HARDY-RAMANUJAN-LITTLEWOOD CIRCLE METHOD SUMMARY

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Abstract. We introduce the circle method and sketch the steps of the process in the case of Goldbach's Weak conjecture and partitions.

1. Introduction

The circle method was initially formulated by Hardy and Ramanujan when they were studying partitions. It was later refined by Hardy and Littlewood.

The general class of problems we will be considering are additive problems. Let A_i for $i \in 1, \dots, k$ be infinite sets of numbers. For example, A_i could be the set of perfect squares or primes. The goal is to determine whether $n \in \mathbb{N}$ can be written as $a_1 + \cdots + a_k$ for $a_i \in A_i$ and find how many ways n can be expressed in this way.

Additive problems covers a broad class of problems. Consider Goldbach's Conjecture of whether all even numbers can be expressed as the sum of two primes. In this case A_1 and A_2 are both the set of primes. Another classic problem is Waring's problem which asks what is the smallest k such that n can be written as a sum of k perfect mth powers. In the language of additive problems, we have A_i is the set of mth powers.

In Section 2, we describe the overarching steps of the circle method. In Section 3, we apply these steps to Goldbach's weak conjecture in a proof sketch. In Section 4, we apply ideas from the circle method to get a formula for the number of partitions of any integer. This proof uses the same idea of a contour integral, but doesn't explicitly use ideas like major and minor arcs.

2. Circle Method Process

Let $e(x) = \exp(2ix)$. First, we convert the additive problem into generating functions by expressing the elements of A_i as

$$F_{A_i}(x) = \sum_{a_i \in A_i} e(a_i x).$$

Notice that

$$F(x) = F_{A_1}(x) \cdots F_{A_k}(x) = \sum_{n=1}^{\infty} r(n,k) e(nx),$$

where $r(n,k)$ is the number of ways n can be expressed as $a_1 + \cdots + a_k$ for $a_i \in A_i$.

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Example 2.1. Let $\mathbb{N}$ be the set of naturals. We then have

$$F(x) = F_{\mathbb{N}}(x) \cdot F_{\mathbb{N}}(x) = \left(\sum_{a=1}^{\infty} e(ax)\right)^2 = \sum_{a=2}^{\infty}(a-1)e(ax),$$

and each coefficient, $a-1$, is the number of ways to write a as the sum of two natural numbers.

Recall that

$$\int_0^1 e(nx)dx = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}.$$

It follows that

$$\int_0^1 F(x)e(-nx)\,dx = \sum_{m=1}^{\infty}\int_0^1 r(m,k)e(mx)e(-nx)\,dx = r(n,k).$$

In reality, convergence issues in the power series makes this a bit more complicated. The way this is resolved is by truncating the generating functions. Let $A_i(N)$ be the set of elements of A_i less than or equal to N. Define $F_{A_i(N)}(x)$ analogously to how $F_{A_i}(x)$ was defined. Define $F_N(x)$ as the product of the $F_{A_i(N)}(x)$. Notice that for $n \leq N$, it is pointless to consider elements of A_i which are larger than N as we want the elements to sum to n. It follows that $F(x)$ and $F_N(x)$ agree for the first N terms, so

$$\int_0^1 F_N(x)e(-nx)\,dx = r(n,k),$$

removing convergence issues. All that is left to do is approximate this integral to find $r(n,k)$. We do this by splitting $[0,1)$ into sections called major arcs $\mathbf{M}$ and sections called minor arcs $\mathbf{m}$ and integrating $F_N(x)e(-nx)$ over the $\mathbf{M}$ and $\mathbf{m}$ separately such that $F_N(x)$ is small on $\mathbf{m}$ and large on $\mathbf{M}$. On the major arcs, we integrate $F_N(x)$ to get an asymptotic formula for $r(n,k)$, and on the minor arcs, we bound the integral to show it has a lower order in terms of N than the integral over the major arcs.

3. Proof Sketch of Goldbach's Weak Conjecture

In this section we provide a high level sketch of the following theorem of Vinogradov closely following [2].

