# Almost-integers $e^{\pi\sqrt{n}}$ and the $j-$function.

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**Abstract**

In this paper, we start from the theory of elliptic functions and show how it gives rise to a natural construction of the $j-$function. After this, we show how $j$ being an integer can explain the observed closeness of $e^{\pi\sqrt{n}}$, for certain values of $n$, to integers. Then we turn briefly to binary quadratic form theory to help state a theorem about algebraic integer values of $j$, as well as the celebrated Baker-Stark-Heegner Theorem, in binary quadratic form terminology. The latter theorem helps us to know arguments $\tau$ for which $j(\tau)$ is an actual integer.

## 1 Introduction

Is it a lucky coincidence that

$$e^{\pi\sqrt{163}} = 262,537,412,640,768,743.999,999,999,999,250\dots$$

is extremely close to an integer, or is something deep going on behind the scenes? The phenomenon, though rare, is not isolated:

$$e^{\pi\sqrt{43}} = 884,736,743.999,777,\dots,$$
$$e^{\pi\sqrt{67}} = 147,197,952,743.999,998,\dots.$$

The explanations (section 3) for these observations are highly non-trivial and rely on quite deep theorems (theorems 3.1 and 4.3) concerning the $j-$function (definition 2.2):

$$j(\tau) = q^{-1} + 744 + 196,884q + 21,493,760q^2 + \dots,$$

a function which arises naturally from homogeneity properties of certain series called *Eisenstein series*, which in turn come naturally from a consideration of elliptic functions.

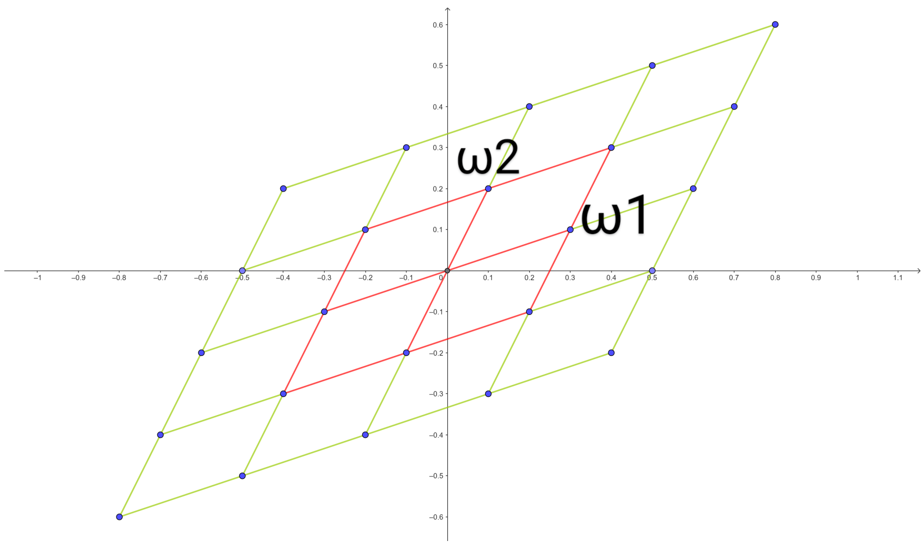
The algebraic integer values of $j$ are intimately tied to binary quadratic forms (theorem 4.3), and the Baker-Stark-Heegner theorem (theorem 4.4) tells us that our method of section 3 will not be able to find an integer $n > 163$ for which $e^{\pi\sqrt{n}}$ is closer to an integer than $e^{\pi\sqrt{163}}$.

# 2 The $j-$function naturally arising from elliptic function theory.

The $j-$function (definition 2.2) is naturally defined as a certain quotient involving Eisenstein series. The latter occur as the coefficients of the Laurent series expansion of the Weierstrass elliptic function $\wp$ about the origin. Accordingly, we begin with a review of elliptic function theory in 2.1. The reader desiring proofs of the statements made there is referred to the excellent texts [3], [11] and [10].
The next important step is to obtain a Fourier series expansion for $j$. In Section 2.2, where we define the Eisenstein series order 2 (definition 2.4 and theorem 2.2), we prove an essential result on the invariance of lattices under $SL_2$ which can be used to establish the periodicity of $j$, after which the Fourier series expansion with integer coefficients can be deduced as in [3]. But we derive the expansion of $j$ in an alternative way in 2.3 where we simultaneously derive an extremely important product development (theorem 2.4) for the discriminant function (definition 2.1).

## 2.1 Elliptic function theory and the $j-$function.

Here we define the Weierstrass $\wp-$function, the Eisenstein series, the discriminant function (definition 2.1), and show how they naturally give rise to the $j-$function (definition 2.2).
Given any non-degenerate parallelogram in the complex plane $\mathbb{C}$, say with vertices at $0, \omega_1, \omega_2$ and $\omega_1 + \omega_2$, where $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$ (this last condition is equivalent to saying that the parallelogram does not degenerate into a line), there is a very simple way to construct a non-trivial function with two fundamental periods (fundamental in the sense that there are no periods of modulus smaller than that of $\omega_1$ or $\omega_2$) given by $\omega_1$ and $\omega_2$. (Note that in this case a complex number is a period, not necessarily fundamental, if and only if it is of the form $m\omega_1 + n\omega_2$, where $m$ and $n$ are integers. These periods are said to form a *lattice* $\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$. The periods themselves are the *lattice points*).




