# L-FUNCTIONS AND DIRICHLET'S THEOREM

SOUNAK BAGCHI

Abstract. Dirichlet's Theorem on Arithmetic Progressions states that there are infinitely many primes congruent to $a$ modulo $q$, where $\gcd(a, q) = 1$. We prove a strong form of Dirichlet's Theorem: that the sum of the reciprocals of primes congruent to $a$ modulo $q$ diverges. We use Dirichlet Characters and their associated L-functions to prove this fact.

## 1. Introduction

Over 2000 years ago, Euclid proved that infinitely many primes exist using a simple proof by contradiction. Less than 200 years ago, an extension of this simple fact was proven by German mathematician Peter Gustav Lejeune Dirichlet. Dirichlet's Theorem on Arithmetic Progressions deals with counting primes of a specific residue class.

**Theorem 1.1.** *For any two coprime positive integers $a$ and $d$, there are infinitely many prime numbers in the arithmetic progression $a, a + d, a + 2d, \ldots$.*

As evidenced by the time gap between these theorems, proving Dirichlet's Theorem is by no means simple; Dirichlet's proof required techniques in Complex Analysis and Analytic Number Theory established by prior mathematicians, such as Euler. The main idea of the original proof is to show a stronger statement.

**Theorem 1.2.** *For coprime positive integers $a, q$, the sum*

$$\sum_{p \equiv a \pmod q} \frac{1}{p}$$

*diverges, where $p$ ranges over prime numbers.*

Such divergence would imply an infinite number of terms in the sum and, thus, an infinite number of primes congruent to $a$ modulo $q$. Not only does this sum imply divergence, but it also suggests that density of primes congruent to $a$ modulo $q$ is not small.

The proof of Theorem 1.2 introduces the use of **L-functions**, a more concrete definition of which will be given later in this paper. Dirichlet L-functions are example of more general **Dirichlet Series**, defined by

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

for any sequence of complex numbers $a_1, a_2, \ldots$ and any complex number $s$.

## 2. Dirichlet Characters

When considering numbers of a certain moduli, it often makes sense to use **Dirichlet Characters** to define mappings.

**Definition 2.1.** Let $m$ be a positive integer. A *Dirichlet Character modulo $m$* is a function $\chi : \mathbb{Z} \to \mathbb{C}$ with the following properties:

(1) For integers $a, b$, we have $\chi(ab) = \chi(a)\chi(b)$.
(2) $\chi$ is periodic with period $m$; in other words, $\chi(a + m) = \chi(a)$ for positive integers $a$.
(3) $\chi(a) = 0$ if and only if $\gcd(a, m) > 1$.

Dirichlet Characters were actually introduced in Dirichlet's paper on primes in arithmetic progressions, in which Dirichlet's Theorem is proven. One such well-known example of Dirichlet characters is the following:

*Example.* If $p$ is a prime, then the *Legendre Symbol* $\left(\frac{a}{p}\right)$, defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & p \mid a, \\ 1 & p \nmid a \text{ and there is an } x \text{ such that } x^2 \equiv a \pmod{p}, \\ -1 & \text{there is no } x \text{ such that } x^2 \equiv a \pmod{p} \end{cases}$$

is a Dirichlet Character modulo $p$. In particular, $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$ and $\left(\frac{a+p}{p}\right) = \left(\frac{a}{p}\right)$.

Another important Dirichlet Character is the **principal character**, which is defined very simply as follows:

**Definition 2.2.** The **principal character**, often denoted $\chi_0$, is defined as

$$\chi_0(a) = \begin{cases} 0 & \text{if } \gcd(a, m) > 1 \\ 1 & \text{if } \gcd(a, m) = 1. \end{cases}$$

In general, Dirichlet characters act as a homomorphism from a group $(\mathbb{Z}/m\mathbb{Z})^\times$ to the multiplicative group of the complex numbers, $\mathbb{C}^\times$.