Theorem 3.1 (Vinogradov)**.** *Any sufficiently large number can be written as the sum of* 3 *primes.*

Let $F_N(x) = \sum_{p\leq N} e(px)\log p$. We add the $\log p$ weights facilitate the asymptotic calculations. This doesn't change the purpose of $F_N(x)$ though as $(F_N(x))^k = \sum_{n=1}^{\infty} r(n,k)e(nx)$, where

$$r(n,k) = \sum_{\substack{p_1+\cdots+p_k=n \\ p_i\leq N}} \log p_1 \cdots \log p_k,$$

and if $r(n,k)$ is positive, there is a way of writing n as a sum of primes.

In order to find the major and minor arcs, we now need to determine when $|F_N(x)|$ takes on large values and small values. To see this, we first need to find an average value for $|F_N(x)|$ on the

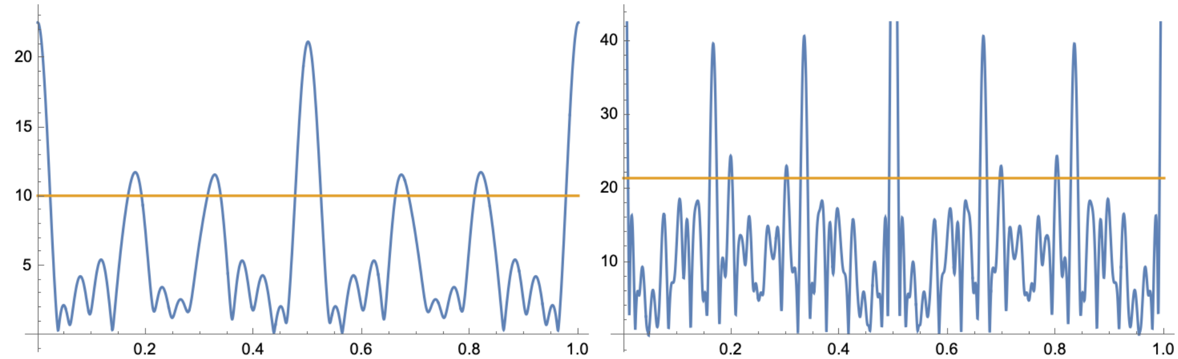

FIGURE 1. Plot of $F_N(x)$ and the line $\sqrt{N \log N}$ for $N = 30$ and 100.

unit circle in the complex plane. It is difficult to do this though, so we instead find the average value of $|F_N(x)|^2$ and take the square root of the result to approximate the average of $|F_N(x)|$.

Lemma 3.2. *The average value of* $|F_N(x)|^2$ *is* $N \log N + o(N \log N)$.

Proof. Using the fact that $|F_N(x)|^2 = F_N(X)\overline{F_N(X)} = F_N(x)F_N(-x)$

$$\int_0^1 |F_N(x)|^2 \, dx = \int_0^1 \sum_{p \le N} e(px) \log p \sum_{q \le N} e(-qx) \log q \, dx$$
$$= \sum_{p,q \le N} \log p \log q \int_0^1 e((p-q)x) \, dx = \sum_{p \le N} \log^2 p,$$

as the integral of $e((p-q)x)$ is 1 if $p = q$ and 0 otherwise. Through Abel summation, it can be shown $\sum_{p \le N} \log^2 p = N \log N + o(N \log N)$. ∎

3.1. **Major Arcs.** Consider Figure 1. The x for which $F_N(x)$ is large seem to be concentrated around fractions with small denominator like $1/2$ and $1/3$. Let us verify this fact. Let $B > 0$ and $Q = \log^B N \ll N$ and fix $q \le Q$ and $a \le q$ relatively prime to q. We then have that

$$F_N\left(\frac{a}{q}\right) = \sum_{p \le N} e\left(\frac{ap}{q}\right) \log p = \sum_{r=1}^{q} \sum_{\substack{p \equiv r \pmod q \\ p \le N}} e\left(\frac{ap}{q}\right) \log p = \sum_{r=1}^{q} e\left(\frac{ar}{q}\right) \sum_{\substack{p \equiv r \pmod q \\ p \le N}} \log p, \tag{3.1}$$

where the last equality comes from the fact that $e(x) = e(x+1)$, so all p equivalent modulo q yield the same value of $e(ap/q)$. By the Prime Number Theorem $\sum_{p \le x} \log p \sim x$. Therefore, it appears that restricting the sum to iterate of $p \equiv r \pmod q$ should just divide the result of the Prime Number Theorem by a factor of $\varphi(q)$ as the primes are distributed roughly even modulo q. The following theorem shows this is indeed true.