In fact, the function

$$\frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda\\ \omega\neq 0}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\},$$

fulfills this requirement. This absolutely convergent series which represents a meromorphic function with poles (double poles) exactly at the lattice points, is called the Weierstrass $\wp$−function so that

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda\\ \omega\neq 0}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\},$$

with respect to the lattice $\Lambda$. The $\wp$−function is an example of an *elliptic function*, which means a meromorphic and doubly periodic function; *doubly periodic*, in turn, means that the function has two periods with non-real ratio, the latter condition avoiding uninteresting cases.

All elliptic functions have the same number of zeros as they have of poles in a fundamental parallelogram, if we count these with multiplicity; this means that the elliptic function $\wp'(z)$, which has a triple pole exactly at each lattice point of $\Lambda$, has three zeros in each fundamental parallelogram. In fact, these zeros are all simple and are located at the half-periods, $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$ and $\frac{\omega_1+\omega_2}{2}$, of $\wp$. It can be shown, by means of Liouville's Theorem in complex analysis, that the Weierstrass $\wp$−function satisfies the cubic first-order differential equation:

$$(\wp'(z))^2 = 4\left(\wp(z) - \wp\left(\frac{\omega_1}{2}\right)\right)\left(\wp(z) - \wp\left(\frac{\omega_2}{2}\right)\right)\left(\wp(z) - \wp\left(\frac{\omega_1+\omega_2}{2}\right)\right).$$

We can expand the Weierstrass $\wp-$function as a Laurent series centred at the origin, and then we obtain the expression

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}\left(\sum_{\omega\in\Lambda^*}\frac{2n+1}{\omega^{2n+2}}\right)z^{2n},$$

where $\Lambda^* = \Lambda\backslash\{0\}$. The series

$$E_{2n}(\omega_1,\omega_2) := \sum_{\omega\in\Lambda^*}\frac{1}{\omega^{2n}}, n \geq 2$$

is called an Eisenstein series with respect to the lattice $\Lambda$. We require that $n \geq 2$ because the series $\sum_{\omega\in\Lambda^*}\frac{1}{\omega^2}$ is not absolutely convergent. Thus the Laurent series expansion of the Weierstrass $\wp-$function centred at the origin is expressible in terms of Eisenstein series as

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(2n+1)E_{2n+2}(\omega_1,\omega_2)z^{2n}.$$

The Eisenstein series $E_{2n}(\omega_1,\omega_2)$ is homogeneous of degree $-2n$ because, for $\lambda \neq 0$, we have

$$E_{2n}(\lambda\omega_1,\lambda\omega_2) = \lambda^{-2n}\sum_{\omega\in\Lambda^*}\frac{1}{\omega^{2n}}.$$

The differential equation for $\wp(z)$ in terms of Eisenstein series becomes:

$$(\wp'(z))^2 = 4(\wp(z))^3 - 60E_4(\omega_1,\omega_2)\wp(z) - 140E_6(\omega_1,\omega_2).$$

But we have shown above that $\wp'(z)$ has only three simple zeros; this means that the roots of the cubic equation in $\wp(z)$,

$$4(\wp(z))^3 - 60E_4(\omega_1,\omega_2)\wp(z) - 140E_6(\omega_1,\omega_2) = 0,$$

are all distinct. In turn, this means that the discriminant of the cubic polynomial in question is always nonzero, no matter the choice of the lattice $\Lambda$ (as long as the ratio of the fundamental periods is non-real), so that

$$(60E_4(\omega_1,\omega_2))^3 - 27(140E_6(\omega_1,\omega_2))^2 \neq 0.$$

We see that the discriminant $(60E_4(\omega_1,\omega_2))^3 - 27(140E_6(\omega_1,\omega_2))^2$, which for brevity, we can denote by $\Delta(\omega_1,\omega_2)$, is homogeneous of degree 12. This means that the function:

$$\frac{E_4^3(\omega_1,\omega_2)}{\Delta(\omega_1,\omega_2)}$$

is homogeneous of degree zero.

**Definition 2.1** (discriminant function $\Delta$)**.**

$$\Delta(\tau) := (60E_4(\tau))^3 - 27(140E_6(\tau))^2.$$

We may therefore scale our lattice by the factor $\frac{1}{\omega_1}$ and, if we denote $\frac{\omega_2}{\omega_1} := \tau$, can consider without loss of generality the function of the single complex variable

$$\frac{E_4^3(1,\tau)}{\Delta(1,\tau)}$$

which we can also write as

$$\frac{E_4^3(\tau)}{\Delta(\tau)}.$$

The last function, when viewed as a function of $\tau$ in the open upper half-plane, i.e. when $\Im(\tau) > 0$, is denoted by

$$\frac{1}{720^3} j(\tau),$$

where $j(t)$ is called the $j-$function i.e.

