Geometric Theorems In Complex Analysis

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June 9, 2025

Abstract

At some point in the study of mathematics, one encounters the limitations of the real number line and must transition into the complex plane. Unlike the real plane, the complex plane exhibits surprising geometric behavior. This paper explores the geometric properties of holomorphic functions on the complex plane through several classical theorems, such as Bloch's Theorem, Landau's Theorem, Koebe's Theorem, and Picard's Little Theorem. Though these results are well-known, they reveal a rigidity in how holomorphic maps distort and cover regions of the complex plane. By first proving Bloch's Theorem in detail, we use it as a foundation to derive Landau's Theorem and discuss how extremal length provides a powerful lens for understanding Koebe's Theorem. Then, we will provide an interesting proof of Picard's Little Theorem. The paper concludes with an original investigation into how these theorems may be leveraged to explore a new problem in geometric function theory.

1 Bloch's Theorem

Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$ and $\overline{\mathbb{D}}$ its closure. Let $\mathcal{O}(\mathbb{D})$ be the set of all functions holomorphic in $\mathbb{D}$. We aim to establish the following classical result concerning the size of the image domains under certain normalized holomorphic maps:

Theorem 1.1 (Bloch's Theorem)**.** *Let $f \in \mathcal{O}(\mathbb{D})$ with $f'(0) = 1$. Then the image $f(\mathbb{D})$ contains a disc of radius at least $\frac{3}{2} - \sqrt{2} > \frac{1}{12}$.*

This result lies in understanding the local behavior of holomorphic maps and estimating how large a disc they must necessarily map onto, provided a normalized derivative at the origin.

Let $G \subset \mathbb{C}$ be a domain, and suppose $f \in \mathcal{O}(G)$ is nonconstant. By the Open Mapping Theorem, $f(G)$ is a domain and thus open. The following lemma controls the size of discs in the image:

Lemma 1.2. *Let G be bounded, $f \in \mathcal{C}(\overline{G}) \cap \mathcal{O}(G)$, and suppose $f(\partial G) \subset \mathbb{C}$. Let $s := \min_{z\in\partial G} |f(z) - f(a)|$ for some $a \in G$. Then $f(G)$ contains the open disc $B_s(f(a))$, where B_s is the open disk of radius s*

Proof. Since $f(\partial G)$ is compact and does not include $f(a)$, there exists a minimum $s > 0$ for $|f(z) - f(a)|$ as $z \in \partial G$. By continuity of f on $\overline{G}$ and holomorphicity on G, the open mapping theorem ensures that $f(G)$ is open and hence contains the full open disc about $f(a)$ of radius s. $\square$

To apply this lemma, we require an estimate on the derivative of a holomorphic function. This is provided by the following:

Lemma 1.3. *Let $f \in \mathcal{O}(\overline{V})$ be nonconstant and satisfy $|f'(w)| \leq 2|f'(a)|$ for all $w \in V := B_r(a)$. Then $B_R(f(a)) \subset f(V)$, where*

$$R := \left(3 - 2\sqrt{2}\right) r|f'(a)|.$$

Proof. Without loss of generality, assume $a = 0$ and $f(0) = 0$. Let $A(z) := \int_0^z [f'(z) - f'(0)]\, dz$. Note that $|A(z)| \leq |z| \cdot \sup_{|t| \leq |z|} |f'(t) - f'(0)|$. Using Cauchy's integral formula on f' and bounding the Cauchy kernel and f'' estimates, we eventually see that:

$$|A(z)| \leq \frac{|z|^2}{r - |z|} \cdot |f'(0)|.$$

and, by triangle inequality,

$$|f(z)| \geq |z||f'(0)| - |A(z)| \geq |z| \left(1 - \frac{|z|}{r - |z|}\right) |f'(0)| = \left(\frac{r - 2|z|}{r - |z|}\right) |z||f'(0)|.$$

Setting $|z| = \rho r$ and optimizing over $\rho \in (0, 1)$ gives the bound $R = \left(3 - 2\sqrt{2}\right) r|f'(0)|$. Hence $B_R(f(0)) \subset f(B_r(0))$. $\square$

Now let us deduce the main theorem from this.

