THE j-FUNCTION

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Abstract. The j-function is a modular function that arises in various areas of mathematics with many interesting properties. In this expository paper, we first define the modular groups and modular functions. We also express the j-function using the Eisenstein series and prove that the j-function is a modular function. We then explain the relationship between the j-function and the remarkable fact that $e^{\pi\sqrt{163}}$ is about 7.5×10^{-13} away from an integer. Finally, we briefly discuss connections between the j-function and the Chudnovsky algorithm, and the monster group.

1. Modular Groups and Functions

Definition 1.1. The *modular group* $\mathrm{SL}_2(\mathbb{Z})$ is the multiplicative group of 2×2 matrices over $\mathbb{Z}$ with determinant 1.

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}$$

We define an action of SL_2 on $\mathbb{H}$ for some matrix A $\in \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and some $\tau \in \mathbb{H}$ as

$$A\tau = \frac{a\tau + b}{c\tau + d}$$

Definition 1.2. A *modular function* is a function $f : \mathbb{H} \to \mathbb{C}$ such that

(1) f is meromorphic.

(2) For any matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2\mathbb{Z}$, we have

$$f(\alpha\tau) = f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau).$$

(3) For all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, the Fourier expansion of $f(\alpha\tau)$ has only finitely many nonzero coefficients for negative exponents. (Letting $a = b = d = 1$, $c = 0$ in (2) gives $f(\tau + 1) = f(\tau)$, so f has a Fourier expansion.)

Condition (3) is often phrased as "f is meromorphic at the cusps". Consider the standard fundamental domain of the modular group $\mathrm{SL}_2(\mathbb{Z})$, $\Omega = \{z \mid \Im(z) > 0, -\frac{1}{2} < \Re(z) < \frac{1}{2}, |z| > \frac{1}{2}\}$. The region Ω has "cusps" at $-\frac{1}{2}$, $\frac{1}{2}$, and ∞. Condition (3) essentially says that f is meromorphic at ∞, where by ∞ we mean $\tau \to \infty$ inside Ω, i.e. $\Im\tau \to \infty$ and $-\frac{1}{2} \leq \Re\tau \leq \frac{1}{2}$.

Remark 1.3. Modular functions are a type of *modular form.* A modular form of weight k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that f is holomorphic at the cusps (so the Fourier expansion of $f(\alpha\tau)$ has no nonzero coefficients for negative exponents) and $f\left(\frac{a\tau+b}{c\tau+d}\right) =$

$(c\tau+d)^k f(\tau)$ when $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2\mathbb{Z}$. (The second condition is a generalization of condition (2), which is the special case where $k=0$).

2. Eisenstein Series

Definition 2.1. For integers $n > 2$, we define the Eisenstein series of weight n as

$$E_n = \sum_{\substack{(a,b) \in \mathbb{Z}^2 \\ (a,b) \neq (0,0)}} \frac{1}{(a+b\tau)^n}.$$

Eisenstein series satisfy several important properties:

Proposition 2.2. *We have*

1. *The series for* E_n *converges absolutely when* $n > 2$.
2. $E_n = 0$ *when* n *is odd.*
3. $E_n(\tau+1) = E_n(\tau)$
4. $E_n(-1/\tau) = \tau^n E_n(\tau)$.
5. $E_n\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^n E_n(\tau)$ *when* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

Proof. Properties (1) through (4) are not hard to prove. We prove (5), as we will use this result later in the paper. To do so, we use the fact that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

So it suffices to prove (5) for these two matrices. We have

$$E_n\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tau\right) = E_n(\tau+1) = E_n(\tau)$$

by (3), and

$$E_n\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tau\right) = E_n(-1/\tau) = r^n E_n(\tau)$$

by (4), as desired, so we're done. ■

Remark 2.3. The Eisenstein series is an example of a modular form. We may show that Eisenstein series is holomorphic using the fact that the limit of a uniformly convergent sequence of holomorphic functions is holomorphic. Condition (5) is exactly the transformation property of modular forms. Lastly, we may show that E_{2n} is "holomorphic at the cusps" either by explicitly working out the coefficients of the Fourier expansion with clever identities (this can be found in [1]), or by showing that E_n converges as $\tau \to i\infty$.

3. j-Function and its properties

Now, we are ready to define the $j-$function.

Definition 3.1. The *$j-$function* $j : \mathbb{H} \to \mathbb{C}$ is defined as

$$j(\tau) = \frac{1728 g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2}$$

where $g_2 = 60E_4$ and $g_3 = 140E_6$.

We will see later why the coefficient 1728 is necessary.

The modular discriminant $\Delta(\tau)$ can be defined as the denominator i.e $g_2(\tau)^3 - 27g_3(\tau)^2$. The discriminant is a modular form of weight 12. It is also the discriminant of $\wp'^2(z) = 4\wp^3 - g_2\wp - g_3$.

Now we proceed to show that $j(\tau)$ is a modular function.