**Definition 2.2** (The $j-$function)**.**

$$j(\tau) := 720^3 \frac{E_4^3(\tau)}{\Delta(\tau)} = \frac{1728(60E_4(\tau))^3}{\Delta(\tau)} = \frac{1728(60E_4(\tau))^3}{\{(60E_4(\tau))^3 - 27(140E_6(\tau)^2\}}, \Im\tau > 0.$$

Sometimes, we also write

**Definition 2.3.**

$$J(\tau) := \frac{1}{1728} j(\tau) = \frac{(60E_4)^3}{\Delta(\tau)}.$$

## 2.2 Lattice Invariance and $j$'s invariance.

Theorem 2.1 proved here can be used to establish periodicity of $j$, after which a Fourier expansion with integer coefficients can be derived for $j$. For this method of derivation, the reader can consult [3].
With any lattice $\Lambda$ with fundamental periods $\omega_1$ and $\omega_2$, we can perform any transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c$ and $d$ are integers such that $ad - bc = \pm 1$, on the periods, and the effect of this transformation on the lattice $\Lambda$ is to leave it completely unaffected except for the fact that the lattice points are permuted among themselves. We can enunciate the following theorem:

**Theorem 2.1.** *Given two complex numbers $\omega_1$ and $\omega_2$, such that $\frac{\omega_2}{\omega_1} \notin \mathbb{R}$, generating the lattice $\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$, if we apply the transformation*

$$\begin{pmatrix} w_2' \\ w_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_2 \\ w_1 \end{pmatrix},$$

*where $a, b, c, d \in \mathbb{Z}$, then the lattice $\Lambda' := \{m\omega_1' + n\omega_2' : m, n \in \mathbb{Z}\}$ is equal to $\Lambda$.*

*Proof.* Since

$$w_2' = aw_2 + bw_1$$
$$w_1' = cw_2 + dw_1$$

we therefore have, for any $m, n \in \mathbb{Z}$, that $mw_2' + nw_1' = (am + nc)w_2 + (mb + nd)w_1$ so that $\Lambda' \subseteq \Lambda$. Now since $ad - bc = \pm 1$ and in particular $\neq 0$, therefore the given matrix is invertible and we obtain

$$\begin{pmatrix} w_2 \\ w_1 \end{pmatrix} = \pm \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} w_2' \\ w_1' \end{pmatrix}$$

from which we similarly deduce that $\Lambda \subseteq \Lambda'$. It follows that $\Lambda = \Lambda'$. □

This means that, under such transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c$ and $d$ are integers such that $ad - bc = \pm 1$, of the fundamental periods $\omega_1$ and $\omega_2$, the (absolutely convergent) Eisenstein series $E_{2n}(\omega_1, \omega_2), n \geq 2$ are unaffected. In particular, since we have already shown that the function

$$\frac{E_4^3(\tau)}{\Delta(\tau)}$$

is homogeneous of degree 0, it is unchanged if we replace $\tau$ by $\frac{a\tau+b}{c\tau+d}$. We note that if $\Im(\tau) > 0$, then $\Im\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{ad-bc}{|c\tau+d|^2}\Im(\tau)$. So that if it is the case that $ad - bc = 1$, then $\Im\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{1}{|c\tau+d|^2}\Im(\tau) > 0$.
Now when $\Im(\tau) > 0$ and $q = e^{2\pi i\tau}$, the Eisenstein series have a particularly nice representation in terms of the sums-of-powers-of-divisors function $\sigma_k(n) = \sum_{d|n} n^k$:

$$E_{2k}(\tau) = 2\zeta(2k) + 2\frac{(-1)^k(2\pi)^{2k}}{(2k-1)!}\sum_{r=1}^{\infty}\sigma_{2k-1}(r)q^r, k \geq 2,$$

where $\zeta$ is the Riemann zeta function which, on $\Re(s) > 1$ is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}.$$

Incidentally, the expression on the right converges even when $k = 2$ and it is thus that $E_2(\tau)$ can be defined. We have therefore

**Definition 2.4** (Eisenstein series of order 2)**.**

$$E_2(\tau) := 2\zeta(2) - 8\pi^2\sum_{r=1}^{\infty}\sigma(r)q^r.$$

Now we recall that $\zeta(2k)$, when $k \in \mathbb{Z}_{>0}$, has been evaluated in closed form by Euler and $\zeta(2k)$ turned out to be a rational number times $\pi^{2k}$. These rational numbers are called Bernoulli numbers. Specifically, the Bernoulli numbers are defined by the coefficients $B_{2n}$ in the power series expansion

$$\frac{x}{2}\cot\frac{x}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n B_{2n}}{(2n)!} x^{2n}.$$

We note that $B_1 := -\frac{1}{2}$ and $B_{2n+1} = 0$ for $n \geq 1$, by definition. Euler has proved, by using the Taylor series and Hadamard product expansions for the sine function, that

$$\zeta(2k) = \frac{(-1)^{k-1} 2^{2k-1} B_{2k}}{(2k)!} \pi^{2k}, k \geq 1.$$

That the Bernoulli numbers are all rational can be seen in by employing the Taylor series expansion for the exponential function in the generating function

