ELLIPTIC PDES

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1. Introduction

A *partial differential equation* (PDE) is an equation involving an unknown function of several variables and its partial derivatives. PDEs arise naturally in the mathematical modeling of phenomena in physics—for instance, in describing heat flow, wave propagation, or electrostatic potential.

A second-order linear PDE in two variables can typically be written in the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + \text{lower-order terms} = 0,$$

where A, B, C are given coefficient functions. The *type* of such a PDE is determined by the discriminant $D = B^2 - AC$. The equation is called

- *elliptic* if $D < 0$,
- *parabolic* if $D = 0$,
- *hyperbolic* if $D > 0$.

Elliptic PDEs describe steady-state or equilibrium configurations—such as the distribution of electric potential in a region with no charge. Among these, the simplest and most fundamental example is the *Laplace equation*.

This paper studies properties of solutions to the Laplace equation, especially in the context of the Dirichlet boundary value problem. We explore the mean value property, the maximum principle, and techniques for solving Laplace's equation using the Poisson kernel and Green's functions.

2. Laplace's Equation

Definition 2.1 (Laplace Equation)**.** A twice continuously differentiable function $u(x,y)$ is said to satisfy the *Laplace equation* in a domain $\Omega \subset \mathbb{R}^2$ if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for all } (x,y) \in \Omega.$$

This is also sometimes written as $\nabla^2 u = 0$ or $\Delta u = 0$ (where Δ is called the *Laplacian*). Such a function u is called *harmonic* on Ω.

One way that this shows up in complex analysis is with the real and imaginary parts of holomorphic functions.

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Theorem 2.2. *Let $f(z) = u(x, y) + iv(x, y)$ be a holomorphic function on an open subset $\Omega \subset \mathbb{C}$, where $z = x + iy$. Then both u and v are harmonic functions on Ω, that is, they satisfy the Laplace equation: $\Delta u = 0$, $\Delta v = 0$.*

Proof. Since f is holomorphic on Ω, it satisfies the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The Laplacian of u is $u_{xx} + u_{yy} = \frac{\partial}{\partial x} u_x + \frac{\partial}{\partial y} u_y = \frac{\partial}{\partial x} v_y + \frac{\partial}{\partial y}(-v_x) = v_{yx} - v_{xy} = 0$. Similarly, for v we get $v_{xx} + v_{yy} = \frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y = \frac{\partial}{\partial x}(-u_y) + \frac{\partial}{\partial y}(u_x) = -u_{yx} + u_{xy} = 0$. This means that $\Delta u = \Delta v = 0$.

□

The problem of solving this differential equation on a domain along with a specified function that the solution should take on values of on the boundary, is called the *Dirichlet problem.*

Definition 2.3 (Dirichlet problem)**.** The *Dirichlet problem* on an open set Ω of finding a function u such that

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

One of the most important properties that come from harmonic functions (and by extension holomorphic functions in the complex plane) is the mean value property which says that a function has the same value as the average of the points around it.

Theorem 2.4 (Mean value property)**.** *Let $u \in C^2\Omega$ where $\Delta u = 0$ and $B_r(x) \in \Omega$. Then,*

$$u(x) = \frac{1}{A_n} \int_{\partial B_r(x)} u \, \mathrm{d}\mathbf{S}$$

where A_n is the surface area of the n-sphere and $\mathbf{S}$ is the normal surface element.

Proof. Define the function

$$\phi(r) = \frac{r^{n-1}}{A_n} \int_{\partial B_1(0)} u(x + r\omega) \mathrm{d}\mathbf{S}_0$$

where ω is a unit vector from 0 to a point on the unit ball. Then the derivative of ϕ by differentiating under the integral sign is

$$\phi'(r) = \frac{r^{n-1}}{A_n} \int_{\partial B_1(0)} \nabla u(x + r\omega) \cdot \omega \, \mathrm{d}\mathbf{S}_0$$

Recall Green's first identity:

$$\int_U (\psi \Delta \varphi + \nabla \psi \cdot \nabla \varphi) \, \mathrm{d}V = \int_{\Delta U} \psi \nabla \varphi \, \mathrm{d}\mathbf{S}$$