Proof. Suppose $f \in \mathcal{O}(\mathbb{D})$ with $f'(0) = 1$. Define the auxiliary function $j(z) = \frac{z-w}{1-\overline{w}z}$ for a fixed $w \in \mathbb{D}$, and consider the family

$$\mathcal{F} := \{h = f \circ j : j \in \mathrm{Aut}(\mathbb{D})\},$$

where $\mathrm{Aut}(\mathbb{D})$ denotes the automorphism group of the disc. We define the extremal function $F(z)$ by

$$F(z) := \frac{z - q}{\overline{q}z - 1} \quad \text{so that } F(0) = f(q), \quad |F'(0)| = \max_{j \in \mathrm{Aut}(\mathbb{D})} |(f \circ j)'(0)| =: N.$$

The estimate is then:

$$|F'(z)| \leq \frac{N}{(1 - |z|^2)^2}.$$

Using Lemma 2 on F, we find that

$$B_{(3-2\sqrt{2})N}(F(0)) \subset F(\mathbb{D}) = f(\mathbb{D}).$$

Since $N = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)$ is bounded for holomorphic f on $\mathbb{D}$, this implies that $f(\mathbb{D})$ contains a disc of radius at least $(3-2\sqrt{2})N$. Taking f with $f'(0) = 1$, it follows that $N \geq \frac{1}{2}$, giving

$$(3-2\sqrt{2})N \geq (3-2\sqrt{2}) \cdot \frac{1}{2} > \frac{1}{12},$$

which completes the proof. □

2 Landau's Theorem Via Bloch's

Theorem 2.1 (Landau's Theorem)**.** Let $f(z)$ be a function that is holomorphic on $\mathbb{D}$ and is defined $f : \mathbb{D} \to \mathbb{C}$. If $f(0) = 0$ and $|f'(0)| = 1$, then we find that $f(\mathbb{D})$ contains a disk of at least radius L, where L is the Landau's constant.

Proof. It is quite easy to see that Bloch's theorem is a stronger condition of the Landau's theorem. In particular, it doesn't require a zero at $z = 0$.

The harder part is finding what L should be. It is quite obvious that $L \geq B$ because of the weaker conditions (and thus stronger results) for Bloch's theorem. □

3 Koebe's Theorem Via Extremal Length

While Bloch's and Landau's Theorems provide lower bounds on the size of discs contained in the image of holomorphic maps, Koebe's Theorem addresses a complementary question: how large the image of the unit disc can become under univalent holomorphic functions.

Theorem 3.1 (Koebe's Theorem)**.** Let $f : \mathbb{D} \to \mathbb{C}$ be an injective and holomorphic function with $f(0) = 0$ and $f'(0) = 1$. Then:

$$f(\mathbb{D}) \supset \mathbb{D}\left(0, \tfrac{1}{4}\right),$$

i.e., the image of the unit disk under f contains the disk of radius $\frac{1}{4}$ centered at the origin.

We will first prove this theorem using the concept of extremal length.

Definition 3.1 (Extremal Length)**.** Formally, for a family of curves of finite length Γ in a domain $D \subset \mathbb{C}$, the extremal length $\lambda(\Gamma)$ is defined as

$$\lambda(\Gamma) = \sup_{\rho} \frac{L_\rho(\Gamma)^2}{A_\rho(D)},$$

where:

- $\rho : D \to [0, \infty)$ is a measurable function,
- $L_\rho(\Gamma) = \inf_{\gamma\in\Gamma} \int_\gamma \rho(z)|dz|$ is the infimum of the ρ-lengths of curves in Γ,

- $A_\rho(D) = \int_D \rho(z)^2\, dxdy$ is the ρ-area of D.