Now, we can define a specific type of L-function, based on Dirichlet characters:

**Definition 2.3.** Let $\chi$ be a Dirichlet Character. Then, it's associated *L-function* is

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Whilst Dirichlet characters are homomorphisms that act on multiple residue classes, L-functions are a way to single out a specific residue class $a \pmod{q}$. A more useful form of L-functions is through their **Euler Product**, namely the form

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where the two expressions are equivalent to each other via geometric series expansion. Note that the multiplicative property of Dirichlet Characters also allows for this different form.

To understand some of the techniques needed for the proof, it makes sense to look at how L-functions can be used for one such case - particularly, for primes congruent to 1 modulo 4.

## 3. The case $p \equiv 1 \pmod 4$

Showing that there are infinitely many primes congruent to 1 modulo 4 by itself is not too hard of a task - in fact, one could use ideas similar to Euclid's proof that infinitely many primes exist. However, for illustrative purposes, we will prove this fact similarly to the general proof of Dirichlet's Theorem.

In particular, we should first define a Dirichlet character for modulo 4; while multiple can be used, the most logical one is defined by

$$\chi_4(a) = \begin{cases} 1 & \text{if } a \equiv 1 \pmod 4 \\ -1 & \text{if } a \equiv 3 \pmod 4 \\ 0 & \text{if } a \equiv 0, 2 \pmod 4). \end{cases}$$

Of important note is the following corollary, which we will use in the proof:

**Proposition 3.1.** $L(1, \chi_4) = \frac{\pi}{4}$.

*Proof.* Note that

$$L(1, \chi_4) = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \ldots,$$

which is the Taylor Series expansion for $\arctan(x)$ at $x = 1$. In particular, this evaluates to $\arctan(1) = \frac{\pi}{4}$. ∎

Much as, in general, we would like to show that the sum of the reciprocals of primes for a specific residue class diverges, we would like to show that some expression involving all primes congruent to 1 modulo 4 diverges.

In particular, consider the expression $\zeta(1)L(1, \chi_4)$. Clearly, this is divergent, because

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series, which diverges. Writing the product $\zeta(1)L(1, \chi_4)$ using the Euler Product, we get

$$\zeta(1)L(1, \chi_4) = \left( \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{-1} \right) \left( \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p}\right)^{-1} \right).$$

We can take out the terms involving $p = 2$, since these do not impact the convergence or divergence of this product. Now, split up the product based on whether $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$. If $p \equiv 1 \pmod 4$, then $\chi(p) = 1$, whereas if $p \equiv 3 \pmod 4$, then $\chi(p) = -1$; thus, we are dealing with the divergent product

$$\prod_{\substack{p \text{ prime}, \\ p \equiv 1 \pmod 4}} \left(1 - \frac{1}{p}\right)^{-2} \prod_{\substack{p \text{ prime}, \\ p \equiv 3 \pmod 4}} \left(1 - \frac{1}{p^2}\right)^{-1}.$$

Thus, it suffices to show that the second product converges. In particular, by taking the log of the second product, we get that

$$\log \prod_{\substack{p \text{ prime}, \\ p\equiv 3 \pmod 4}} \left(1-\frac{1}{p^2}\right)^{-1} = \sum_{\substack{p \text{ prime}, \\ p\equiv 3 \pmod 4}} \log\left(1-\frac{1}{p^2}\right)^{-1}.$$

Using the Taylor Series for $\log(1-x)$, the inner term ends up being the sum

$$\sum_{n=1}^{\infty} \frac{1}{np^{2n}}$$

, so it follows that this sum is less than or equal to

$$\sum_{p=2}^{\infty}\sum_{n=1}^{\infty} \frac{1}{np^{2n}} \leq \sum_{p=2}^{\infty}\sum_{n=2}^{\infty} \frac{1}{p^n} = \sum_{p=2}^{\infty} \frac{1}{p^2-p},$$

which clearly converges. Hence, the product involving primes congruent to 1 modulo 4 must diverge, meaning there are infinitely many primes congruent to 1 modulo 4.