Proposition 3.2. *The j-function is a modular function.*

Proof. We first need to prove that the j-function is holomorphic. We will not do this in detail, but a sketch of the proof is as follows: $\Delta(\tau)$ is nonvanishing since the roots of $\wp'^2(z) = 4\wp^3 - g_2\wp - g_3$ are distinct, therefore its discriminant is nonzero. Now we only need to prove that $g_2(\tau)$ and $g_3(\tau)$ are holomorphic. We know that $g_2(\tau)$ converges absolutely, so we prove uniform convergence on compact subsets of $\mathbb{H}$ using the fact that $g_2(\tau+1) = g_2(\tau)$. The proof for $g_3(\tau)$ is similar. Thus, $j(\tau)$ is holomorphic.

Next, we prove that modular functions are invariant under the action of $\mathrm{SL}_2(\mathbb{Z}$. Consider an arbitrary $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Using properties of Eisenstein series, we get

$$j\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{1728((c\tau + d)^4 g_2(\tau))^3}{((c\tau + d)^4 g_2(\tau))^3 - 27((c\tau + d)^6 g_2(\tau))^2} = \frac{1728 g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2} = j(\tau)$$

Both modular forms of weight 12 in the numerator and denominator change by the same proportion, so the j-function is invariant under the action of $\mathrm{SL}_2\mathbb{Z}$. ■

We now prove a theorem about the set of rational functions of $j(\tau)$.

Theorem 3.3. *The set of modular functions is the same as the set of rational functions of $j(\tau)$.*

It suffices to prove that all rational functions of $j(\tau)$ are modular functions and all modular functions are rational functions of $j(\tau)$. The first part of the statement is trivial as for any rational function of $j(\tau)$, $f(\tau) = \frac{P(j(\tau))}{Q(j(\tau))}$, f is meromorphic and $f(A\tau) = f(\tau)$ for $A \in \mathrm{SL}_2(\mathbb{Z})$ so its invariant under $\mathrm{SL}_2(\mathbb{Z})$ and thus a modular function.

To prove that all modular functions are rational functions of $j(\tau)$, we first take an arbitrary modular function $f(\tau)$ and get rid of its poles. The function $j(\tau) - j(\tau_0)$ has a zero at τ_0, so multiplying this function by f at the poles with multiplicity will remove all of the finitely many poles of f on the fundamental domain. Specifically, the function

$$g(\tau) = f(\tau) \prod_k (j(\tau) - j(\tau_k))^{m_k}$$

where τ_k is each pole on the fundamental domain of j with multiplicity m_k has no poles on the fundamental domain, and therefore in $\mathbb{H}$. Thus $g(\tau)$ is holomorphic in $\mathbb{H}$ and can be expressed as

$$g(\tau) = a_{-n}q^{-n} + a_{-n+1}q^{-n+1} + \cdots$$

The j-function is periodic, so it has a Fourier expansion. The Fourier expansion of the j-function has only one negative power of q, where $q = e^{2\pi i r}$, i.e q^{-1}. So there exists a polynomial $P(j(\tau))$ such that $h(j(\tau) = g(\tau) - P(j(\tau))$ has no terms with nonpositive power of q. Thus, $h(i\infty) = \lim_{\Im(\tau)\to\infty} f(\tau) = 0$. We can further prove that $h(\mathbb{H}\cup\infty)$ is compact, so by the Maximum Modulus Principle, h is constant and $h(\tau) = 0$. A good way to prove that $h(\mathbb{H}\cup\infty)$ is compact is by considering $f(\tau_k)$, a sequence of points in the image; we just find a subsequence that converges to a point of the form $f(\tau)$ where $\tau \in \mathbb{H}\cup\infty$. More on this can be found in [2]. Therefore, $g(\tau)$ is a polynomial in $j(\tau)$ and so $f(\tau)$ is a rational function in $j(\tau)$.

4. The Fourier expansion of the j-function

Since $j(\tau+1) = j(\tau)$, the j-function is periodic and has a Fourier expansion. We will also refer to Fourier expansions as q-expansions, where $q = e^{2\pi i\tau}$.

This approach is due to [1]. We recall the q-expansions of $E_4(\tau)$ and $E_6(\tau)$:

$$E_4(\tau) = \frac{\pi^4}{45}\left(1 + 240\sum_{r=1}^{\infty}\sigma_3(r)q^r\right),$$
$$E_6(\tau) = \frac{2\pi^6}{945}\left(1 - 504\sum_{r=1}^{\infty}\sigma_5(r)q^r\right).$$

For convenience, we use I to denote additional terms of an arbitrary power series in q with integer coefficients.

$$E_4(\tau) = \frac{\pi^4}{45}\left(1 + 240q + I\right),$$
$$E_6(\tau) = \frac{2\pi^6}{945}\left(1 - 504q + I\right).$$

and

$$g_2(\tau) = \frac{4\pi^4}{3}\left(1 + 240q + I\right),$$
$$g_3(\tau) = \frac{8\pi^6}{27}\left(1 - 504q + I\right).$$

Now we may prove the main theorem of this section.

Theorem 4.1.

$$j(\tau) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c_n q^n = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots$$

where the c_n are integer coefficients and $q = e^{2\pi i\tau}$.