Theorem 3.3 (Siegel-Walfisz)**.** *Let $C, B > 0$ and let a and q be relatively prime. Then*

$$\sum_{\substack{p \equiv r \pmod q \\ p \leq x}} \log p = \frac{x}{\varphi(q)} + O\left(\frac{x}{\log^C x}\right),$$

where $\varphi(q)$ is Euler's totient function, $q \leq \log^B x$, and the constant for the O-notation is independent of $r, q,$ and x.

Thus let us approximate the sum of $\log p$ at the end of Equation (3.1) with $N/\varphi(q)$. Because this sum does not depend on r, it can be factored out of the sum. The remaining sum is

$$\sum_{\substack{r=1 \\ (r,q)=1}}^{q} e\left(\frac{ar}{q}\right) = \sum_{\substack{r=1 \\ (r,q)=1}}^{q} e\left(\frac{r}{q}\right),$$

which has large magnitude for small q but as q increases the roots of unity $e(r/q)$ become more dispersed throughout the unit circle, causing them to cancel each other. But for small q, the sum becomes some large constant K (which needs to be bounded more precisely in terms of q) and $F_N(a/q)$ is roughly $KN/\varphi(q)$ which is much greater than $\sqrt{N \log N}$. Plugging the Siegel-Walfisz theorem into equations is the more rigorous way of determining the magnitude of $F_N(x)$ in the major arcs.

Let $B > 0$ and let $Q = \log^B N$. Define the following set

$$\mathbf{M}_{a,q} = \left\{x \in [0,1]|\colon \left|x - \frac{a}{q}\right| < \frac{Q}{N}\right\},$$

where the absolute value wraps around if needed, e.g., 0.9 and 0.1 have an absolute difference of 0.2. Let the major arc $\mathbf{M}$ be the union of $\mathbf{M}_{a,q}$ for all $q \leq Q$ and $a \leq q$ relatively prime to q. These arcs are chosen to simplify asymptotics and because they are centered around fractions with small denominator which are the areas where $F_N(x)$ is large. For large enough N, these arcs are disjoint. When we integrate $F_N(x)^3 e(-Nx)$ to find $r(N, 3)$ which is related to the number of ways of expressing N as the sum of 3 primes modulo the $\log p$ weights, we will find

$$\int_{\mathbf{M}} F_N(x)^3 e(-Nx)\, dx \approx \left[2Q^3 \sum_{q=1}^{Q} \frac{1}{\varphi(q)^3} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\sum_{\substack{r=1 \\ (r,q)=1}} e\left(\frac{ar}{q}\right)\right)^3 e\left(\frac{-Na}{q}\right)\right] N^2,$$

through the Siegel-Walfisz theorem. It remains to lower bound the coefficient of N^2 as if it is too small, the integral over the major arcs could be canceled by the integral over the minor arcs.

3.2. **Minor Arcs.** Using $\mathbf{m} = [0,1] \setminus \mathbf{M}$, we can bound the contribution of the integral over the minor arcs.

$$\left|\int_{\mathbf{m}} F_N(x)^3 e(-Nx)\, dx\right| \leq \int_0^1 |F_N(x)^3|\, dx \leq (\max_{x \in \mathbf{m}} |F_N(x)|) \int_0^1 |F_N(x)^2|\, dx = (\max_{x \in \mathbf{m}} |F_N(x)|) N \log N.$$

Integrating over $[0,1]$ instead of over $\mathbf{m}$ does not change the asymptotics by very much as $\mathbf{m}$ cover the majority of $[0,1]$ already. We already have factored out the $\max_{x \in \mathbf{m}} |F_N(x)|$ term which accounts for the lower magnitude of $F_N(x)$ on $\mathbf{m}$. Vinogradov showed $\max_{x\in\mathbf{m}}|F_N(x)|\ll \frac{N}{\log^D N}$, which shows that the minor arcs cannot possibly cancel the major arcs which are on the order of N^2 for large N. It follows that any large enough N can be written as the sum of three primes.

