QUATERNIONIC ANALYSIS

NIKHIL REDDY AND SAMARTH DAS

Abstract. In this paper we discuss extending complex analysis to quaternions, a four dimensional analogue of complex numbers. We explore natural generalizations of differentiability, and eventually arrive at the notion of a regular function, which allows us to extend the Cauchy Integral theorem and other results to quaternions.

1. Introduction

On Monday, October 16, 1843, a carving was made on the Brougham Bridge in Dublin, Ireland by the mathematician William Rowan Hamilton. It read

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

This was the fundamental theorem defining the quaternions $\mathbb{H}$. It was fundamental in the development of four-dimensional analytical systems and spawned numerous other hypercomplex field discoveries. Due to this, the development of complex analysis as a rigorous mathematical system in both education and research generated an interest in doing something similar for quaternions. We will discuss some of these topics in this paper.

2. Quaternion Primer

Definition 2.1. If $a, b, c, d \in \mathbb{R}$, then $q = a + bi + cj + dk$ is called a quaternion, where i, j, k are basis vectors in four-dimensional space, along with 1. The space of all quaternions is denoted by $\mathbb{H}$.

It's easy to see that we add quaternions term wise. For multiplication, we simply distribute each term and use the rule in the introduction. Because of this, however, multiplication with quaternions is not necessarily commutative.

Example. Consider $q_1 = 1 + i + j + k$ and $q_2 = 2 + i - j + 2k$. Then

$$q_1 q_2 = (1 + i + j + k)(2 + i - j + 2k) = 6i + 2k,$$

while

$$q_2 q_1 = (2 + i - j + 2k)(1 + i + j + k) = 2j + 6k.$$

Similar to complex numbers, quaternions also have a conjugation operation:

$$\overline{q} = \overline{a + bi + cj + dk} = a - bi - cj - dk.$$

Again similarly to complex numbers, this allows us to define a modulus:

$$|q| = \sqrt{q\overline{q}} = \sqrt{\overline{q}q} = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

From this, we're able to see that the every quaternion has an inverse:

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$$q^{-1} = \frac{\overline{q}}{|q|^2}.$$

3. Differentiability

From complex analysis, a function $f : \mathbb{C} \to \mathbb{C}$ is differentiable at a point z if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. We can define differentiability for quaternionic functions similarly, although we have to be careful about non-commutativity.

Definition 3.1. A function $f : \mathbb{H} \to \mathbb{H}$ is left quaternion differentiable at q if the limit

$$\lim_{h \to 0} h^{-1}[f(q+h) - f(q)]$$

exists. Similarly, a function $f : \mathbb{H} \to \mathbb{H}$ is right quaternion differentiable at q if the limit

$$\lim_{h \to 0} [f(q+h) - f(q)]h^{-1}$$

exists.

Example. The function $f(q) = q$ is both left and right differentiability with derivative 1.

Example. The function $\mathbf{i}q + 1$ is right differentiable, but not left differentiable.

In fact, we have the following theorem:

Theorem 3.2. *Let $f : U \to \mathbb{H}$ be quaternion differentiable on the left, where U is a connected open subset of $\mathbb{H}$. Then there are constants $a, b \in \mathbb{H}$ such that*

$$f(q) = a + qb$$

for all $q \in U$.

Proof. the proof is at `https://www.google.com/url?sa=t&rct=j&q=&esrc=s&source=web&cd=&cad=rja&uact=8&ved=2ahUKEwjr2c3x59-NAxVF_8kDHfi3PYkQFnoECCEQAQ&url=https%3A%2F%2Fwww.diva-portal.org%2Fsmash%2Fget%2Fdiva2%3A1637307%2FFULLTEXT01.pdf&usg=AOvVaw1867aSpkd6Z76gcKDackwA&opi=89978449`

Unfortunately, this means that the natural extension of differentiability from complex analysis isn't very interesting in the quaternion setting. However, we can extend another definition of differentiability, namely the Cauchy-Riemann equations.

In order for a complex function $f(z) = u(x, y) + iv(x, y)$ to be differentiable, the two equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

need to hold. Note that we can write this more succinctly as

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0.$$

Thus we again have a natural generalization to quaternions.