Proof. Using our expansions for g_2 and g_3, we have

$$g_2(\tau)^3 = \frac{64\pi^{12}}{27}(1 + 720q + I)$$

and

$$\Delta(\tau) = \frac{64\pi^{12}}{27}(1728q(1 - 24q + I)).$$

Thus

$$\begin{aligned}
j(\tau) &= 1728 \cdot \frac{(64\pi^{12}/27)(1 + 720q + I)}{(64\pi^{12}/27)(1728q(1 - 24q + I))} \\
j(\tau) &= \frac{1 + 720q + I}{q(1 - 24q + I)} \\
j(\tau) &= \frac{1}{q}(1 + 720q + I)(1 + 24q + I) \\
j(\tau) &= \frac{1}{q}(1 + 744q + I) \\
j(\tau) &= \frac{1}{q} + 744 + I
\end{aligned}$$

which is the desired form. (This is why 1728 is present in the definition of the j-function: to make the coefficients of the q-expansion integers.) We can explicitly compute terms of the expansions of g_2 and g_3 to find the coefficients c_n, which gives

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots .$$

■

Remark 4.2. We have $c_n \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{\frac{3}{4}}}$ as $n \to \infty$, as proved in [3].

5. Why $e^{\pi\sqrt{163}}$ is almost an integer

We unfortunately must black-box the meat of this result, as it is quite technical. A proof can be found in [2].

Theorem 5.1. *We have*

$$j\left(\frac{1 + \sqrt{-163}}{2}\right) = -640320^2.$$

Note that if $\tau = \frac{1+\sqrt{-163}}{2}$, then $q = e^{2\pi i((1+\sqrt{-163})/2)} = -e^{-\pi\sqrt{163}}$. So we have

$$-640320^2 = -e^{\pi\sqrt{163}} + 744 + \sum_{n=1}^{\infty} c_n q^n$$

and thus

$$e^{\pi\sqrt{163}} = 640320^2 + 744 \sum_{n=1}^{\infty} c_n q^n.$$

But $\sum_{n=1}^{\infty} c_n q^n$ is very small, since c_n grows subexponentially and $q = e^{-\pi\sqrt{163}} \approx 3.808980937 \times 10^{-18}$ is very small. Thus $e^{\pi\sqrt{163}}$ is very close to an integer.

Proving this theorem requires a significant amount of abstract algebra, including Galois theory, and is beyond the scope of this paper. A proof can be found in [2]. We remark that it relies on the fact that 163 is a (in fact, the largest) *Heegner number*.

Definition 5.2. A *Heegner number* is a square-free integer d such that the ring of algebraic integers $\mathbb{Q}[\sqrt{-d}]$ is a unique factorization domain.

The other Heegner numbers are 1, 2, 3, 7, 11, 19, 43, and 67. As a result, similar near-equalities can be obtained from 43 and 67:

$$e^{\pi\sqrt{43}} = 884736743.999777466\ldots \approx 960^3 + 744$$

and

$$e^{\pi\sqrt{67}} = 147197952743.999998662454\ldots \approx 5280^3 + 744.$$

The smaller Heegner numbers do not work as well, as $e^{\pi\sqrt{163}}$ is so close to an integer in part because $e^{-\pi\sqrt{163}}$ is so small, and replacing 163 with a significantly smaller number results in a larger difference with the nearest integer.

6. Connections

The *Chudnovsky algorithm* for computing π, used for the main computation of all recent records (including the computation of 3×10^{14} digits on April 2nd, 2025), is based on the following rapidly convergent formula:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(545140134k + 13591409)}{(3k)!(k!)^3 (640320)^{3k+3/2}}$$

Note the appearance of $j\left(\frac{1+i\sqrt{-163}}{2}\right) = 640320^3$—this is not a coincidence!

The j-function is one half of *monstrous moonshine*, the surprising connection between the j-function and the *monster group* M, the largest simple sporadic group. Take a look at sequence A001379 in the OEIS, "Degrees of irreducible representations of Monster group M.":

$$1, 196883, 21296876, 842609326, \ldots$$

Now let's see a few terms of the q-expansion for the j-function:

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \ldots$$

Letting $r_1 = 1, r_2 = 196883, r_3 = 21296876$, and $r_4 = 842609326$, we have the remarkable relationships

$$\begin{aligned} 1 &= r_1, \\ 196884 &= r_1 + r_2, \\ 21493760 &= r_1 + r_2 + r_3, \\ 864299970 &= 2r_1 + 2r_2 + r_3 + r_4. \end{aligned}$$

Similar nontrivial relationships can be observed in further terms. This connection was first observed in 1979 by John McCay. In 1992, Richard Borcherds won the Fields Medal in large part for explaining the connection by proving the Conway-Norton moonshine conjectures.

References

[1] Tom A. Apostol. *Modular functions and Dirichlet series in number theory*. Springer, 1976.

[2] David A Cox. *Primes of the Form x2+ ny2: Fermat, Class Field Theory, and Complex Multiplication. with Solutions*. Vol. 387. American Mathematical Soc., 2022.

[3] Hans Petersson. "Über die Entwicklungskoeffizienten der automorphen Formen". In: (1932).