4. Partitions

We now provide a more complete use of the circle method in determining a formula for partitions closely following [1], i.e., the number of ways of a number can be expressed as a sum of positive integers disregarding the order of the sum.

Example 4.1. The number of partitions for 4 is 5 : we have 4, $1+3$, $2+2$, $1+1+2$, $1+1+1+1$.

The generating function for the partitions is

$$F(x) = \prod_{n=1}^{\infty} \frac{1}{1-e(nx)} = 1 + \sum_{m=1}^{\infty} p(n)e(mx),$$

where $p(n)$ is the number of ways of partitions for n.

This proof will be split into three main parts. First, we will we will justify a different path of integration using Ford circles and Farey fractions. This will lead to the Dedekind η function which is roughly the reciprocal of $F(x)$. Finally, we will use the properties of the Dedekind η function to bound terms in the circle method integral and reduce it to a main term whose integral can be found through the theory of Bessel functions.

4.1. Path of Integration.

A key idea in the circle method is to find paths of integration where the function can be easily approximated. Around singularities, functions generally have one dominant term which can be used to approximate them. The singularities of $F(x)$ are at all rational numbers. Therefore, we want a path of integration around which moves around these rational numbers. The rational numbers with smallest denominator are also the most significant as around a rational number a/b, the term $(1-e(bx))^{-1}$ is large but if b increased, as x shifts away from a/b, the term $(1-e(bx))^{-1}$ decreases faster than if b was smaller. This leads us to Farey fractions and Ford circles.

Define the sequence F_N to be a list of rational numbers in $[0,1]$ whose denominator in simplified form is less than or equal to N in order of smallest to largest.

Example 4.2. We have the following Farey sequences

- $F_1 = \frac{0}{1}, \frac{1}{1}$.
- $F_2 = \frac{0}{1}, \frac{1}{2}, \frac{1}{1}$.
- $F_3 = \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}$.
- $F_4 = \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}$.
- $F_5 = \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$.

Define the mediant of two fractions a/b and c/d to be $(a+c)/(b+d)$. Notice that the mediant of two rational numbers always lies in between the two rational numbers. We also have the following proposition.

Proposition 4.3. *Given $0 \leq a/b < c/d \leq 1$ and $bc - ad = 1$, the fractions a/b and c/d are consecutive in F_N if and only if* $\max(b, d) \leq n \leq b + d - 1$.

Proof. The lower bound on n follows from the fact that for $n < \max(b, d)$ at least one of a/b and c/d is not in F_N. For the upper bound, assume h/k lies between a/b and c/d. Then $bh - ak \geq 1$ and $ck - dh \geq 1$. We also have that

$$k = (bc - ad)k = b(ck - dh) + d(bh - ak) \geq b + d.$$

Therefore for $n \leq b + d - 1$, there is no h/k between a/b and c/d. When $n = b + d$ though, the mediant of a/b and c/d lies between them in F_N. ∎

As shown in the example above, consecutive Farey fractions a/b and c/d satisfy $bc - ad = 1$. This can be shown through induction as from the previous proposition, the first fraction between a/b and c/d if $bc - ad = 1$ is $(a + c)/(b + d)$, and $b(a + c) - a(b + d) = 1$ and $(b + d)c - (a + c)d = 1$. Therefore, for all F_N consecutive Farey fractions satisfy $bc - ad = 1$.

Farey fractions lead to Ford circles. Define the circle $C(h, k)$ in the upper halfplane to be the circle tangent to the real axis at h/k with radius $1/(2k^2)$. See Figure 2 for an image of Ford circles. Ford circles are closely related fo Farey fractions because of the following proposition.