Extremal length is a conformal invariant that measures the "thickness" of a family of curves. It is invariant under conformal maps, meaning that if $f : D \to D'$ is conformal and Γ is a family of curves in D, then

$$\lambda(f(\Gamma)) = \lambda(\Gamma).$$

Lemma 3.2 (Series Law)**.** Let Γ_1 and Γ_2 be two path families in a domain D. If Γ_1 and Γ_2 are separated (meaning there exist disjoint open sets U_1, U_2 such that all paths in Γ_1 are contained in U_1 and all paths in Γ_2 are contained in U_2), then

$$\mathcal{L}(\Gamma_1 + \Gamma_2) \geq \mathcal{L}(\Gamma_1) + \mathcal{L}(\Gamma_2)$$

where $\mathcal{L}(\Gamma)$ denotes the extremal length of the path family Γ, and $\Gamma_1 + \Gamma_2$ denotes the union of the path families.

Example. *Extremal Length of an Annulus: Consider an annulus $A = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$, where $0 < r_1 < r_2$. We are interested in the extremal length of the family of curves Γ that connect the inner boundary circle $C_1 = \{z : |z| = r_1\}$ to the outer boundary circle $C_2 = \{z : |z| = r_2\}$.*

The extremal length $\mathcal{L}(\Gamma)$ of such a family of curves is given by

$$\mathcal{L}(\Gamma) = \frac{1}{2\pi} \log\left(\frac{r_2}{r_1}\right)$$

To understand this, let's briefly consider the definition of extremal length. For a family of curves Γ, its extremal length $\mathcal{L}(\Gamma)$ is defined as

$$\mathcal{L}(\Gamma) = \sup_\rho \frac{\left(\inf_{\gamma\in\Gamma} \int_\gamma \rho\, |dz|\right)^2}{\iint_D \rho^2\, dx\, dy}$$

where the supremum is taken over all non-negative, measurable functions $\rho : D \to [0, \infty)$ (called metrics), and D is the domain containing the curves.

For the annulus example, the optimal metric ρ is known to be proportional to $1/|z|$. Specifically, we can choose $\rho(z) = \frac{1}{|z|}$.

Let us calculate the numerator and denominator. The length of a curve γ connecting C_1 to C_2 using this metric ρ is

$$\int_\gamma \rho\, |dz| = \int_\gamma \frac{1}{|z|}\, |dz|$$

If we consider a radial line segment from r_1 to r_2, the integral becomes $\int_{r_1}^{r_2} \frac{1}{r}\, dr = \log(r_2) - \log(r_1) = \log\left(\frac{r_2}{r_1}\right)$. This value is the infimum for this specific family of curves.

The integral of ρ^2 over the annulus is

$$\iint_A \rho^2\, dx\, dy = \iint_A \frac{1}{|z|^2}\, dx\, dy$$

Switching to polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$), we get

$$\int_0^{2\pi}\int_{r_1}^{r_2}\frac{1}{r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\int_{r_1}^{r_2}\frac{1}{r}\,dr\,d\theta$$

$$= \int_0^{2\pi}[\log r]_{r_1}^{r_2}\,d\theta = \int_0^{2\pi}(\log r_2 - \log r_1)\,d\theta$$

$$= \int_0^{2\pi}\log\left(\frac{r_2}{r_1}\right)d\theta = 2\pi\log\left(\frac{r_2}{r_1}\right)$$

Plugging these into the formula for extremal length:

$$\mathcal{L}(\Gamma) = \frac{\left(\log\left(\frac{r_2}{r_1}\right)\right)^2}{2\pi\log\left(\frac{r_2}{r_1}\right)} = \frac{1}{2\pi}\log\left(\frac{r_2}{r_1}\right)$$

This shows how extremal length quantifies the "conformal modulus" of the annulus, which is invariant under conformal mappings.