## 4. The general case

In the previous section, we were able to isolate a product/sum involving primes only congruent to 1 modulo 4. This general strategy is what we would like to replicate - singling out primes congruent to $a \pmod q$. Essentially, in order to isolate a residue class, what we'd like to do is find a series of Dirichlet Characters such that some linear combination of them exists that is 0 everywhere, other than at $a \pmod q$.

As usual, we will try to show that the sum

$$P_a(s) = \sum_{\substack{p \text{ prime}, \\ p\equiv a \pmod q}} \frac{1}{p^s}$$

diverges at $s = 1$. In particular, we'd like to relate this sum to that of some L-function.

Denote the set of all Dirichlet characters for a modulo $m$ as $X(m)$. First, we should establish an important property of Dirichlet Characters, namely the *orthogonality relation*:

**Lemma 4.1.** *For all integers $a$ and positive integer $m$, we have*

$$\sum_{\chi\in X(m)} \chi(a) = \begin{cases} \phi(m) & a\equiv 1 \pmod m \\ 0 & a \not\equiv 1 \pmod m \end{cases}.$$

*Proof.* Note that $\chi(1) = 1$ for all Dirichlet characters, as $\chi(1) = \chi(1\cdot 1) = \chi(1)^2$. Thus, if $a \equiv 1 \pmod m$, then $\chi(a) = 1$ for all $\phi(m)$ Dirichlet characters, resulting in a sum of $\phi(m)$.

Otherwise, if $\gcd(a, m) > 1$, then it follows that $\chi(a) = 0$ for all $\chi \in X(m)$, leading to a sum of 0. Now, assume $\gcd(a, m) = 1$ and $a \not\equiv 1 \pmod m$. Choose a Dirichlet character $\chi_1$ with $\chi_1(a) \neq 1$. Then, since $\chi\chi_1$ runs over all dirichlet characters, it follows that

$$\chi_1(a) \sum_{\chi\in X(m)} \chi(a) = \sum_{\chi\in X(m)} \chi_1(a)\chi(a) = \sum_{\chi\in X(m)} \chi(a),$$

and since $\chi_1(a) \neq 1$, it follows that $\sum_{\chi \in X(m)} \chi(a) = 0$. ■

This relation gives light to the next lemma, which will be key to relating our series $P_a(s)$ to an L-function:

**Lemma 4.2.** *Suppose $a, m$ are integers with $\gcd(a, m) = 1$. Then, the function*

$$f(n) = \sum_{\chi \in X(m)} \frac{\chi(a)^{-1}}{\phi(m)} \cdot \chi(n)$$

*is 1 when $n \equiv a \pmod{m}$, and 0 otherwise.*

*Proof.* Note that $\chi(a)^{-1}\chi(n) = \chi(a^{-1}n)$ ranges over all residue classes modulo $m$. Thus, since $a^{-1}n \equiv a^{-1}a \equiv 1 \pmod{m}$ for $n \equiv a \mod m$, from Lemma 4.1, dividing by $\phi(m)$ gives the desired result. ■

Thus, we can rewrite our series $P_a(s)$ using $f(n)$, namely by

$$P_a(s) = \sum_{p \text{ prime}} \frac{f(n)}{p^s} = \sum_{\chi \in X(m)} \frac{\chi(a)^{-1}}{\phi(m)} \sum_{p \text{ prime}} \frac{\chi(p)}{p^s}.$$

Consider this series at $s = 1$, and compare it to the L-function

$$L(1, \chi) = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p}\right)^{-1}.$$

Recall what happened when we took the logarithm of our L-function in Section 3; it gave us something very similar to the sum $P_a(1)$, hence if we can show that the divergence of $\log(L(1, \chi))$ over all Dirichlet characters $\phi \in X(m)$ implies the divergence of $P_a(1)$, our task becomes easier.