Proposition 4.4. *Two Ford circles $C(a, b)$ and $C(c, d)$ are tangent if and only if $bc - ad = \pm 1$.*

Proof. The Ford circles are tangent if the distance between their centers is the sum of their radii. The square of the distance between the centers of $C(a, b)$ and $C(c, d)$ is

$$D^2 = \left(\frac{c}{d} - \frac{a}{b}\right)^2 + \left(\frac{1}{2d^2} - \frac{1}{2b^2}\right).$$

Solving, we eventually see that $D^2 = (1/(2b)^2 + 1/(2d)^2)^2$ yields

$$\frac{(bc - ad)^2 - 1}{b^2 d^2} = 0,$$

so $bc - ad = \pm 1$. ∎

Therefore, the sequence of Farey fractions F_N will correspond to a sequence of tangent Ford circles. We can find an explicit form for the points of tangency between Ford circles $C(a, b)$ and $C(c, d)$ with $bc - ad = 1$. We omit the proof of the following proposition.

Proposition 4.5. *The tangent point between $C(a, b)$ and $C(c, d)$ if $bc - ad = 1$ is*

$$\alpha = \frac{c}{d} - \frac{b}{d(b^2 + d^2)} + \frac{i}{b^2 + d^2} = \frac{a}{b} + \frac{d}{b(b^2 + d^2)} + \frac{i}{b^2 + d^2}.$$

We can now describe the path of integration $P(N)$. Consider consecutive Farey fractions $a/b, h/k$, and c/d in F_N. Let $C(a, b)$ and $C(c, d)$ be tangent to $C(h, k)$ at tangent points $\alpha_1(h, k)$ and $\alpha_2(h, k)$ respectively. We have that $\alpha_1(h, k)$ and $\alpha_2(h, k)$ split $C(h, k)$ into a upper arc and lower arc. Let the upper arc be $\gamma_{h,k}$. The path of integration $P(N)$ is the union of these arcs for all Farey fractions in F_N. Let $\gamma_{0,1}$ only consist of points with nonnegative real part and $\gamma_{1,1}$ only consist of points with real part less than or equal to 1 so that $P(N)$ is a path from i to $i + 1$. Notice that this path moves around the Farey fractions through the Ford circles so there will be simplifications due to a dominating term around singularities.

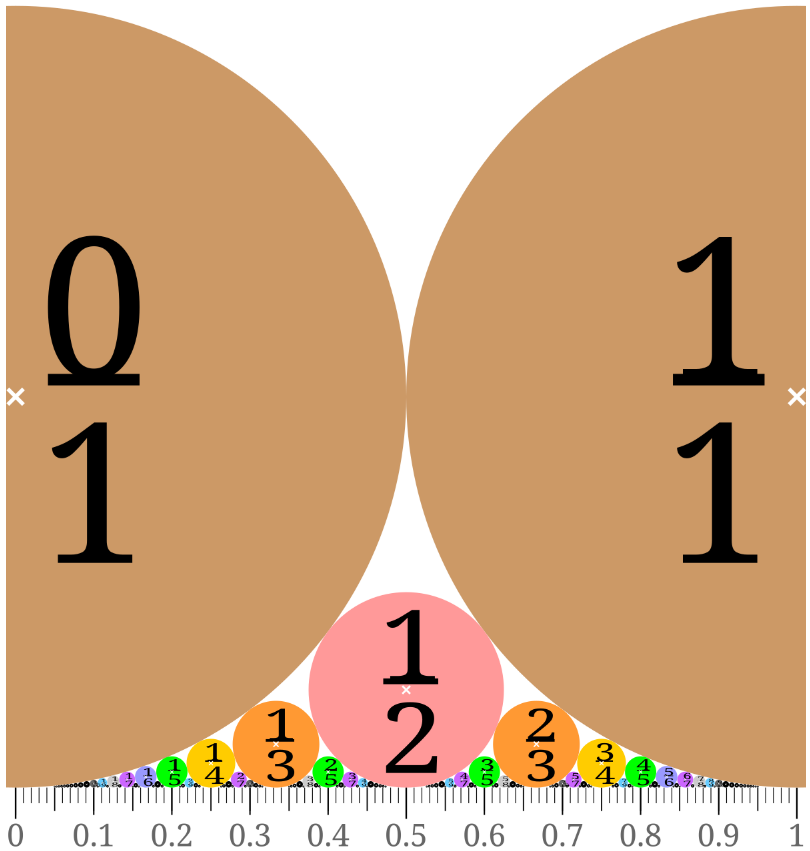

FIGURE 2. Ford circles in the complex plane. Image from `https://en.wikipedia.org/wiki/Ford_circle`.