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n z^n}{n!}, |z| < 2\pi,$$

performing cross-multiplication and equating coefficients on both sides to recursively deduce that $B_n \in \mathbb{Q}$ for all $n$. The first few Bernoulli numbers are $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, \ldots$. The Bernoulli numbers make the aforementioned expressions for the Eisenstein series particularly nice: we have

$$\begin{aligned} E_{2k}(\tau) &= 2\zeta(2k) + 2\frac{(-1)^k (2\pi)^{2k}}{(2k-1)!} \sum_{r=1}^{\infty} \sigma_{2k-1}(r) q^r \\ &= \frac{(2\pi)^{2k}(-1)^{k-1} B_{2k}}{(2k)!} \left\{ 1 - \frac{4k}{B_{2k}} \sum_{r=1}^{\infty} \sigma_{2k-1}(r) q^r \right\}, k \geq 1, \Im\tau > 0, q^{2\pi i \tau}. \end{aligned}$$

It will be useful shortly to explicitly state the expressions for the first three Eisenstein series(these are the most important of all, because all other Eisenstein series (of higher order than 6) can be expressed as polynomial in $E_4(\tau)$ and $E_6(\tau)$ with rational coefficients. Thus

**Theorem 2.2.**

$$\begin{aligned} E_2(\tau) &= \frac{\pi^2}{3} \left\{ 1 - 24 \sum_{r=1}^{\infty} \sigma(r) q^r \right\}, \\ E_4(\tau) &= \frac{\pi^4}{45} \left\{ 1 + 240 \sum_{r=1}^{\infty} \sigma_3(r) q^r \right\}, \\ E_6(\tau) &= \frac{2\pi^6}{945} \left\{ 1 - 504 \sum_{r=1}^{\infty} \sigma_5(r) q^r \right\}. \end{aligned}$$

## 2.3 Expansion of $j-$function in powers of $q$.

It is possible to use the invariance of $j(\tau)$ under $SL_2(\mathbb{Z})$, i.e. under transformations of the type

$$\tau \to \frac{a\tau + b}{c\tau + d}, a, b, c, d \in \mathbb{Z}, ad - bc = 1,$$

as well as the Eisenstein series expansions in terms of $\sigma_k$ to derive a Fourier expansion for $\Delta(\tau)$; after which it is possible to demonstrate the remarkable result that

$$\frac{1}{1728}J(\tau) = j(\tau) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c(n)q^n,$$

where $c(n)$ are all positive integers. For this method of proof, the interested reader is referred to pp. 17-21 of [3]. However, this result can also be demonstrated by first showing that

$$\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

For then

$$\begin{aligned} 1728J(\tau) = j(\tau) &= \frac{1728(60E_4(\tau))^3}{\Delta(\tau)} \\ &= \frac{60^3}{4096} \frac{1728\left(1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n\right)^3}{q\pi^{12}\prod_{n=1}^{\infty}(1 - q^n)^{24}} \\ &= \frac{\left(1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n\right)^3}{q\prod_{n=1}^{\infty}(1 - q^n)^{24}}. \end{aligned}$$

and from this representation it is immediately clear that

$$j(\tau) = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n)q^n,$$

where $c(n)$ is a positive integer because of the expansions

$$\prod_{n=1}^{\infty} (1 - q^n)^{-1} = \prod_{n=1}^{\infty} \left(1 + q^n + q^{2n} + \dots\right),$$

giving

$$j(\tau) = q^{-1}\left(1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n\right)^3 \cdot \prod_{n=1}^{\infty} (1 + q^n + q^{2n} + \dots).$$

We must therefore prove that $\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty}(1 - q^n)^{24}$. There are several proofs of this fact. One of them depends on a well-known transformation formula of the Dedekind $\eta-$function of which Siegel has given a short proof in [8]. This is also the subject of Chapter 3 in [3] where, in Theorem 3.3, Apostol proves that $\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty}(1-q^n)^{24}$, using a simple but non-elementary method. However, it is possible to prove that $\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty}(1-q^n)^{24}$ using only elementary methods of Ramanujan. The following lemma is proved in [4] using elementary methods.

**Lemma 2.3.**

$$240 \sum_{r=1}^{\infty} r\sigma_3(r)q^r = \frac{\{1-24\sum_{k=1}^{\infty}\sigma(r)q^r\}\{1+240\sum_{k=1}^{\infty}\sigma_3(r)q^r\} - \{1-504\sum_{r=1}^{\infty}\sigma_5(r)q^r\}}{3}$$

$$-504 \sum_{r=1}^{\infty} r\sigma_5(r)q^r = \frac{\{1-24\sum_{k=1}^{\infty}\sigma(r)q^r\}\{1-504\sum_{r=1}^{\infty}\sigma_5(r)q^r\} - \{1+240\sum_{r=1}^{\infty}\sigma_3(r)q^r\}^2}{2}$$

*Proof.* The proof is given in Theorem 4.2.3 in [4]. □

Now we can prove the following theorem, which is Theorem 4.2.4 in [4]:

**Theorem 2.4** (Product development of $\Delta$)**.**

$$\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty}(1-q^n)^{24}, q = e^{2\pi i\tau}, \Im(\tau) > 0.$$