Definition 3.3. A function $f : \mathbb{H} \to \mathbb{H}$ is left regular if

$$\frac{\partial f}{\partial t} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = 0.$$

A function is right regular if

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k = 0.$$

We let $\overline{\partial}_l$ denote the operator

$$\frac{1}{4}\left(\frac{\partial}{\partial t} + i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)$$

and we let $\overline{\partial}_r$ denote the operator

$$\frac{1}{4}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right).$$

Example. Let $f(q) = q + ix + jy + kz$. Then

$$\overline{\partial}_l f = \frac{1}{4}(1 + ii + jj + kk) = \frac{1}{4}(1 - 1 - 1 - 1) = -\frac{1}{2} \neq 0,$$

so f is not left regular.

From the definition of regular, we have the following two assertions:

Proposition 3.4. *If $f, g : \mathbb{H} \to \mathbb{H}$ are left regular functions, then $f + g$ is also left regular.*

Proof. This simply follows from the linearity of differentiation, since we have

$$\overline{\partial}_l(f + g) = \overline{\partial}_l f + \overline{\partial}_l g = 0 + 0 = 0.$$

Proposition 3.5. *If $f : \mathbb{H} \to \mathbb{H}$ is left regular and $a \in \mathbb{H}$, then fa is left regular.*

Proof. We have

$$\overline{\partial}_l(fa) = \frac{1}{4}\left(\frac{\partial(fa)}{\partial t} + i\frac{\partial(fa)}{\partial x} + j\frac{\partial(fa)}{\partial y} + k\frac{\partial(fa)}{\partial z}\right) = \frac{1}{4}\left(\frac{\partial f}{\partial t} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right)a = \overline{\partial}_l(f)a = 0.$$

4. Quaternion-Valued Forms

Definition 4.1. If $f : \mathbb{H} \to \mathbb{H}$ is a differentiable function, then

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

Theorem 4.2. *Let $f : \mathbb{H} \to \mathbb{H}$ be a differentiable function. Then f is left regular at q if an only if*

$$Dq \wedge df = 0,$$

where Dq is the form

$$dx \wedge dy \wedge dz - idt \wedge dy \wedge dz - jdt \wedge dz \wedge dx - kdt \wedge dx \wedge dy.$$

Proof. Distribute the wedge products and expand, factor what's left to obtain

$$-\left(\frac{\partial}{\partial t}+i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(dt\wedge dx\wedge dy\wedge dz)=-4\overline{\partial}_l f(dt\wedge dx\wedge dy\wedge dz).$$

For the full calculation, see `https://www.google.com/url?sa=t&rct=j&q=&esrc=s&source=web&cd=&cad=rja&uact=8&ved=2ahUKEwjr2c3x59-NAxVF_8kDHfi3PYkQFnoECCEQAQ&url=https%3A%2F%2Fwww.diva-portal.org%2Fsmash%2Fget%2Fdiva2%3A1637307%2FFULLTEXT01.pdf&usg=AOvVaw1867aSpkd6Z76gcKDackwA&opi=89978449`

5. Integral Theorem and Series Results

Theorem 5.1 *For a quaternionic function $F : H \to H$ of quaternionic variable q, we have*

$$\int_{\partial\Omega}[dS]\overline{q}=\int_{\Omega}dV.$$

Proof. Recall that the definition of $\mathbb{Q}q$ is

$$\mathbb{Q}q=q_i\frac{\partial}{\partial x_i}q=q_1\frac{\partial q}{\partial x_1}+q_2\frac{\partial q}{\partial x_2}+q_3\frac{\partial q}{\partial x_3}+q_4\frac{\partial q}{\partial x_4},$$

or the outwardly directed surface element of Ω. We then define $[dS](Q_1, -Q_2, -Q_3, -Q_4)$, a row vector with the components of $\mathbb{Q}q$. Let M be the 4×4 matrix representation of F. Then, by the divergence theorem, we have

$$\int_{\partial\Omega}[dS]\overline{M}=\int_{\Omega}\text{div }M\,dV.$$

Because of the definition of divergence, we also have

$$\text{div }M=0\Rightarrow\int_{\Omega}\text{div }M=0,$$

proving the theorem. ■

Theorem 5.2 (Cauchy's theorem). *If f is regular on every point of a 4-parallelepiped C, then*

$$\int_{\partial C}[dS]f=0.$$

Proof. The proof can be found in `https://dougsweetser.github.io/Q/Stuff/pdfs/deavours.pdf` ■

This theorem says: If f is a right-regular function and is a left-regular function, we have that

$$\int_{\partial\Omega}[dS]q=0.$$

Now that we have developed enough context and prerequisites, we come to the last main theorem of our paper, the Cauchy-Fueter integral formula, providing an analogue (but more powerful) of Cauchy's integral formula from complex function theory.