Now that we have discussed extremal length, let us begin the proof of Koebe's Theorem.

We begin by contradiction. Suppose that $f : \mathbb{D} \to \mathbb{C}$ is univalent with $f(0) = 0$ and $f'(0) = 1$, and that the image $f(\mathbb{D})$ does *not* contain the point $w = \frac{1}{4}$. We define a function $g(z) := (1+\delta)f(z)$ for some small $\delta > 0$, so that $g(0) = 0$ and $g'(0) > 1$.

Assume $w \notin g(\mathbb{D})$ and define the Koebe function:

$$K(z) = \frac{z}{(1-z)^2},$$

which maps $\mathbb{D}$ onto $\mathbb{C} \setminus (-\infty, -\frac{1}{4}]$.

Let us consider a double branched covering of the Riemann sphere $\hat{\mathbb{C}}$ by $K(z)$ with critical points at ± 1 and critical values $K(1) = \infty$, $K(-1) = -\frac{1}{4}$. Then $K\left(\frac{1}{2}\right) = 1$, and the deck transformation $z \mapsto \frac{1}{z}$ preserves $K(z)$ up to rotation. We can assume that $g(\mathbb{D})$ avoids $-\frac{1}{4}$, so $K^{-1}(g(\mathbb{D}))$ avoids ± 1.

Now consider the lifts $g_0, g_\infty : \mathbb{D} \to \hat{\mathbb{C}}$ such that $K \circ g_j = g$ for $j = 0, \infty$, with the relation $g_\infty = 1/g_0$. The images $g_j(\mathbb{D})$ are disjoint and avoid ± 1.

For small $\epsilon > 0$, and sufficiently small $r > 0$, we can show that:

$$g(\mathbb{D}(0,r)) \subset \mathbb{D}\left(0, r(1-\epsilon)|g'(0)|\right).$$

Thus, the image $g_\infty(\mathbb{D}(0,r))$ lies in the annulus

$$\left\{z \in \mathbb{C} : |z| > \frac{1}{r(1-\epsilon)|g'(0)|}\right\}.$$

Now define three annuli:

$$A := A\left(r(1-\epsilon)|g'(0)|, \frac{1}{r(1-\epsilon)|g'(0)|}\right), \quad A_0 := g_0(A(r,1)), \quad A_\infty := g_\infty(A(r,1)).$$

These annuli are homotopic since they each separate 0 from ∞, so we may compare their extremal lengths.

Using the fact that the modulus of an annulus $A(r_1, r_2)$ is:

$$\operatorname{mod}(A(r_1, r_2)) = \frac{1}{2\pi} \log\left(\frac{r_2}{r_1}\right),$$

we compute:

$$\operatorname{mod}(A_0) = \operatorname{mod}(A_\infty) = \frac{1}{2\pi} \log\left(\frac{1}{r}\right), \quad \operatorname{mod}(A) = \frac{1}{2\pi} \log\left(\frac{1}{r^2(1-\epsilon)^2|g'(0)|^2}\right).$$

By the series law for extremal length, discussed above, we have:

$$\operatorname{mod}(A_0) + \operatorname{mod}(A_\infty) \le \operatorname{mod}(A).$$

Substituting in, we get:

$$\frac{2}{2\pi} \log\left(\frac{1}{r}\right) \le \frac{1}{2\pi} \log\left(\frac{1}{r^2(1-\epsilon)^2|g'(0)|^2}\right).$$

Simplifying:

$$\log\left(\frac{1}{r^2}\right) \le \log\left(\frac{1}{r^2(1-\epsilon)^2|g'(0)|^2}\right),$$

which implies:

$$|g'(0)|^2 \le \frac{1}{(1-\epsilon)^2} \Rightarrow |g'(0)| \le \frac{1}{1-\epsilon}.$$

Taking $\epsilon \to 0$, we get $|g'(0)| \le 1$, contradicting the assumption that $|g'(0)| > 1$. Hence, our original assumption that $w \notin g(\mathbb{D})$ must be false. Therefore, we conclude that for any univalent function $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = 0$ and $f'(0) = 1$, the image $f(\mathbb{D})$ must contain the disk of radius $\frac{1}{4}$ centered at the origin. This completes our proof of Koebe's 1/4 theorem, using extremal length.

