AN INTROCUTION TO MODULAR FORMS

SIDDHARTH KOTHARI

Definition 0.1. Let $k \in \mathbb{Z}$. A *modular form of weight k for the group* $\mathrm{SL}_2(\mathbb{Z})$ is a function $f : \mathbb{H} \to \mathbb{C}$ such that

- f is holomorphic on $\mathbb{H}$,
- (modularity condition)$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$,
- $|f(\tau)|$ is bounded as $\mathrm{Im}(\tau) \to \infty$.

Remark 0.2. Note that the zero function is a modular form of any weight. Further, observe that there are no non-zero modular forms of odd weight: Set $a = -1 = d$ and $b = 0 = c$ in the modularity condition. Then $f(\tau) = (-1)^k f(\tau) \implies f(\tau) = -f(\tau) \implies f(\tau = 0)$ for all $\tau \in \mathbb{H}$.

This definition poses many questions. Why is the function define only on $\mathbb{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$, the *upper-half plane*? Why is f composed with the function $(a\tau + b)(c\tau + d)^{-1}$, and what does $\mathrm{SL}_2(\mathbb{Z})$ have to do with any of this? And, most importantly, why would one care to learn more about these kind of objects? Where do they fit in the landscape of mathematics? We aim to answer, at least partially, these questions here.

1. The Modularity Condition and Homogeneous Functions on Lattices

We start our discussion with a lattice, which generalizes the 'discreteness' of the relation between $\mathbb{Z}$ to $\mathbb{R}$, in an attempt to 'draw out' requiremnt number 2 from the definition of a modular form.

Definition 1.1. Let ω_1 and ω_2 be complex numbers. Define the *lattice generated* by ω_1 and ω_2, by $\Lambda(\omega_1, \omega_2) := \{a_1\omega_1 + a_2\omega_2 : a_i \in \mathbb{C}\}$. We require ω_1 and ω_2 to be linearly independent, that is, $\omega_1/\omega_2 \notin \mathbb{R}$, so that $\Lambda(\omega_1, \omega_2)$ is not a line. Any set of this form is called a *lattice.*

A trivial way in which we can create another lattice out of a pre-existing one, say Λ, is simply by scaling it by some non-zero complex number λ: $\lambda\Lambda := \{\lambda a : a \in \Lambda\}$. With this in mind, consider a function $F : \mathcal{L} \to \mathbb{C}$, where $\mathcal{L}$ is the set of all lattices. If F is such that $F(\lambda\Lambda) = \lambda^k F(\Lambda)$ for all $\lambda \in \mathbb{C}^\times$ and $\Lambda \in L$ for some integer k, then it does seem plausible to call it a *generalization* of a linear map, since the λ comes out of the ride, albeit by some factor k that may not be one. We call such an F homogeneous (on lattices) of weight k.

For any linearly independent ω_1 and ω_2, we can consider the function $G(\omega_1, \omega_2) := F(\Lambda(\omega_1, \omega_2))$. The homogeneity of F implies that

$$G(\alpha\omega_1, \alpha\omega_2) = F(\Lambda(\alpha\omega_1, \alpha\omega_2)) = F(\alpha\Lambda(\omega_1, \omega_2)) = \alpha^k G(\omega_1, \omega_2).$$

Taking α to be ω_1^{-1}, we have $G(1, \omega_2/\omega_2) = \omega_1^k G(\omega_1, \omega_2) = F(\Lambda(1, \omega_2/\omega_1))$, so G is completely determined by where F sends lattices generated by $\{1, \tau\}$ for some $\tau \in \mathbb{C}$. This allows us to consider a uni-variate function $f(\tau) := G(1, \tau) = F(\Lambda(1, \tau))$.

However, there are constraints on f: given two complex numbers $\tau_1 \neq \tau_2$, we may have $\Lambda(1, \tau_1) = \Lambda(1, \tau_2)$, meaning we have $f(\tau_1) = f(\tau_2)$. And when does this happen?