By using this the above equation becomes

$$\phi'(r) = \frac{r^{n-1}}{A_n} \int_{\partial B_1(0)} \nabla u(x + r\omega) \cdot \omega \, \mathrm{d}\mathbf{S}_0 = \frac{r^{n-1}}{A_n} \int_{B_1(0)} \nabla \cdot \nabla u(x + r\omega) \mathrm{d}V = \frac{r^{n-1}}{A_n} \int_{B_1(0)} \Delta u \mathrm{d}V$$

Since u is harmonic, $\phi'(r) = 0$ meaning ϕ is constant. If we plug in $r = 0$ we get $\phi(0) = u(x)$. Combining this with the previous fact $\phi(r) = \phi(0) = u(x)$.

□

Corollary 2.4.1. *Integrating the previous result from* 0 *to* r *gives*

$$u(x) = \frac{1}{V_n} \int_{B_r(x)} u \, \mathrm{d}V$$

From this, something called the *maximum principle* follows.

Theorem 2.5 (Maximum principle)**.** *Suppose u is harmonic on Ω, which is connected, and attains a maximum in Ω, then u is constant on Ω.*

Proof. Define the set X as the set of points $x_0 \in \Omega$ such that $u(x_0) = M := \sup_{x\in\Omega} u(x)$. Then X must be closed because it is the pre-image of a singleton set on a continuous function. We also have another definition for X being open, which is that for every $x_0 \in X$, there exists some $\delta > 0$ such that $B_\delta(x_0) \subset X$. Take some radius r such that $B_r(x_0) \in \Omega$. Then because $u(x_0)$ is the maximum, $\frac{1}{V_n}\int_{B_r(x_0)} u \, \mathrm{d}V \leq u(x_0)$. But we also know from the Mean Value Property that these must be exactly equal. This means that u takes on the value M on the entirety of $B_r(x_0)$. Meaning $B_r(x_0) \subset X$. Taking $\delta < r$, we get that X is open. Since X is clopen, it must be either Ω or the empty set. □

3. The Poisson Kernel on a Disk

Let's see how to solve the Dirichlet problem where $\Omega \in \mathbb{R}^2$ is the unit disk with a boundary condition $u(\cos(\theta), \sin(\theta)) = f(\theta)$ for some f. Since we are dealing with circles, it will be beneficial to use polar coordinates. Say that u is a function of r, θ and the polar coordinate version of the Laplacian is

$$0 = \Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

We can perform a separation of variables with $u(r, \theta) = \rho(r)\phi(\theta)$. Plugging this back into the equation and simplifying the derivatives we get

$$\phi(\theta)\rho''(r) + \frac{1}{r}\phi(\theta)\rho'(r) + \frac{1}{r^2}\phi''(\theta)\rho(r) = 0$$

Multiplying both sides by $\frac{r^2}{u}$ and rewriting we will have

$$r^2\frac{\rho''(r)}{\rho(r)} + r\frac{\rho'(r)}{\rho(r)} = -\frac{\phi''(\theta)}{\phi(\theta)} = \lambda$$

which gives two easy to solve ODEs:

$$\phi'' + \lambda\phi = 0$$

$$r^2\rho'' + r\rho' - \lambda\rho = 0$$

To make sure that ϕ is 2π periodic we need that $\lambda = n^2$ for some $n \in \mathbb{N}$. Then we get the solution $\phi_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta)$. The general solution for the equation for ρ is

$$\rho(r) = \begin{cases} c_1 + c_2 \log r, & n = 0 \\ c_1 r^n + c_2 r^{-n}, & n > 0 \end{cases}$$

Since we need the function to be continuous at 0 and not to blow up near there we will take c_2 to be 0. For the same reason, we can't use the log for the $n = 0$ case. This means our final solution will be

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]$$

If we plug in $r = 1$ we can see that we just get the Fourier series of the boundary function, so a_n and b_n must be said Fourier coefficients. We get the coefficients as

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \mathrm{d}t$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \mathrm{d}t$$

We can rewrite the function as

$$u(r, \theta) = a_0 + \Re \sum_{n=1}^{\infty} r^n (a_n - ib_n) e^{in\theta} = a_0 + \Re \sum_{n=1}^{\infty} (a_n - ib_n) z^n$$

where $z = re^{i\theta}$. This means that $u(r, \theta)$ is the real part of a holomorphic function on the unit disk, which is consistent with the fact that real parts of holomorphic functions are harmonic.