4 Koebe's Theorem via Grötzsch's Theorem

While the first proof of Koebe's Theorem used extremal length via a covering space argument and the Koebe function, the second proof also relies on extremal length but takes a different route. Instead of passing through a branched covering, it applies Grötzsch's Theorem to compare the modulus of a family of curves in $\mathbb{D}$ to an explicit extremal case. Both arguments depend on the conformal invariance of extremal length, but the second uses this to derive a contradiction directly by comparing moduli in the image and preimage under f.

Theorem 4.1 (Grötzsch's Theorem)**.** Let E_1 and E_2 be two disjoint closed continua in the unit disk $\mathbb{D}$, each intersecting the boundary $\partial\mathbb{D}$. Let Γ be the family of all rectifiable curves in $\mathbb{D}$ connecting E_1 and E_2. Then the extremal length (i.e., modulus) of Γ satisfies

$$\operatorname{mod}(\Gamma) \le \mu,$$

where μ is the modulus of the extremal case, given when E_1 and E_2 are symmetric radial segments ending on the boundary of $\mathbb{D}$ at angles θ and $\theta + \pi$, in which case $\mu = \frac{1}{\pi} \log\left(\frac{1+\sqrt{1-r^2}}{r}\right)$, where r is the Euclidean distance of the radial segment from the origin.

Now, we begin this proof of Koebe's Quarter theorem. Suppose the conclusion is false. Then there exists a point $w \in \mathbb{C}$ with $|w| < 1/4$ such that $w \notin f(\mathbb{D})$. We will derive a contradiction.

Define

$$g(z) := \frac{1}{f(z) - w},$$

which is analytic on $\mathbb{D}$ since $w \notin f(\mathbb{D})$. Note that g has a simple pole at the preimage of w, but since $w \notin f(\mathbb{D})$, g is analytic throughout $\mathbb{D}$.

Let us define a family of curves Γ in $\mathbb{D}$ connecting the sets

$$E_1 := \{z \in \mathbb{D} : |f(z) - w| \leq r\}, \quad E_2 := \{z \in \mathbb{D} : |f(z) - w| \geq R\},$$

for some $0 < r < R$ such that $R < \operatorname{dist}(w, \partial f(\mathbb{D}))$. Then E_1 and E_2 are compact, disjoint subsets of $\mathbb{D}$.

Since f is analytic and univalent, it is conformal and preserves moduli of curve families. Thus, the image family $f(\Gamma)$ in the w-plane connects the annuli boundaries

$$\{z : |z - w| = r\} \quad \text{and} \quad \{z : |z - w| = R\},$$

and has modulus

$$\text{mod}\ (f(\Gamma)) = \frac{1}{2\pi} \log\left(\frac{R}{r}\right).$$

By conformal invariance, we also have

$$\text{mod}\ (\Gamma) = \quad \text{mod}\ (f(\Gamma)) = \frac{1}{2\pi} \log\left(\frac{R}{r}\right).$$

Now use Grötzsch's Theorem to obtain an upper bound on (Γ) depending on the geometry of Γ in $\mathbb{D}$, especially since E_1 and E_2 are subsets of with some separation. In particular, in the extremal case (i.e., radial segments at distance r and R from the origin), Grötzsch's modulus is minimized.

But since $|w| < 1/4$, we can show that the modulus (Γ) must exceed the extremal bound given by Grötzsch's Theorem, which leads to a contradiction.