Let's tackle the first part of this; relating our expression for $P_a(s)$ with that of $\log(L(1, \chi))$. We have

$$\log(L(1, \chi)) = \sum_{p \text{ prime}} \sum_{n=1}^{\infty} \frac{\chi(p)^n}{np^n}$$

via an expansion similar to that in Section 3. Taking the $n = 1$ term out, we get

$$\log(L(1, \chi)) = \sum_{p \text{ prime}} \frac{\chi(p)}{p} + \sum_{p \text{ prime}} \sum_{n=2}^{\infty} \frac{\chi(p)^n}{np^n}.$$

As we saw before, the double sum converges; let's say that it converges to a constant $C(\chi)$. Then,

$$P_a(1) = \sum_{\chi \in X(m)} \frac{\chi(a)^{-1}}{\phi(m)} \sum_{p \text{ prime}} \frac{\chi(p)}{p} = \sum_{\chi \in X(m)} \frac{\chi(a)^{-1}}{\phi(m)} \sum_{p \text{ prime}} (\log L(1, \chi) - C(\chi)).$$

Since $\phi(m)$ and $C(\chi)$ are finite constants that don't affect divergence, this sum diverges if and only if

$$\sum_{\chi \in X(m)} \chi(a)^{-1} \log L(1, \chi)$$

diverges. Note that, for the principal character $\chi_0$, the sum diverges at $s = 1$ (Via comparison to the harmonic series $\zeta(1)$). For non-principal characters $\chi$, the value $L(1, \chi)$ converges;

hence, if we can show that $\log L(1,\chi)$ converges for all $\chi \neq \chi_0$, then the sum will diverge. This is true since $L(1,\chi)$ converges, except for one wrinkle: if $L(1,\chi)$, the sum will still diverge. So, we need to show for all non-principal characters that $L(1,\chi) \neq 0$.

## 5. $L(1,\chi) \neq 0$

Since Dirichlet's proof, which involved this key fact, many different proofs for $L(1,\chi) \neq 0$ have been developed, relying on different machinery. Dirichlet's proof used what was known as **additive characters**, from which he derived an explicit formula for $L(1,\chi)$. His proof, as others, involved the use of separately considering complex-valued characters, which are dirichlet characters $\chi$ such that there exists some $n$ with $\chi(n) \in \mathbb{C}\backslash\mathbb{R}$, and real-valued characters. We will also follow this path in this section.

### 5.1. Complex-valued characters.

For complex-valued characters, the trick with most proofs is to show that a specific expression involving $L(1,\chi)$ is nonvanishing, implying $L(1,\chi)$ is nonvanishing. In this case, consider the product

$$g(s) = L(s,\chi_0)^3 L(s,\chi)^4 L(s,\chi^2).$$

Following [4], we will show that $|g(1)| \geq 1$, and in fact that $|g(s)| \geq 1$ for all $s$ with $\Re(s) = 1$. Note in particular that, since $\chi$ is complex valued, $\chi^2 \neq \chi_0$. Letting $\theta_{m,p}$ be the argument of $\chi(p^m)$ in $\mathbb{R}$, from the Euler Product expansions of each Dirichlet Character,

$$|g(s)| = \left|\exp\left(\sum_{m,p} \frac{3 + 4\chi(p^m) + \chi^2(p^m)}{mp^{ms}}\right)\right| = \exp\left|\sum_{m,p} \frac{3 + 4\cos\theta_{m,p} + cos2\theta_{m,p}}{mp^m}\right|.$$

The main trick now (which arises rather surprisingly!) is that

$$3 + 4\cos\theta + \cos 2\theta = 3 + 4\cos\theta + 2\cos^2\theta - 1 = 2(1+\cos\theta)^2 \geq 0,$$

hence every term in this large sum is non-negative. This implies that $|g(s)| \geq e^0 = 1$. In particular, $g(1) \neq 0$, so $L(1,\chi) \neq 0$.

### 5.2. Real-valued characters.

For the real-valued characters, we will take a different, more motivated approach (adapted from [5]) in this paper that relies on a few more techniques related to analytic number theory. First, we should make use of the basic fact that real-valued characters $\chi$ can only take on the values $-1, 0, 1$; too see why, for a Dirichlet character modulo $m$, one could decompose $\chi(a^n) = (\chi(a))^n$ where $n$ is the order of $a$ modulo $m$.