Proposition 1.2. *Two pairs $(1, \tau_1)$ and $(1, \tau_2)$ are such that $\Lambda(1, \tau_1) = \Lambda(1, \tau_2)$ if and only if there exists integers a, b, c and d such that $ad - bc = 1$ and*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau_1 \\ 1 \end{pmatrix} = \begin{pmatrix} \tau_2 \\ 1 \end{pmatrix}.$$

Indeed, these 'change of basis' like matrices have a special name.

Definition 1.3. The special linear group over $\mathbb{Z}$ of degree 2, denoted by $\mathrm{SL}_2(\mathbb{Z})$, is defined to be the set of 2×2 matrices with integer entries and with determinant 1.

Remark 1.4. $\mathrm{SL}_2(\mathbb{Z})$ forms a group under matrix multiplication.

Therefore, we have

$$\begin{aligned} f(\tau_1) = F(\Lambda(1,\tau_1)) &= F(\Lambda(c\tau_1 + d, a\tau_1 + b)) \\ &= F\left((c\tau_1+d)\Lambda\left(1, \frac{a\tau_1+b}{c\tau_1+d}\right)\right) \\ &= (c\tau_1 + c)^k F\left(\Lambda\left(1, \frac{a\tau_1+b}{c\tau_1+d}\right)\right) \\ &= (c\tau_1+d)^k f\left(\frac{a\tau_1+b}{c\tau_1+d}\right), \end{aligned}$$

meaning that f satisfies the modularity condition!

Remark 1.5. Note that the map $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ defines a left group action on $\mathbb{H}$.

Indeed, the formula

$$\Im\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{(ad-bc)\Im(\tau)}{|c\tau+d|^2},$$

applicable for all $a,b,c,d \in \mathbb{R}$ and $\tau \in \mathbb{C}$ such that $\tau \neq -d/c$, means that $\gamma\tau \in \mathbb{H}$ for all $\tau \in \mathbb{H}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. It can also be shown that $\gamma_1(\gamma_2\tau) = (\gamma_1\gamma_2)\tau$ for $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$ simply by expanding everything out.

Before we end this section, we introduce the first non-trivial modular form.

Definition 1.6. The Eisenstein series of weight k for $k \geq 1$ is a function from $\mathbb{H}$ to $\mathbb{C}$ defined by

$$E_k(\tau) = \sum_{(m,n)\in\{\mathbb{Z}^2-(0,0)\}} \frac{1}{(m+n\tau)^k}.$$

This definition is quite *latticy*. Indeed,

$$E_k(\tau) = \sum_{\omega\in\Lambda(1,\tau)^*} \frac{1}{\omega^k},$$

where $\Lambda(1,\tau)^*$ denotes the non-zero elements of $\Lambda(1,\tau)$. From our previous work, it should not come as a suspire that E_k satisfies the modularity condition. Spieling out the details,

Proof. We have

$$\begin{aligned} E_k\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{\left(m+n\left(\frac{a\tau+b}{c\tau+d}\right)\right)^k} &= \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{(c\tau+d)^k}{(m(c\tau+d)+n(a\tau+b))^k} \\ &= (c\tau+d)^k \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{((md+nb)+\tau(cm+an))^k} \\ &= (c\tau+d)^k \sum_{(m',n')=(m,n)\begin{pmatrix} d & c \\ b & a \end{pmatrix}} \frac{1}{(m'+n'\tau)^k}. \end{aligned}$$

As $ad - bc = 1$, the map from $\mathbb{Z}^2$ to $\mathbb{Z}^2$ defined by $(m,n) \mapsto (m,n)\begin{pmatrix} d & c \\ b & a \end{pmatrix}$ is a bijection, which completes the proof. Eisenstein series satisfy curious identities involving $\zeta(s)$ and $\sigma(n)$. Also, note that $E_4^2 = E_8$ and $E_6E_4 = E_{10}$, and in general, any E_{2k} can be expressed as a polynomial expression in E_4 and E_6, a fact whose generalization we will explore later.

■

2. $q-$series for a modular form

Recall the Fouries series of a function, defined below.

Proposition 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a $2\pi-$periodic function. Then we have that*

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx},$$

where

$$f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx,$$

for all $n \in \mathbb{Z}$.