The path $P(N)$ can be smoothly deformed into a straight line between i and $i+1$. Usually, we integrate $F(x)$ from 0 to 1, but integrating from i to $i+1$, has the same property that

$$\int_i^{i+1} e(nx)dx = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}.$$

So we have

$$\begin{aligned} p(n) &= \int_i^{i+1} F(x)e(-nx)\,dx = \int_{P(N)} F(x)e(-nx)\,dx, \\ &= \sum_{k=1}^{N} \sum_{\substack{h=0 \\ (h,k)=1}}^{k} \int_{\gamma_{h,k}} F(x)e(-nx)\,dx = \sum_{h,k} \int_{\gamma_{h,k}} F(x)e(-nx)\,dx, \end{aligned}$$

where $\sum_{h,k}$ is shortened notation. Therefore, it suffices to determine $\int_{\gamma_{h,k}} F(x)\,dx$. For bounding purposes, it is useful to do the following transformation which "normalizes" $C(h,k)$ to the circle K

centered at 1/2 with radius 1/2.

$$z = -ik^2\left(x - \frac{h}{k}\right) \Rightarrow x = \frac{h}{k} + i\frac{z}{k^2} \Rightarrow dx = \frac{i}{k^2}dz.$$

For any point z on or inside K, the real part of z is less than or equal to 1 and the real part of z^{-1} is greater than or equal to 1. We need to find how the points of tangency bounding $\gamma_{h,k}$ change as a result of this transformation. We omit the proof of the following proposition.

Proposition 4.6. *Let $a/b, h/k$ and c/d be consecutive Farey fractions in F_N. The transformation to z maps $\alpha_1(h,k)$ and $\alpha_2(h,k)$ to*

$$z_1(h,k) = \frac{k^2}{k^2+b^2} + i\frac{kb}{k^2+b^2},$$
$$z_2(h,k) = \frac{k^2}{k^2+d^2} - i\frac{kb}{k^2+d^2}.$$

The arc $\gamma_{h,k}$ is mapped to the arc on K between $z_1(h,k)$ and $z_2(h,k)$ not including the origin.

Critically, as N groes large, for fixed h and k, we have that b and d grow large. Thus $z_1(h,k)$ and $z_2(h,k)$ move closer to the origin. We can bound the distance between them and their distance to the origin with the following proposition. We omit its proof.

Proposition 4.7. *For the Farey sequence F_N, we have*

$$|z_1(h,k)| = \frac{k}{\sqrt{k^2+b^2}} \qquad |z_2(h,k)| = \frac{k}{\sqrt{k^2+d^2}}.$$

For any z on the chord between $z_1(h,k)$ and $z_2(h,k)$,

$$|z| < \frac{\sqrt{2}k}{N}.$$

The length of the chord is less than $2\sqrt{2}k/N$.

As a result, when we are integrating an error term over the arc between $z_1(h,k)$ and $z_2(h,k)$, we can deform it to integrate over the chord between $z_1(h,k)$ and $z_2(h,k)$ and use the ML inequality. Adding the transformation to z, we now have

$$\begin{aligned} p(n) &= \sum_{h,k} \int_{z_1(h,k)}^{z_2(h,k)} F\left(\frac{h}{k} + i\frac{z}{k^2}\right) \frac{i}{k^2} e\left(-n\left(\frac{h}{k} + i\frac{z}{k^2}\right)\right) dz, \\ &= \sum_{h,k} \frac{i}{k^2} e\left(\frac{-nh}{k}\right) \int_{z_1(h,k)}^{z_2(h,k)} F\left(\frac{h}{k} + i\frac{z}{k^2}\right) e\left(\frac{-inz}{k^2}\right) dz. \end{aligned}$$

But this brings up a new problem as we do not know what $F(h/k + iz/k^2)$ is. Fortunately, $F(x)$ works well with Mobius transformations as we will describe in the next section. This will allow us to transform $F(h/k + iz/k^2)$ into any expression of F which is easily approximated.