*Proof.* Firstly, let $\Im(\tau) \geq t > 0$. Then, for $n \in \mathbb{Z}_{>0}$, if we take the principal branch of the logarithm,

$$\begin{aligned}
|\log(1-q^n)| &= \left| -q^n - \frac{q^{2n}}{2} - \frac{q^{3n}}{3} - \frac{q^{4n}}{4} - \dots \right| \\
&\leq |q|^n + \frac{|q|^{2n}}{2} + \frac{|q|^{3n}}{3} + \frac{|q|^4}{4} + \dots \\
&\leq e^{-2\pi nt} + \frac{e^{-4\pi nt}}{2} + \frac{e^{-6\pi nt}}{3} + \frac{e^{-8\pi nt}}{4} + \dots \\
&< e^{-2\pi nt} + e^{-4\pi nt} + e^{-6\pi nt} + e^{-8\pi nt} + \dots \\
&= \frac{e^{-2\pi nt}}{1-e^{-2\pi nt}} \\
&= \frac{1}{e^{2\pi nt}-1}.
\end{aligned}$$

But now $\sum_{n=1}^{\infty} \frac{1}{e^{2\pi nt}} < \infty$ as this is a convergent geometric series; so that, by the limit comparison test, for example, $\sum_{n=1}^{\infty} \frac{1}{e^{2\pi nt}-1} < \infty$. Hence, by the Weierstrass $M-$test, $\sum_{n=1}^{\infty} \log(1-q^n)$ converges absolutely and uniformly on every compact subset of the open upper half-plane $\Im(\tau) > 0$. (so it is holomorphic on $\Im\tau > 0$). Since, moreover, the term-wise differentiated series $\sum_{n=1}^{\infty} \frac{-nq^{n-1}}{1-q^n} < \infty$ (because it equals $\sum_{n=1}^{\infty} \sigma(n)q^n$ and $\sigma(n) < 1+2+\dots+n = \frac{n^2+n}{2} < n^3$ for sufficiently large $n$, and the power series $\sum_{n=1}^{\infty} n^3 q^n$ in $q$ has radius of convergence 1, i.e. exactly when $\Im\tau > 0$) it follows that logarithmic term-wise differentiation is permitted so that

$$q\frac{d}{dq}\left\{\log\left(q\prod_{n=1}^{\infty}(1-q^n)^{24}\right)\right\} = q\left\{\frac{1}{q} - \sum_{n=1}^{\infty}\frac{nq^{n-1}}{1-q^n}\right\}$$

$$= 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

But now

$$(60E_4(\tau))^3 - 27(140E_6(\tau))^2 = \pi^{12}\left(\frac{4}{3}\right)^3\left(1+240\sum_{r=1}^{\infty}\sigma_3(r)q^r\right)^3$$

$$- 27\pi^{12}\left(\frac{280}{945}\right)^2\left(1-504\sum_{r=1}^{\infty}\sigma_r(5)q^r\right)^2$$

$$= \frac{64}{27}\pi^{12}\left\{\left(1+240\sum_{r=1}^{\infty}\sigma_3(r)q^r\right)^3 - \left(1-504\sum_{r=1}^{\infty}\sigma_r(5)q^r\right)^2\right\}$$

We should therefore equivalently be proving that

$$\left(1+240\sum_{r=1}^{\infty}\sigma_3(r)q^r\right)^3 - \left(1-504\sum_{r=1}^{\infty}\sigma_5(r)q^r\right)^2 = 1728q\prod_{n=1}^{\infty}(1-q^n)^{24.}$$

By logarithmic differentiation, we can at least try to show that

$$q\frac{d}{dq}\log\left(\left(1+240\sum_{r=1}^{\infty}\sigma_3(r)q^r\right)^3 - \left(1-504\sum_{r=1}^{\infty}\sigma_5(r)q^r\right)^2\right) = 1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

The left side is equal to

$$\frac{720\left(1+240\sum_{r=1}^{\infty}\sigma_3(r)q^r\right)^2\sum_{r=1}^{\infty}r\sigma_3(r)q^{r-1} + 1008\left(1-504\sum_{r=1}^{\infty}\sigma_5(r)q^r\right)\sum_{r=1}^{\infty}r\sigma_5(r)q^{r-1}}{\left(1+240\sum_{r=1}^{\infty}\sigma_3(r)q^r\right)^3 - \left(1-504\sum_{r=1}^{\infty}\sigma_5(r)q^r\right)^2}$$

Direct substitution of the results in the previous lemma shows that the last expression is equal to

$$1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

We have therefore successfully shown that

$$\frac{d}{dq}\log\left(\left(1+240\sum_{r=1}^{\infty}\sigma_3(r)q^r\right)^3 - \left(1-504\sum_{r=1}^{\infty}\sigma_5(r)q^r\right)^2\right) = \frac{d}{dq}\left\{\log\left(q\prod_{n=1}^{\infty}(1-q^n)^{24}\right)\right\}.$$