The key take-away here was to analyze a possibly complex function by breaking it up into several manageable parts. More precisely, we expressed f as a linear combination of the exponential functions, e^{inx}, which is itself a combination of the fundamental, the most basic, periodic functions: $\sin(nx)$ and $\cos(nx)$. It terns out we can do something similar for modular forms (Observe that modular forms have 'a lot' of periodicity built into them. In particular, $f(\tau+1) = f(\tau)$.) as well, as illustrated in the next theorem.

Theorem 2.2. *Let $f : \mathbb{H} \to \mathbb{C}$ be a function such that*

- *f is holomorphic,*
- *$f(\tau+1) = f(\tau)$ for all $\tau \in \mathbb{H}$,*
- *and $|f(\tau)|$ is bounded as $\mathfrak{J}(\tau) \to \infty$.*

Then there exist complex numbers $(a_i)_{i=0}^{\infty}$ such that

$$f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n\tau}.$$

Before going into the formal proof, note that the above series resembles a power series expansion in $e^{2\pi i\tau}$. Thus, we define the function $q(\tau) := e^{2\pi i\tau}$ from $\mathbb{H}$ to the punctured unit disk.

The proof is broken up into a series of lemmas.

Lemma 2.3. *The function defined by $q(\tau) := e^{2\pi i\tau}$ for all $\tau \in \mathbb{H}$ is locally invertible and maps surjectively to the punctured disk $D' := \{z \in \mathbb{C} : 0 < |z| < 1\}$.*

Proof. Write $\tau = x + yi$ where $y > 0$. Then $e^{2\pi i\tau} = e^{2\pi i(x+yi)} = e^{2\pi ix - 2\pi y} = e^{-2\pi y}e^{2\pi ix}$. Thus, $|q(\tau)| = |e^{-2\pi y}| = e^{-2\pi y} \in (0,1)$. Therefore, $q(\tau) \in D'$. Next, being locally invertible is equivalent to having a non-zero derivative throughout a function's domain. We have $q'(\tau) = 2\pi i e^{2\pi i\tau}$, which is clearly non-zero as the complex exponential is never zero. Conversely, a $z \in D'$ can be written as $z' = au$, where $1 > a > 0$ is such that $a = |z|$, and hence can be written in the form $e^{-2\pi y}$ for $y > 0$, and u is a unit complex number, which can also be written as $e^{2\pi ix}$ for some real x. ■

Lemma 2.4. *Let the function $g : D' \to \mathbb{C}$ be defined by $g(q) = f(\tau)$ where $e^{2\pi i\tau} = q$. Then g is well-defined, holomorphic and bounded.*

Next, we compute the $q-$series of the Eisenstein series.

Theorem 2.5. *For $k \geq 4$, we have*

$$E_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n}.$$

To prove this, we assume the Poisson summation formula.

3. The Vector space of Modular forms

Let M_k denote the vector space of all modular forms of weight k of $\mathrm{SL}_2(\mathbb{Z})$.

Lemma 3.1. *If $k < 0$, then $M_k = \{0\}$.*

Theorem 3.2. *Every M_k is finite dimensional, and $\dim M_k$ is $[k/12] + 1$ if $k \not\equiv 2$ modulo 12 and $\dim M_k$ is $[k/12]$ if k is 2 modulo 12.*

Proposition 3.3. *Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function. Next, let S be the set*

$$S_k = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \text{ for all } \tau \in \mathbb{H} \right\}.$$

Then S is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

4. The Weierstrass Elliptic Function

4.1. **Periodic Functions in Real-Land.** When we studied calculus, we were exposed to a complete zoo of functions, things like x^n, $\log x$, e^x, $\sin x$ and $\cos x$. Out of these characters, one set that particularly stood out were the so-called *periodic* functions, like the trigonometric golden duo $\sin x$ and $\cos x$. On the definition level, a function $f : \mathbb{R} \to \mathbb{R}$ is said to be periodic if there exists a $p \in \mathbb{R}$ such that $f(x) = f(x + p)$ for all $x \in \mathbb{R}$. If we choose such a $p > 0$ to be minimal, then we call p the *period*, and more importantly, f is completely determined by the values it takes on in the interval $I = [0, p]$, which is called the fundamental domain for f, in the sense that one can construct the whole of f by duplicating and translating I along the real line.

