\right| < \frac{1}{q_k q_{k+1}}$$

The other inequalities follow from $q_k < q_{k+1}$ ■

Theorem 3.2. *Voor twee opeenvolgende convergenten $p_{k-1}/q)k-1$; p_k/q_k van een irrationaal getal x geldt:*

$$\left|x - \frac{p_{k-1}}{q_{k-1}}\right| < \frac{1}{2q_{k-1}^2} \quad \textit{of} \quad \left|x - \frac{p_k}{q_k}\right| < \frac{1}{2q_k^2}$$

Proof. It would follow from

$$\left|\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}}\right| = \left|x - \frac{p_k}{q_k}\right| + \left|x - \frac{p_{k-1}}{q_{k-1}}\right|$$

and the assumption that the statement is false, that

$$\frac{1}{q_{k-1}q_k} = \left|\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}}\right| \geq \frac{1}{2q_k^2} + \frac{1}{2q_{k-1}^2}$$

which is equivalent to

$$\left(q_k - q_{k-1}\right)^2 \leq 0$$

this is a contradiction as $q_k > q_{k-1}$ for $k \geq 2$ ■

Theorem 3.3. *If a fraction p/q satisfies $0 < q \leq qk$ for some convergent p_k/q_k of x then (Dajani & Kraaikamp, 2002)*

$$\left|\frac{p}{q} - \frac{p_k}{q_k}\right| > \frac{1}{q_k}$$

but

$$\left|x - \frac{p_k}{q_k}\right| < \frac{1}{2q_k}$$

so

$$\left|x - \frac{p_k}{q_k}\right| < \left|x - \frac{p}{q}\right|$$

Suppose that $q_{k-1} < q < q_k$; let integers e , f be defined by

$$e = \left(qp_{k-1} - pq_{k-1}\right), \quad f = \left(pq_k - qp_k\right)$$

then $f \neq 0$ and

$$ep_k + fp_{k-1} = p\left(p_{k-1}q_k - p_k q_{k-1}\right) = \pm p$$
$$eq_k + fq_{k-1} = q\left(p_{k-1}q_k - p_k q_{k-1}\right) = \pm q$$

Theorem 3.4. *If p/q satisfies*

$$\left|x - \frac{p}{q}\right| < \frac{1}{2q^2}$$

then

$$\frac{p}{q} = \frac{p_k}{q_k}$$

for some convergent p_k/q_k of x (Austin, 2013)

Proof. Expand p/q in a finite continued fraction of odd length n; then $P/q = p_n/q_n$ and

$$\frac{p_n}{q_n} - x = \frac{\delta}{q_n^2}, \quad \delta < \frac{1}{2}$$

There exists y ¿ 0 such that

$$x = \frac{yp_n + p_{n-1}}{yq_n + q_{n-1}}$$

and then

$$\frac{\delta}{q_n^2} = \frac{p_n}{q_n} - x = \frac{p_n q_{n-1} - p_{n-1} q_n}{q_n (yq_n + q_{n-1}} = \frac{(-1)^{n+1}}{q_n (yq_n + q_{n-1}})$$

so

$$\delta = \frac{q_n}{yq_n + q_{n-1}}$$

implying

$$y = \frac{1}{\delta} - \frac{q_{n-1}}{q_n} > 1$$

■

Theorem 3.5. *If*

$$x = \frac{py + r}{qy + s}$$

With $y \in \mathbb{R}$ and $p, q, r, s \in \mathbb{Z}$ such that $y > 1, \quad q > s > 0, \quad ps - qr = \pm 1$ (Allouche & Jean-Paul, 2003)

Then there exists $n \geq 0$ with

$$y = x_{n+1}, \quad \frac{p}{q} = \frac{p_n}{q_n}, \quad \frac{r}{s} = \frac{p_{n-1}}{q_{n-1}}$$

Where $x = [a_0; a_1, \ldots], x_i = [a_i; a_{i+1}, \ldots]$ and $p_i/q_i = [a_0; a_1, \ldots, a_i]$
For $i \geq 0$

Proof. Expand p/q in a continued fraction so that $p/q = [A_0; A_1, \ldots, A_n]$

This is also equivalent to v_n/w_n, and let $v_{n-1}/w_{n-1} = [A_0; A_1, \ldots, A_{n-1}]$

In this way, the continued fraction has been chosen, so that $(-1)^{n+1} = v_n w_{n-1} - v_{n-1} w_n = ps - qr = \pm 1$

This then means that $v_n w_{n-1} - v_{n-1} w_n = v_n s - v_n r$

Hence $v_n (w_{n-1} - s) = w_n (v_{n-1} - r)$

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4. Conclusion and future steps

This paper has given a method for calculating continued fraction for both the finite and infinite cases through the use of several theorems. Future steps however should involve the calculation of Mobius transformations, which are more detailed expansion of fractions, and represent positive integers. Part of the application of these theorems is on the (Gear ratios). In this case, Christian Huygens utilized proceeded with part convergents in his development of a planetarium, a model of the solar based framework as it was known at the time. Utilizing a solitary drive shaft and apparatuses with quantities of teeth in painstakingly picked proportions, every realized planet ought to rotate with sensible precision around the sun in this model. The proportions would compare to apportion between the length of the year on every planet and that on earth. To have the option to make a physical model with actual gears, the number of teeth could be neither very huge nor excessively little. Huygens found, for instance, for the deepest planet, Mercurius, a proportion of 25335=105190. Its proceeded with division is [0; 4; 6; 1; 1; 2; 1; 1; 1; 1; 7; 1; 2] and at first Huygens utilized the fifth concurrent [0; 4; 6; 1; 1; 2] = 33 137. Afterward, he understood that, in spite of the fact that utilizing the ninth merged would require such a large number of teeth: [0; 4; 6; 1; 1;

2; 1; 1; 1; 1] = 204 847, yet since 204 = 12 17 en 847 = 7 121 this estimate can be utilized and gives a superior outcome, when utilizing 4 gears with 12; 17; 7, and 121 teeth, two of these fixed to a similar shaft.

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