Theorem 5.3 (Cauchy-Fueter integral formula). *If F is regular and sufficiently differentiable on every point of the hypersurface $\partial\Omega$ and q_0 is a point within $\partial\Omega$, then*

$$F(q_0)=\frac{1}{8\pi^2}\int_{\partial\Omega}\frac{F(q)}{|q-q_0|^4}(q-q_0)^{-1}dS.$$

To prove this, we first establish another result:

Theorem 5.5. *For a hypersurface $\partial\Omega$ in $\mathbb{R}^4$ with $q \in \partial\Omega$, we have*

$$\int_{\partial\Omega} (q - q_0)^{-1}\overline{\mathbb{Q}}q = -8\pi^2.$$

Proof. Note that since translation preserves regularity, we need only prove this result for $q_0 = 0$, and later containing 0. Because $\overline{\mathbb{Q}}q$ is regular except at 0, by Theorem 5.3 we have

$$\int_{\partial\Omega} (q - q_0)^{-1}\overline{\mathbb{Q}}q = \int_{\Omega} \mathbb{Q}(q^{-1})\, dV.$$

This is easily calculated to be $-8\pi^2$ using the fact that $q^{-1} = \frac{\bar{q}}{|q|^2}$, since $|q| = 1$ in our integral.

For the unit sphere in $\mathbb{R}^4$ (4-dimensional Euclidean space), we have

$$\int_{S^3} q^{-1}\overline{\mathbb{Q}}q = \int_{S^3} -8\pi^2 dS.$$

See `https://dougsweetser.github.io/Q/Stuff/pdfs/Quaternionic-analysis-memo.pdf` for details. Putting this all together, we have

$$\int_{\partial\Omega} (q - q_0)^{-1}\overline{\mathbb{Q}}q = -8\pi^2. \quad \blacksquare$$

With this step proven, we proceed to the proof of the Cauchy-Fueter type integral formula (5.3).

Proof. Using Theorem 5.5 again, we see that because F and $\mathbb{Q}(q^{-1})$ are regular in the region between $|q - q_0| = \epsilon$ (for sufficiently small epsilon) and $\partial\Omega$, we have

$$\int_{\partial\Omega} \frac{F(q)}{|q - q_0|^4}(q - q_0)^{-1} dS = \int_{|q-q_0|=\epsilon} \frac{F(q)}{|q - q_0|^4}(q - q_0)^{-1} dS.$$

Then, $\overline{\mathbb{Q}}q$ can be found by re-modifying the expression as:

$$\overline{\mathbb{Q}}q = (q - q_0)^{-1} dS.$$

Our function F is to be sufficiently differentiable (we must check that $|q - q_0| > 0$), Then, we evaluate

$$\frac{1}{8\pi^2}\int_{|q-q_0|=\epsilon} \frac{F(q)}{|q - q_0|^4}(q - q_0)^{-1} dS = \lim_{\epsilon\to 0}\frac{1}{8\pi^2}\int_{|q-q_0|=\epsilon} \frac{F(q)}{\epsilon^4}\frac{\overline{q - q_0}}{\epsilon^2} dS$$
$$= \frac{1}{8\pi^2}\lim_{\epsilon\to 0}\frac{1}{\epsilon^6}\int_{|q-q_0|=\epsilon} F(q)\overline{(q - q_0)}\, dS.$$

Assuming F to be sufficiently differentiable (again), this leads to

$$= \frac{1}{8\pi^2}\lim_{\epsilon\to 0}(F(q_0) + O(\epsilon))\int_{|q-q_0|=\epsilon} \overline{(q - q_0)}\, dS = F(q_0).$$

This completes the proof. $\blacksquare$