In general,

Definition 4.1. Let ω_1 and ω_2 be two complex numbers linearly independent over $\mathbb{R}$. Then, the Weierstrass elliptic function corresponding to the lattice $\Lambda(\omega_1, \omega_2)$ generated by ω_1 and ω_2, denoted by $\wp_{\Lambda(\omega_1,\omega_2)}$, is defined by

$$\wp_{\Lambda(\omega_1,\omega_2)}(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda(\omega_1,\omega_2)\setminus\{0\}} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right),$$

for all $z \in \mathbb{C} - \Lambda(\omega_1, \omega_2)$.

Remark 4.2. The sum above is actually finite, that is, it converges. To see this, we rewrite

$$\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} = \frac{\lambda^2 - (z+\lambda)^2}{\lambda^2 (z+\lambda)^2} = -\frac{z(2\lambda + z)}{\lambda^2(z+\lambda)^2}.$$

Now, for a fixed z, we have that as $|\lambda| \to \infty$,

Remark 4.3. Sometimes, we may write $\wp(z; \omega_1, \omega_2)$ to denote $\wp_{\Lambda(\omega_1,\omega_2)}(z)$.

4.2. **Extracting Modular Forms from the Weierstrass Elliptic Function.** The Laurent series expansion for $\wp_{\Lambda(\omega_1,\omega_2)}$ about $z = 0$ (where it has a double pole, as at every other lattice point), has the form

$$\frac{1}{z^2} + \sum_{k=1}^{\infty} c_{2k}(\omega_1, \omega_2) z^{2k},$$

where c_{2k} is a complex-valued function that takes as input two linearly independent complex numbers for each $k \in \mathbb{N}$. It can be easily checked from the definition that $\wp_{\Lambda(\alpha\omega_1,\alpha\omega_2)}(\alpha z) = \alpha^{-2}\wp_{\Lambda(\omega_1,\omega_2)}(z)$ for all $\alpha \in \mathbb{C}^*$. Putting this relationship into the Laurent series, we have

$$\begin{aligned}\wp_{\Lambda(\alpha\omega_1,\alpha\omega_2)}(\alpha z) = \frac{1}{(\alpha z)^2} + \sum_{k=1}^{\infty} c_{2k}(\alpha\omega_1, \alpha\omega_2)(\alpha z)^{2k} &= \alpha^{-2}\left(\frac{1}{z^2} + \sum_{k=1}^{\infty} \alpha^{2k+2} c_{2k}(\alpha\omega_1, \alpha\omega_2) z^{2k}\right) \\ &= \alpha^{-2}\left(\frac{1}{z^2} + \sum_{k=1}^{\infty} c_{2k}(\omega_1, \omega_2) z^{2k}\right),\end{aligned}$$

meaning that $c_{2k+2}(\alpha\omega_1, \alpha\omega_2) = \alpha^{-(2k+2)} c_{2k}(\omega_1, \omega_2)$ for all $\alpha \in \mathbb{C}^*$. This means that c_{2k} is a homogeneous function on lattices of weight $2k + 2$. From our previous work, we know that the function defined by $f_{2k}(\tau) = c_{2k}(1, \tau)$ for all $\tau \in \mathbb{H}$ is a modular form of weight $2k + 2$! We've created a huge number of modular forms of different weights in a jiffy! All that remains to be done is to calculate these coefficients!

Proposition 4.4. *Using the notation above, we have that*

$$c_{2k}(\omega_1, \omega_2) = (2k + 1)E_{2k+2}(\omega_1, \omega_2),$$

where

$$E_n(\omega_1, \omega_2) := \sum_{\lambda \in \Lambda(\omega_1,\omega_2) - \{0\}} \frac{1}{\lambda^k},$$

for all $n \geq 3$.