By integrating both sides and exponentiation, we find that

$$\left(1+240\sum_{r=1}^{\infty}\sigma_3(r)q^r\right)^3-\left(1-504\sum_{r=1}^{\infty}\sigma_5(r)q^r\right)^2=cq\prod_{n=1}^{\infty}(1-q^n)^{24},$$

for some constant $c$. By equating coefficients of $q$ on both sides we find that

$$3(240)+2(504)=1728=c.$$

This concludes the proof.
Now we know that, in the series development,

$$j(\tau)=q^{-1}+744+\sum_{n=1}^{\infty}c(n)q^n,$$

the $c(n)$'s are all positive integers. By actual expansion

$$j(\tau)=\frac{(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n)^3}{q\prod_{n=1}^{\infty}(1-q^n)^{24}}=q^{-1}\left(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)^3\cdot\prod_{n=1}^{\infty}(1+q^n+q^{2n}+\dots),$$

for example, we could even calculate the first few values as

$$c(1)=196,884,c(2)=21,493,760,\dots,$$

so that

$$j(\tau)=q^{-1}+744+196,884q+21,493,760q^2+\dots.$$

□

# 3 Connection between $j$ and almost-integers $e^{\pi\sqrt{n}}$.

By the way we have defined the function $j(\tau)$ so far, it might come as a surprise that

**Theorem 3.1** (Explicit values of $j$)**.**

$$j\left(\frac{1+\sqrt{-47}}{2}\right)=-(960)^3$$
$$j\left(\frac{1+\sqrt{-67}}{2}\right)=-(5280)^3$$
$$j\left(\frac{1+\sqrt{-163}}{2}\right)=-(640320)^3$$

*Proof.* Reference to a proof can be found in [9]. □

These are essentially what explain the closeness of $e^{\pi\sqrt{47}}$, $e^{\pi\sqrt{67}}$ and $e^{\pi\sqrt{163}}$. For example, the spectacular string of twelve consecutive 9's after the decimal point, of

$$e^{\pi\sqrt{163}} = 262,537,412,640,768,743.999999999999250$$

can be seen as follows: letting $\tau = \frac{-1+i\sqrt{163}}{2}$ in definition 2.2, so that $q = e^{\pi i(-1+i\sqrt{163})} = -e^{-\pi\sqrt{163}}$, we have, using theorem 3.1

$$j(\tau) = e^{-\pi i\tau} + 744 + 196,884e^{\pi i\tau} + 21,493,760e^{2\pi i\tau} + \dots$$
$$\therefore (640320)^3 = e^{\pi\sqrt{163}} - 744 + \left(196,884e^{-\pi\sqrt{163}} - 21,493,760e^{-2\pi\sqrt{163}} - \dots\right).$$

The bracketed terms evaluate to an extremely small positive number because the coefficients of powers of $e^{-\pi\sqrt{163}}$ grow sufficiently slowly compared to those powers, since if we put

$$j(\tau) = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n)q^n$$

then

**Theorem 3.2** (rate of growth of coefficients of $j$)**.**

$$c(n) \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{\frac{3}{4}}}, n \to \infty.$$

*Proof.* This is due to Petersson and Rademacher independently. For a reference consult [3]. □

We therefore have, to a very close approximation,

$$-(640320)^3 \approx e^{\pi\sqrt{163}} + 744 - 196884e^{-\pi\sqrt{163}},$$

making $e^{\pi\sqrt{163}}$ only very slightly smaller than a positive integer.

# 4 Binary Quadratic Forms and $j$.

Can we make $e^{\pi\sqrt{n}}$ as close as desired to a positive integer, by making suitable choices of $n$? This is not the case, at least not by the previous procedures involving the $j$-function. In order to explain why this is so, we need to pay a short visit to binary quadratic forms first. The reader who is interested in learning more about binary quadratic forms is referred to [5] and [1].
The remarkable theorem connecting the $j-$function and binary quadratic forms is theorem 4.3 below. The Baker-Stark-Heegner theorem (theorem 4.4), makes 163 very special in that we cannot hope, by the present methods at least, to find a positive integer value of $n$ which makes $e^{\pi\sqrt{n}}$ closer to a positive integer than $n = 163$.

## 4.1 Terminology

A function

$$f(x, y) = ax^2 + bxy + cy^2$$

for some numbers $a, b$ and $c$ (in this account, $a, b$ and $c$ will always be integers) is called a *binary quadratic form* (we will use the abbreviation *BQF* for short throughout.)
An integer is *represented* by a BQF if there are integers $x$ and $y$ such that

$$f(x, y) = n.$$

$n$ is *properly represented* if, in addition to being represented by $f(x, y)$, $\gcd(x, y) = 1$.
A BQF is said to be *positive semidefinite* if it represents only nonnengative integers.
It is *negative semidefinite* if it represents only nonpositive integers.
If a BQF is semidefinite (either positive or negative semidefinite), it is said to be *definite* if the following implication is true:

$$f(x, y) = 0 \implies x = y = 0.$$

A BQF is *indefinite* if it represents both positive and negative integers. For example, $x^2 + y^2$ is positive definite while $-x^2 - y^2$ is negative definite.

## 4.2 The Discriminant.

**Definition 4.1.** The number $b^2 - 4ac$ is called the discriminant of $f(x, y)$ and is often denoted by $d$ or $\text{Disc}(f)$.

Disc($f$) is immensely important, as we shall see.