We will then have

$$a_n - ib_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \mathrm{d}t$$

and then write the function as

$$u(r, \theta) = a_0 + \frac{1}{\pi} \Re \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} f(t) r^n e^{in(\theta - t)} \mathrm{d}t = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \Re \left(1 + 2 \sum_{n=1}^{\infty} r^n e^{in(\theta - t)}\right) \mathrm{d}t$$

Definition 3.1. Define the *Poisson kernel* for the unit disk to be $P_r(\varphi) = \Re\left(1 + 2\sum_{n=1}^{\infty} r^n e^{in\varphi}\right)$. We can evaluate this to be

$$\Re\left(1 + 2 \sum_{n=1}^{\infty} r^n e^{in\varphi}\right) = \Re\left(1 + 2 \cdot \frac{re^{i\varphi}}{1 - re^{i\varphi}}\right) = \Re\left(\frac{1 + re^{i\varphi}}{1 - re^{i\varphi}}\right) = \frac{1 - r^2}{1 - 2r\cos\varphi + r^2}$$

Finally, we have

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) P_r(\theta - t) \mathrm{d}t = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r^2) f(t)}{1 - 2r\cos(\theta - t) + r^2} \mathrm{d}t$$

This is called the *Poisson integral.*

4. The Dirichlet Problem with Conformal Maps

Let's now consider the problem on the strip $\Omega = \{x + iy : x \in \mathbb{R}, 0 < y < 1\}$ where the boundary is considered $\mathbb{R}$ and $\mathbb{R} + i$ with functions f_0 and f_1 such that $u(x, 0) = f_0(x)$, $u(x, 1) = f_1(x)$, and both functions vanish at $|x| \to \infty$. We can solve this by mapping this region to the unit disk and using the Poisson kernel from before and then mapping it back. Consider the following two maps: $F : \mathbb{D} \to \Omega$

$$F(z) = \frac{1}{\pi} \log \left(i \frac{1 - z}{1 + z} \right)$$

and $G : \Omega \to \mathbb{D}$

$$G(z) = \frac{i - e^{\pi z}}{i + e^{\pi z}}$$

Looking at the behavior of these maps we can see that if $A = \{e^{i\theta}, 0 < \theta < \pi\}$ we get that $F(A) = \mathbb{R}$ and if $B = \{e^{i\theta}, \pi < \theta < 2\pi\}$ then $F(B) = \mathbb{R} + i$. Let's define the mapped boundary functions $\tilde{f}_1(\theta) = f_1(F(e^{i\theta}) - i)$ and $\tilde{f}_0(\theta) = f_0(F(e^{i\theta}))$. The reason we made f_0, f_1 vanish at the ∞s is so that $\tilde{f}$ would be continuous on $\partial\mathbb{D}$. Now we can use the formula from before to get that

$$\tilde{u}(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{f}(t) P_r(\theta - t) \mathrm{d}t = \frac{1}{2\pi} \int_{-\pi}^{0} \tilde{f}_1(t) P_r(\theta - t) \mathrm{d}t + \frac{1}{2\pi} \int_{0}^{\pi} \tilde{f}_0(t) P_r(\theta - t) \mathrm{d}t$$

Now we can apply the G map to get our final solution which will be

$$u(z) = \tilde{u}(G(z))$$

The same thing can be done with any open set Ω if a suitable conformal map is found.

5. The Poisson Equation and Green's Function

Next, we introduce a very similar PDE to the Laplace equation, which is the *Poisson equation.* This is defined as

$$-\Delta u = f \quad \text{when } x \in U$$
$$u = g \quad \text{when } x \in \partial U$$

To solve these, we can introduce a technique called *Green's functions.* Say that we have a linear differential operator L (in this case it would be Δ) and a differential equation $Lu = f$. Then the Green's function G that satisfies

$$LG = \delta$$

where δ is the Dirac delta function which is defined to be 0 on $\mathbb{R}^n \setminus \{0\}$ and $\int_{\mathbb{R}^n} \delta(x) \mathrm{d}x = 1$. Then, u becomes $u(x) = \int_U G(x, y) f(y) \, \mathrm{d}y$.