Proposition 4.5. *The series defined by* $E_n(\omega_1, \omega_2)$ *converges absolutely for all* $n \geq 3$ *and* $\mathbb{R}-$*linearly independent* $\omega_1, \omega_2 \in \mathbb{C}$.

Definition 4.6. Let $\tau \in \mathbb{H}$. We define the *Dedekind eta function* $\eta(\tau)$ by

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q := e^{2\pi i \tau}$.

Definition 4.7. Define the complex valued function P by

$$P(\tau) = (2\pi)^{12} \eta^{24}(\tau),$$

for all $\tau \in \mathbb{H}$.

To show that P is a modular form of weight 12, we need the following proposition that governs how η transforms when it is composed with the map $\tau \mapsto -\frac{1}{\tau}$.

Theorem 4.8. *We have that*

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\eta(\tau),$$

for all $\tau \in \mathbb{H}$.

The proof of this functional equation will be split into several parts. First, we relate η to ϑ using Euler's pentagonal number theorem. Then, we show that ϑ satisfies a particular functional equation to get the result.

Lemma 4.9. *Let a be a positive real number and $x, b \in \mathbb{R}$. Then we have that*

$$\sum_{n=-\infty}^{\infty} e^{-2\pi i (x+n) b} e^{-\pi a (x+n)^2} = \frac{1}{\sqrt{a}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{a}(n+b)^2} e^{2\pi i x n}. \tag{4.1}$$

Proof. Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = e^{-2\pi i b x} e^{-\pi a x^2}$ for all $x \in \mathbb{R}$. Notice that f is a Schwartz function (see the appendix) for $a > 0$, so we can use the Poisson summation formula on f. To do so, we first calculate $\hat{f}(u)$.

We have

$$\begin{aligned}
\hat{f}(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i u x}\, dx &= \int_{-\infty}^{\infty} e^{-\pi a x^2 - 2\pi i b x - 2\pi i u x}\, dx \\
&= \int_{-\infty}^{\infty} e^{-\pi(a x^2 + 2i(u+b)x)}\, dx \\
&= \int_{-\infty}^{\infty} e^{-\pi((x\sqrt{a})^2 + 2i(u+b)x + ((u+b)i/\sqrt{a})^2 - ((u+b)i/\sqrt{a})^2)}\, dx \\
&= e^{-\frac{(u+b)^2\pi}{a}} \int_{-\infty}^{\infty} e^{-\pi(x\sqrt{a} + (u+b)i/\sqrt{a})^2}\, dx.
\end{aligned}$$

Now, set $s = \sqrt{\pi}(x\sqrt{a} + (u+b)i/\sqrt{a})$ so that $dx = \frac{1}{\sqrt{\pi a}} ds$ and

$$\hat{f}(u) = \frac{e^{-\frac{(u+b)^2\pi}{a}}}{\sqrt{\pi a}} \int_{-\infty+(u+b)i/\sqrt{a}}^{\infty+(u+b)i/\sqrt{a}} e^{-s^2}\, ds = \frac{e^{-\frac{u^2\pi}{a}}}{\sqrt{\pi a}} \int_C e^{-z^2}\, dz,$$

where C is the contour defined by $\gamma(t) = t + (u+b)i/\sqrt{a}$ for $t \in \mathbb{R}$. Note that the function e^{-z^2} is an entire function, so $\int_C e^{-z^2}\, dx = \int_{C'} e^{-z^2}\, dz$, where C' is the real axis. The latter is just the famous Gaussian integral, whose value is $\sqrt{\pi}$. Therefore,

$$\hat{f}(u) = \frac{1}{\sqrt{a}} e^{-\frac{(u+b)^2\pi}{a}},$$

for all $u \in \mathbb{R}$. Lastly, we have that

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} e^{-2\pi i(x+n)b} e^{-\pi a(x+n)^2} \quad \text{and} \quad \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi ixn} = \frac{1}{\sqrt{a}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{a}(n+b)^2} e^{2\pi ixn}.$$

Invoking the Poisson summation formula gives the desired result. ■

We can Notice that the multivariate complex valued function defined by

$$G_1(z, w, \tau) := \sum_{n=-\infty}^{\infty} e^{-2\pi i(z+n)w} e^{\pi i\tau(z+n)^2},$$

for $(z, w, \tau) \in \mathbb{C} \times \mathbb{C} \times \mathbb{H}$ is absolutely convergent on compact subsets of its domain, and hence holomorphic on its domain.