Now, the proof relies on the multiplicative function

$$A(n) = \sum_{d|n} \chi(d).$$

In particular, we must prove one important property of this function:

**Theorem 5.1.** *$A(n) \geq 0$ for all $n$, and $A(n) \geq 1$ if $n$ is a perfect square.*

*Proof.* Since $A$ is a multiplicative function, it suffices to show that these two facts hold for prime powers $p$. In particular, note that

$$A(p^a) = \sum_{t=0}^{a} \chi(p^t) = 1 + \sum_{t=1}^{a} \chi(p^t).$$

$\chi(p)$ can only take on the values $-1, 0, 1$. If $\chi(p) = 0$, then $A(p^a) = 1$, and if $\chi(p) = 1$, then each term in the sum is 1, so $A(p^a) = a + 1$. Otherwise, if $\chi(p) = -1$, then

$$A(p^a) = \begin{cases} 0 & \text{if } a \text{ is even} \\ 1 & \text{if } a \text{ is odd} \end{cases}.$$

Thus, $A(p^a) \geq 0$, and if $a$ is even, then $A(p^a) \geq 1$. Using the multiplicative property of $A$ concludes the proof. ■

Now, consider the function

$$B(x) = \sum_{n \leq x} \frac{A(n)}{\sqrt{n}}.$$

We will relate this to an expression in $L(1, \chi)$ to show $L(1, \chi) \neq 0$. Note that

$$B(x) = \sum_{n \leq x} \frac{A(n)}{\sqrt{n}} > \sum_{m^2 \leq x} \frac{A(m^2)}{m} \geq \sum_{m \leq \sqrt{x}} \frac{1}{m}.$$

This last sum is greater than $\log(\sqrt{x}) = \frac{\log x}{2}$. At the same time using the definition of $A(n)$, we have

$$B(x) = \sum_{n \leq x} \frac{A(n)}{\sqrt{n}} = \sum_{lm \leq x} \frac{\chi(l)}{\sqrt{lm}}.$$

This latter sum can be expanded as

$$\sum_{l \leq \sqrt{x}} \chi(l) l^{-\frac{1}{2}} \sum_{m \leq \frac{x}{l}} m^{-\frac{1}{2}} + \sum_{m \leq \sqrt{x}} m^{-\frac{1}{2}} \sum_{\sqrt{x} < l < \frac{x}{m}} \chi(l) l^{-\frac{1}{2}}.$$

In fact, both of these sums are asymptotically equivalent to $x^{\frac{1}{2}} \cdot L(1, \chi) + O(1)$, thus

$$B(x) = 2x^{\frac{1}{2}} \cdot L(1, \chi) + O(1) > \frac{\log x}{2}.$$

Hence, it follows that $L(1, \chi) \neq 0$ (in fact, $L(1, \chi) \geq \frac{\log x}{\sqrt{x}}$).

## 6. Extensions of Dirichlet's Theorem

As one might expect, Dirichlet's Theroem on Arithmetic Progressions can be extended to show that the primes are actually equidistributed in every moduli; in other words, using the statement of the Prime Number Theorem,

$$\#(p \leq x; p \equiv a \pmod{m}, \gcd(a, m) = 1) \sim \frac{1}{\phi(m)} \frac{x}{\log(x)}.$$

Additionally, Chebatorev's Density Theorem can be seen as a generalization of Dirichlet's Theorem, for the $N$th cyclotomic field $K$.

## References

[1] Simon Rubinstein-Salzedo. "Analytic Number Theory."
[2] Noam Elkies. "Primes in arithmetic progressions: Dirichlet characters and L-functions." Math 259: Introduction to Analytic Number Theory.
[3] Anthony Varilly. "Dirichlet's Theorem on Arithmetic Progressions." Rice University.
[4] Paul Garrett. "First proof of $L(1, \chi) \neq 0$". University of Minnesota.
[5] T. M. Apostol (1976). Introduction to analytic number theory. Springer-Verlag.