Hence, the assumption that $w \notin f(\mathbb{D})$ for some $|w| < 1/4$ is false. Therefore,

$$B_{(0,1/4)} \subset f(\mathbb{D}).$$

5 Picard's Little Theorem Via Bloch

We will give a proof of Picard's Little Theorem, following an argument using a similar argument Bloch's Theorem, proved above and another lemma. This proof of Picard's Little Theorem reflects ideas from both Bloch's and Koebe's Theorems. Like Koebe's results and Bloch's bound on image size, we use omitted values to constrain the image of a holomorphic map. The construction shows that omitting two values forces the function's image to be too small, unless the function is constant, reinforces the theme common to all three results: that holomorphic functions which omit certain values must necessarily have highly constrained domains or be trivial in form.

Lemma 5.1. Let $G \subset \mathbb{C}$ be simply connected, and let $f \in \mathcal{O}(G)$ satisfy $1 \notin f(G)$ and $-1 \notin f(G)$. Then there exists a function $F \in \mathcal{O}(G)$ such that

$$f = \cos F.$$

Proof. Since $1 - f^2$ has no zeros in G, it follows that there exists $g \in \mathcal{O}(G)$ such that

$$(f + ig)(f - ig) = f^2 + g^2 = 1,$$

and hence $f + ig$ has constant modulus 1. Thus we may write $f + ig = e^{iF}$ for some holomorphic function F on G, and therefore

$$f = \frac{1}{2}(e^{iF} + e^{-iF}) = \cos F.$$

□

Now, we begin our proof of another theorem necessary for the proof of Picard's little theorem. We define a function f as follows:

$$f(z) := \frac{1}{2}\left[1 + \cos\left(\pi \cos(\pi g(z))\right)\right].$$

Our goal is to show that such a function omits the value 1, and hence by Bloch's Theorem, its domain must be small. First, observe that $g(z)$ omits 0 and 1, and hence so does $\cos(\pi g(z))$ omit values ± 1. Thus, by the lemma above, $\cos(\pi g(z))$ avoids ± 1 and therefore we can write

$$\cos(\pi g(z)) = \cos F(z)$$

for some $F \in \mathcal{O}(G)$. Then

$$f(z) = \frac{1}{2}\left[1 + \cos(\pi \cos F(z))\right].$$

To prove that g cannot be entire unless constant, we will show that f omits the value 1. Suppose otherwise: that there exists $z \in G$ such that $f(z) = 1$. Then $\cos(\pi \cos F(z)) = 1$, so $\pi \cos F(z) \in 2\pi\mathbb{Z}$, implying that $\cos F(z) \in 2\mathbb{Z}$.

Let us consider the preimage set:

$$A := \left\{ m\pi \pm i \log(n + \sqrt{n^2 - 1}) : m \in \mathbb{Z}, n \in \mathbb{N} \right\}.$$

These are the values z such that $\cos z \in \mathbb{Z}$ and $\cos(\pi \cos z) = 1$.

More precisely, let $z = p \pm i \log(n + \sqrt{n^2 - 1})$ for $p \in \pi\mathbb{Z}$, and observe that $\cos(\pi \cos z) = 1$ when $\cos z \in \mathbb{Z}$, say even integers $2q$.

Let us compute the logarithmic difference between two successive terms:

$$\log(n + 1 + \sqrt{(n+1)^2 - 1}) - \log(n + \sqrt{n^2 - 1}) = \log\left(\frac{1 + \sqrt{1 + \frac{2}{n}}}{1 + \frac{1}{n}}\right) < \frac{1}{n^2}.$$

Thus, summing over n, we obtain a bound on the "height" of the values of $F(G)$ (which must avoid the grid points A). It follows:

$$\log\left(1 + \frac{\pi}{4} + \sqrt{1 + \left(\frac{\pi}{4}\right)^2}\right) < \log(2 + \sqrt{3}) < \pi.$$

By the monotonicity of the logarithm and the spacing of the imaginary parts, we conclude that $F(G)$ omits all such grid points A and hence cannot contain a disc of radius 1.