Indeed, assume that $|\Re z| \le A_R, |\Im z| \le A_I$, $|\Re w| \le B_R, |\Im w| \le B_I$ and $|\Re\tau| \le C_R, |\Im\tau| \le C_I$. Then we have that

$$\left|e^{-2\pi i(z+n)w} e^{\pi i\tau(z+n)^2}\right| = \left|e^{-2\pi izw} e^{-2\pi iwn} e^{\pi i\tau z^2} e^{2\pi i\tau nz} e^{\pi i\tau n^2}\right| \le e^{2\pi(\Im(zw)+n\Im(w)-n\Im(\tau z))-n^2\pi\Im(\tau)-\pi\Im(\tau z^2)}.$$

Notice that $\Im(zw)$, $\Im(\tau z)$ and $\Im(\tau z^2)$ are bounded. Thus,

$$\sum_{n=-\infty}^{\infty} \left|e^{-2\pi i(z+n)w} e^{\pi i\tau(z+n)^2}\right| \le e^{A'} \sum_{n=-\infty}^{\infty} e^{B'n-C'n^2},$$

where A', B' and C' are positive constants, which can be expressed in terms of the I's and the R's. The fact that G_1 is absolutely convergent now follows as the sum on the right is convergent.

Similarly, the function defined by

$$G_2(z, w, \tau) := (-i\tau)^{-1/2} \sum_{n=-\infty}^{\infty} e^{2\pi izn-\pi i(n+w)^2/\tau},$$

for all $(z, w, \tau) \in \mathbb{C}\times\mathbb{C}\times\mathbb{H}$ is also absolutely convergent and holomorphic. Recall that $G_1(x, b, ia) = G_2(x, b, ia)$ for all $x, b \in \mathbb{R}$ and $a > 0$, meaning that G_1 and G_2 agree on the set $S = \{(m, k, n) : m, k \in \mathbb{R} \text{ and } \Im(n) > 0, \Re(n) = 0\}$ which is a subset of the domain $\mathbb{C} \times \mathbb{C} \times \mathbb{H}$. Therefore, by the multi-variable identity theorem (see appendix), G_1 and G_2 agree everywhere on their domain.

Corollary 4.10. *Let $z, w \in \mathbb{C}$ and $\tau \in \mathbb{H}$. Then we have that*

$$\sum_{n=-\infty}^{\infty} e^{-2\pi i(z+n)w} e^{\pi i\tau(z+n)^2} = (-i\tau)^{-1/2} \sum_{n=-\infty}^{\infty} e^{2\pi izn-\pi i(n+w)^2/\tau}. \tag{4.2}$$

We can use this to prove a related functional equation satisfied by the ϑ function.

Corollary 4.11. *Let $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$. Then we have that*

$$\vartheta\left(z, -\frac{1}{\tau}\right) = \sqrt{\frac{\pi}{i}} e^{\pi i\tau z^2} \vartheta(z\tau, \tau).$$

We now massage the expression for η a bit in order to use equation 4.2.

Lemma 4.12. *We have that*

$$\eta(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi in} e^{3\pi i\tau(n+1/6)^2}.$$

Proof. Recall the pentagonal number theorem, which states that

$$\prod_{n=1}^{\infty}(1 - e^{2\pi in\tau}) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i\tau(3n^2+n)}.$$

Multiplying both sides by $e^{\frac{\pi i\tau}{12}}$ and invoking the definition of η yields

$$\eta(\tau)=\sum_{n=-\infty}^{\infty}(-1)^n e^{\pi i\tau(1/12+3n^2+n)}=\sum_{n=-\infty}^{\infty}(-1)^n e^{3\pi i\tau(n^2+n/3+1/36)}=\sum_{n=-\infty}^{\infty}e^{\pi in}e^{3\pi i\tau(n+1/6)^2},$$

as $(-1)^n=e^{\pi in}$. ■

And now for the main proof!