**Theorem 4.1.** *(a) If* $d > 0$*, then* $f$ *is indefinite.*

*(b) If* $d = 0$*, then* $f$ *is semidefinite.*

*(c) If* $d \neq 0$ *and is not a perfect square, then* $f$ *is definite.*

*(d) If* $d < 0$*, then* $f$ *is definite.*

*Proof.* For proofs see [6]. □

Now, we will turn to transformations of binary quadratic forms. Given a BQF

$$f(x, y) = ax^2 + bxy + cy^2$$

we can replace $x$ and $y$ by linear forms $\alpha x + \beta y$ in $x$ and $y$ to obtain a transformation of the BQF. In this way, we can put

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$, and then

$$f(x', y') = Ax'^2 + Bx'y' + Cy'^2$$

where $A, B$ and $C$ are functions of $\alpha, \beta, \gamma$ and $\delta$.
Note that the transformations are invertible provided that

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \alpha\delta - \beta\gamma = \pm 1.$$

Now we will consider the question of equivalence of binary quadratic forms. Two BQFs

$$g(x, y) \text{ and } f(x, y)$$

are said to be *equivalent* if

$$g(x, y) = f(\alpha x + \beta y, \gamma x + \delta y),$$

where

$$\alpha\delta - \beta\gamma = \pm 1.$$

If $\alpha\delta - \beta\gamma = 1$, then $g$ and $f$ are *properly equivalent.*
If $\alpha\delta - \beta\gamma = -1$, then $g$ and $f$ are *improperly equivalent.*
Equivalent binary quadratic forms represent exactly the same integers, because one can always be transformed into the other.
A question may arise: given a BQF $f$ and a transformation of $f$ into another BQF $g$, how is $\text{Disc}(g)$ related to $\text{Disc}(f)$? Suppose that

$$f(x, y) = ax^2 + bxy + cy^2$$

so that

$$\text{Disc}(f) = b^2 - 4ac.$$

Also suppose that $f(x', y')$ is transformed into $g(x, y)$ by the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$. Then

$$\begin{aligned} g(x, y) &= f(\alpha x + \beta y, \gamma x + \delta y) \\ &= a(\alpha x + \beta y)^2 + b(\alpha x + \beta y)(\gamma x + \delta y) + c(\gamma x + \delta y)^2 \\ &= (a\alpha^2 + b\alpha\gamma + c\gamma^2)x^2 + (2a\alpha\beta + b(\alpha\delta + \beta\gamma) + 2c\gamma\delta)xy \\ &+ (\alpha\beta^2 + b\beta\delta + c\delta^2)y^2 \\ \therefore \text{Disc}(g) &= (2a\alpha\beta + b(\alpha\delta + \beta\gamma) + 2c\gamma\delta)^2 - 4(a\alpha^2 + b\alpha\gamma + c\gamma^2)(\alpha\beta^2 + b\beta\delta + c\delta^2) \end{aligned}$$

Now if one expands this out and cancel as many terms as possible, one obtains

$$\text{Disc}(g) = b^2\alpha^2\delta^2 - 2b^2\alpha\beta\gamma\delta + b^2\beta^2\gamma^2 - 4ac\alpha^2\delta^2 + 8ac\alpha\beta\gamma\delta - 4ac\beta^2\gamma^2$$

and if one looks carefully one finds that this is expressible as

$$\mathrm{Disc}(g) = (b^2 - 4ac)(\alpha^2\delta^2 - 2\alpha\beta\gamma\delta + \beta^2\gamma^2) = (b^2 - 4ac)(\alpha\delta - \beta\gamma)^2.$$

So we end up with the following result:

**Theorem 4.2.** *If $f$ and $g$ are two BQF and if $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$, transforms $f$ into $g$, then*

$$\mathrm{Disc}(g) = \mathrm{Disc}(f)(\alpha\delta - \beta\gamma)^2.$$

As a corollary we find that

**Corollary.** If $f$ and $g$ are two equivalent binary quadratic forms (meaning that $\alpha\delta - \beta\gamma = \pm 1$), then

$$\mathrm{Disc}(f) = \mathrm{Disc}(g)$$

We note that the converse is false, i.e. if

$$\mathrm{Disc}(f) = \mathrm{Disc}(g)$$

it does not necessarily imply that $f$ and $g$ are equivalent. For example, the BQF $x^2 + y^2$ and $-x^2 - y^2$ have the same discriminant $-4$ but are clearly not equivalent because equivalent forms represent the same numbers but $x^2 + y^2$ is positive definite while $-x^2 - y^2$ is negative definite, and so they clearly cannot represent the numbers, and cannot therefore be equivalent.
We also note the following: suppose $f$ can be transformed into $g$, and $g$ turns out to have the same discriminant as $f$, and moreover this common discriminant is nonzero. Then, the equation

$$\mathrm{Disc}(g) = \mathrm{Disc}(f)(\alpha\delta - \beta\gamma)^2$$

shows that

$$(\alpha\delta - \beta\gamma)^2 = 1$$

and so the BQF $f$ and $g$ are actually equivalent
It turns out that if $d < 0$ is any integer, then we can find only finitely many positive definite binary quadratic forms $f, g, \ldots$ such that $\mathrm{Disc}(f) = \mathrm{Disc}(g) = \ldots$ and $f, g, \ldots$ are pairwise inequivalent.
Naturally, we now want to count exactly how many pairwise inequivalent positive definite binary quadratic forms of discriminant $d < 0$ we can find. Before doing this, however, it turns out to be useful to define a last notion:
If $f(x, y) = ax^2 + bxy + cy^2$ is a binary quadratic form, we say that $f$ is *primitive* if $\gcd(a, b, c) = 1$.
Now we define the class number:

**Definition 4.2.** The number of primitive positive definite binary quadratic forms whose discriminant is $d < 0$, is denoted by $h(d)$, and is called the *class number* of $d < 0$.