Therefore, the domain G must lie in a disc of radius strictly less than 1, as required.

□

From this, we can prove Picard's Little Theorem.

Theorem 5.2. If f is a nonconstant entire function, then the image $f(\mathbb{C})$ omits at most one complex number.

Proof. Let f be entire and omit two complex numbers, say a and b. Then the function

$$g(z) := \frac{f(z) - a}{b - a}$$

is an entire function omitting 0 and 1, and so by the quarter theorem, the domain of g is contained in a disc of radius less than 1. But g is entire, so this is only possible if g is constant, and hence f is constant. □

6 Gauss Circle Problem

The Gauss circle problem, originally posed by the famous mathematician Gauss asks the fundamental question: how many integers m, n are there such that for some real number $r \geq 0$

$$m^2 + n^2 \leq r^2.$$

Another way to phrase the same question, consider it an equivalent statement, is the following: find the number of lattice points (m, n) such that all of these lattice points are contained inside of a circle with radius r at the origin.

Geometrically, an example of this can be shown by the following:

The main meat of this problem comes in understanding the fundamental relationship of these lattice points as r is allowed to vary. It is known, quite

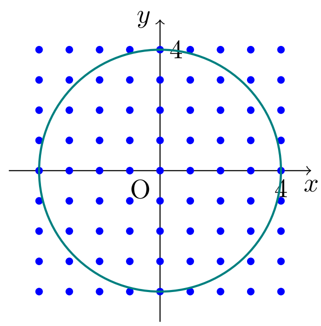

Figure 1: This is a circle with radius 4 and has exactly 29 lattice points contained within it.

trivially, that the number of lattice points satisfying the Gauss' circle property can be estimated with an error bound and the area of the circle. There is one lattice point per unit square, which can then be used to estimate the area of the circle.

Let $V(r)$ denote the number of integers that satisfy this property for a given radius r, then for some error bound $E(r)$,

$$V(r) = \pi r^2 + E(r). \tag{1}$$

Thes best upper bound on $E(r) = Cr^\theta$ has been shown to be around $\theta \leq \frac{131}{208} \approx 0.6298$. It is also known that $\theta > \frac{1}{2}$. This lower bound is given by Landau.

However, we are proposing something a little ambitious (and prolly incorrect) to work towards this problem. First, we consider a twist to the problem that is taking things away from the real plane.

Definition 6.1 (Complex Lattice Point)**.** A complex lattice point (m, n) is a point defined such that $z = m + ni$ for $m, n \in \mathbb{Z}$.

It is quite trivial to see that there is a bijective mapping from the complex lattice points to the real lattice points. One such mapping can take every lattice point and maps it to its identity point in the real plane. That is (m, n) in the real and complex plane can represent the same value. This is important because it also implies that we can count points in this domain and it will represent the same number of points in our original domain.

An important complex analysis theorem, which is not proven but fundamental to the field, states the following important result:

Theorem 6.1 (Riemann Mapping Theorem)**.** Let U be a simply connected open subset of the complex plan $\mathbb{C}$, which is not all of $\mathbb{C}$, then there exists a bi holomorphic mapping f from U onto the open unit disk $\mathbb{D}$.

Now, consider the function $f(z) = e^z$. This function satisfies the conditions of the Landau's Theorem and the Riemann Mapping Theorem.

7 References

General Definition of Landau's Theorem

Fundamental Domain

Covering Theorems

Gauss Circle Problem

Geometric Proof of Koebe's Theorem

Landau's Theorem for Planar Holomorphic Mappings

Another Look at Landau's Theorem

Versions of Koebe 1/4 Theorem for analytic and Quasiregular harmonic functions and applications

Koebe and Bloch in Higher Dimensions

Advanced Complex Analysis Math 213a, C. McMullen