To solve the Poisson equation let's look at the solution of an easier problem, namely

$$-\Delta \Phi = \delta$$

where Φ is called the *fundamental solution* of the Laplacian. Then we can set u to be

$$u = (\Phi * f)(x) = \int \Phi(x-y)f(y)\mathrm{d}y$$

While the proof is more involved, we can see intuitively that $\Delta u = (-\Delta\Phi) * f = \delta * f = f$. We can try and find the function Φ. We can see that the δ function is radially symmetric so Φ must only depend on r. This means we can set $u(x) = \varphi(r)$ where $r = |x|$. Then the differential equation just becomes, for $x \neq 0$ for $x \in \mathbb{R}^n$,

$$\Delta u = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}(r^{n-1}\varphi) = 0$$

This comes from the radial version of the Laplacian. Since φ is only dependent on r this is once again a fairly easy ODE which has the solution

$$\varphi(r) = \begin{cases} a\log r + b & n = 2 \\ \frac{a}{r^{n-2}} + b & n \geq 3 \end{cases}$$

We must then choose suitable constants a and b such that $-\Delta\Phi = \delta$. This means that $\int_V \Delta\Phi(x)\mathrm{d}V = -1$. Using Stokes' Theorem, we get $\int_S \nabla\Phi\cdot\hat{\mathbf{n}}\,\mathrm{d}S = -1$. In the $n \geq 3$ case this becomes $a(2-n)r^{1-n}\cdot S_n r^{n-1} = -1$ where S_n is the surface area of the n-unit-sphere so $a = \frac{1}{(n-2)S_n}$. In the $n = 2$ case we get $\frac{a}{r}\cdot 2\pi = -1$ so $a = -\frac{1}{2\pi}$. This gives the fundamental solution as

$$\Phi(r) = \begin{cases} -\frac{1}{2\pi}\log r & n = 2 \\ \frac{1}{(n-2)S_n}\cdot\frac{1}{r^{n-2}} & n \geq 3 \end{cases}$$

The issue with this right now, is that this only deals with the interior of U, and the boundary condition of $u = g$ on ∂U is not satisfied. Because of this, we define the Green's function $G(x,y)$ as

$$G(x,y) = \Phi(x-y) - h(x,y)$$

where $h(x,y)$ is the solution to the boundary problem

$$\begin{cases} \Delta_x h(x,y) = 0 & \text{on } U \\ h(x,y) = \Phi(x-y) & \text{on } \Delta U \end{cases}$$

We do this because in the interior, $\Delta_x h(x,y)$ will go to 0 meaning we get the correct behavior from the Φ term. On the boundary, h and Φ will cancel each other out so when we take the integral the solution on that region will be 0, allowing us to add back the correct boundary term. To finally get u we recall Green's second identity:

$$\int_U (u\Delta v - v\Delta u)\mathrm{d}y = \int_{\partial U}\left(u\frac{\partial v}{\partial\nu} - v\frac{\partial u}{\partial\nu}\right)\mathrm{d}S$$

We plug in $G(x,y)$ as v and recall that $\Delta_x G(x,y) = -\delta(x-y)$ meaning that the lefthand side will become

$$-u - \int_U G(x,y)f(y)\mathrm{d}y$$

and since $u = g$ and $G(x, y) = 0$ on ∂U the right hand side will become

$$\int_{\partial U} \left(g \frac{\partial G}{\partial \nu} \right) \mathrm{d}S$$

Simplifying, we get

$$\boxed{u = -\int_U G(x, y) f(y) \mathrm{d}y - \int_{\partial U} \left(g \frac{\partial G}{\partial \nu} \right) \mathrm{d}S}$$

References

[1] G. Teschl, *Partial Differential Equations: An Introduction*, Graduate Studies in Mathematics, Vol. 130, American Mathematical Society, 2012. https://www.mat.univie.ac.at/~gerald/ftp/book-pde/