### 4.3 Connection between $j$ and Binary Quadratic Forms

There is an extremely remarkable and unexpected connection between the $j-$functions which we have seen arises naturally out of a discussion of elliptic functions, and binary quadratic forms. To understand the connection, we make a few definitions.

**Definition 4.3.** An *algebraic number* is a number that is a root of a nonzero polynomial with integer coefficients.

**Definition 4.4.** An *algebraic integer* is an algebraic number that is a root of a polynomial whose highest-degree term has a 1 as coefficient.

**Definition 4.5.** The *degree* of an algebraic integer is the degree of the lowest-degree polynomial with leading coefficient 1 satisfied by the algebraic integer.

**Example.** (a) An integer $n$ is an algebraic integer of degree 1 because $x - n = 0$ is the polynomial of smallest degree satisfied by $n$. (notice the uniqueness of this polynomial)

(b) A surd $\sqrt{d}$, where $d$ is an integer which is not a perfect square, is an algebraic integer of degree 2 because its minimal polynomial is $x^2 - d = 0$. (again polynomial is unique)

Here is the remarkable connection between the $j-$function and binary quadratic forms:

**Theorem 4.3** (algebraic integer values of $j$)**.** *Let $d < 0$ be a negative integer. And let $f(x, y) = ax^2 + bxy + cy^2$ be a primitive positive definite binary quadratic form of discriminant $d$.( Note that $a, c > 0$ necessarily.) Then*

$$j\left(\frac{-b+\sqrt{d}}{2a}\right)$$

*is an algebraic integer of degree exactly $h(d)$.*

*Proof.* References to proofs can be found in [9]. Proofs are also found in [7] and [2]. □

This makes finding values of the negative discriminant $d$ such that $h(d) = 1$ i.e. negative discriminant of class number exactly one, extremely exciting because then the coefficients $a, b$ and $c$ of the corresponding binary quadratic form $ax^2 + bxy + cy^2$ will be such that

$$j\left(\frac{-b+\sqrt{d}}{2a}\right)$$

will be an algebraic integer of degree 1, in other words an actual integer!
So how do we find all negative discriminants of class number 1?. Are there

| $d$ | BQF |
|---|---|
| $-3$ | $x^2 + xy + y^2$ |
| $-7$ | $x^2 + xy + 2y^2$ |
| $-11$ | $x^2 + xy + 3y^2$ |
| $-19$ | $x^2 + xy + 4y^2$ |
| $-27$ | $x^2 + xy + 7y^2$ |
| $-43$ | $x^2 + xy + 11y^2$ |
| $-67$ | $x^2 + xy + 17y$ |
| $-163$ | $x^2 + xy + 41y^2$ |
| $-4$ | $x^2 + y^2$ |
| $-8$ | $x^2 + 2y^2$ |
| $-12$ | $x^2 + 3y^2$ |
| $-16$ | $x^2 + 4y^2$ |
| $-28$ | $x^2 + 7y^2$ |

Table 1: Examples of primitive positive definite binary quadratic forms of discriminants $d$ with class number 1.

finitely or infinitely many? It turns out that they are finite in number and they are:

$$d = -3, -4, -7, -8, -11, -12, -16, -19, -27, -28, -43, -67 \text{ and } -163.$$

We give a table of quadratic forms of negative discriminants $d$ whose class number is $h(d) = 1$: We see from the table that $d = -163$ is the largest discriminant in absolute value. By the manner in which we have used the $j-$function previously to demonstrate that $e^{\pi\sqrt{163}}$ is extremely close to an integer we now see that, by this method at least, we cannot find another value of the positive integer $n$ which makes $e^{\pi\sqrt{n}}$ even closer to an integer,

## 4.4 Baker-Stark-Heegner Theorem

That there are finitely many negative discriminants of class number 1 was initially a conjecture of Gauss. It was first solved by Heegner, although it took a while for the mathematical community to finally recognise his contribution. It was later reproved by Stark and Baker.
It turns out that the most important discriminants of class number 1 to consider are $d = -3, -4, -7, -8, -11, -19, -43, -67, \text{ and } -163$. These are called the *fundamental discriminants* of class number 1. We can state the *Baker-Stark-Heegner* theorem (although in a slightly uncommon form) as

**Theorem 4.4** (Baker-Stark-Heegner)**.** *There are exactly nine fundamental discriminants $d < 0$ for which the class number $h(d)$ is 1. They are*

$$d = -3, -4, -7, -8, -11, -19, -43, -67, \text{ and } -163.$$

**Theorem 4.5.** *References to several proofs can be found in [9].*

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