Proof. Replacing τ with $\tau/3$, setting $w=1/6$ and $z=1/2$ in equation 4.2, we get that

$$\sum_{n=-\infty}^{\infty}e^{\pi in}e^{-3\pi i(n+1/6)/\tau}=(-i\tau/3)^{1/2}e^{-i\pi/6}\sum_{n=-\infty}^{\infty}e^{-\pi in/3}e^{\pi i\tau(1/2+n)^2/3}.$$

Notice that as n runs through the integers, so do $3n-1$, $3n$ and $3n+1$. Therefore,

$$\eta(-1/\tau)(-i\tau/3)^{-1/2}e^{i\pi/6}=\sum_{n=-\infty}^{\infty}e^{-\pi i(3n-1)/3}e^{\pi i\tau(3n-1/2)^2/3}+\sum_{n=-\infty}^{\infty}e^{-\pi in}e^{\pi i\tau(3n+1/2)^2/3}+I,$$

where $I=\sum_{n=-\infty}^{\infty}e^{-\pi i(3n+1)/3}e^{(3n+3/2)^2/3}$.

First, notice that as n runs through the integers, so does $-n-1$. Hence,

$$\begin{aligned}I=\sum_{n=-\infty}^{\infty}e^{-\pi i(3(-n-1)+1)/3}e^{(3(-n-1)+3/2)^2/3}&=e^{2\pi i/3}\sum_{n=-\infty}^{\infty}e^{i\pi n}e^{(-3n-3/2)^2/3}\\&=e^{2\pi i/3}e^{\pi i/3}\sum_{n=-\infty}^{\infty}e^{-i\pi n-i\pi/3}e^{(3n+3/2)^2/3}\\&=-I,\end{aligned}$$

which implies $I=0$. Second, note that for the other part $S=\sum_{n=-\infty}^{\infty}e^{-\pi i(3n-1)/3}e^{\pi i\tau(3n-1/2)^2/3}+\sum_{n=-\infty}^{\infty}e^{-\pi in}e^{\pi i\tau(3n+1/2)^2/3}$ we have,

$$\begin{aligned}S&=e^{i\pi/3}\sum_{n=-\infty}^{\infty}e^{-\pi in}e^{\pi i\tau(3n-1/2)^2/3}+\sum_{n=-\infty}^{\infty}e^{-\pi in}e^{\pi i\tau(3n+1/2)^2/3}\\&=e^{i\pi/3}\sum_{n=-\infty}^{\infty}e^{\pi in}e^{\pi i\tau(-3n-1/2)^2/3}+\sum_{n=-\infty}^{\infty}e^{\pi in}e^{\pi i\tau(3n+1/2)^2/3}\\&=(3/2+i\sqrt{3}/2)\sum_{n=-\infty}^{\infty}e^{\pi in}e^{\pi i\tau(n+1/6)^2}\\&=(3/2+i\sqrt{3}/2)\eta(\tau).\end{aligned}$$

Therefore, $\eta(-1/\tau)=(-i\tau/3)^{1/2}(e^{-\frac{\pi i}{6}})(3/2+i\sqrt{3}/2)\eta(\tau)=(-i\tau)^{1/2}\eta(\tau)$, as $e^{-\pi i/6}(3/2+i\sqrt{3}/2)=\sqrt{3}$, which completes the proof. ■

Lemma 4.13. *We have that*

$$\eta(\tau+1)=e^{\frac{\pi i}{12}}\eta(\tau).$$

Proof. We have that

$$\eta(\tau+1)=e^{\frac{\pi i(\tau+1)}{12}}\prod_{n=1}^{\infty}(1-e^{2\pi in(\tau+1)})=e^{\frac{\pi i}{12}}e^{\frac{\pi i\tau}{12}}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}e^{2\pi in})=e^{\frac{\pi i}{12}}\eta(\tau),$$

as $e^{2\pi in}=1$. ■

Corollary 4.14. *The function P as defined above is a modular form of weight 12.*