4.2. **Modular Functions and The Dedekind η Function.** In this subsection, we provide an rough overview of Modular functions to see how Mobius transformations affect $F(x)$. See [1] for a more complete overview. Define a Mobius transformation on a complex number z to be

$$z \to \frac{az+b}{cz+d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z.$$

where $ad-bc=1$ and the matrix times complex number is shorter notation for the transformation. Composing Mobius transformations is analogous to multiplying matrices, i.e., for Mobius transformations $f(z)$ and $g(z)$ with corresponding matrices A and B, we have $f(g(z)) = A(BZ) = (AB)z$.

Definition 4.8. The modular group Γ is the group of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $ad-bc=1$ and the operation is matrix multiplication.

Mobius transformations also map the upper half-plane to itself. To see this, notice that any matrix can be expressed as the product of the generators

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The intuition for the generators is that starting with any matrix $A \in \Gamma$, we first make the bottom left entry A_{22} small by multiplying by S^n (n can be negative in which case we are multiplying by the inverse of S), but this leaves the first column of the matrix unchanged. We then multiply by T to flip the columns and repeat the process making all of the entries smaller until the matrix we are left with is the identity matrix. Undoing the operations, we recover A as a product of the generators. The Mobius transformations corresponding to S and T are $z+1$ and $-z^{-1}$ respectively, both of which map the upper half-plane to itself. Thus any composition of the two will as well.

We can now define a modular function,

Definition 4.9. A function f is *modular* if it satisfies all of the following

(1) f is meromorphic in the upper half-plane.
(2) $f(Ax) = f(x)$ for all $A \in \Gamma$ and x in the upper half-plane.
(3) The Fourier expansion of f is of the form

$$f(x) = \sum_{i=-m}^{\infty} a_i e(ix).$$

Modular functions have the following key property

Theorem 4.10. *If a function f is modular and f has no zeroes, then f is constant.*

The proof of this theorem involves looking at a shape called the fundamental region denoted R_Γ and showing that the number of poles and zeroes in R_Γ is equal through the argument principal as long as f is not identically 0. Because $f(x)-c$ is a modular function for all c, if f is never 0, it has no poles, so it cannot have any solutions for $f(x)=c$ for all c unless $f=c$. Thus, f is constant.

An important function related to modular functions is $\Delta(x)$. Recall from the theory of elliptic functions that an elliptic function f is a meromorphic function which is doubly periodic, i.e.,

$f(x) = f(x+\omega_1) = f(x+\omega_2)$ for some ω_1 and ω_2 and for all x. Let Λ be the set of integer combinations of ω_1 and ω_2, and let $\Lambda^\star$ be $\Lambda \setminus \{0\}$. Elliptic functions can be expressed as a rational polynomial in terms of

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^\star} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \qquad \wp'(z) = \sum_{\omega \in \Lambda} -2\frac{1}{(z-\omega)^3}.$$

The Weierstrass $\wp$ function has the following Laurent expansion

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(2n+1)E_{2n+2}z^{2n},$$

where

$$E_n = \sum_{\omega \in \Lambda^\star} \frac{1}{\omega^n}.$$

The Weierstrass $\wp$ function also satisfies the following differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3.$$

where $g_2 = 60E_4$ and $g_3 = 140E_6$. In reality, g_2 and g_3 are functions of ω_1 and ω_2. The right hand side of the equation looks like a cubic, so we can define its discriminant $\Delta(\omega_1,\omega_2) = g_2(\omega_1,\omega_2)^3 - 27g_3(\omega_1,\omega_2)^2$. Notice that Δ is homogenous in the sence that

$$\Delta(\lambda\omega_1, \lambda\omega_2) = \lambda^{-12}\Delta(\omega_1,\omega_2).$$

Letting $\lambda = \omega_1^{-1}$ and $\omega_2\lambda = x$, we then have

$$\Delta(1,x) = \omega_1^{12}\Delta(\omega_1,\omega_2).$$

We replace shorten $\Delta(1,x)$ to $\Delta(x)$. Because the discriminant of a cubic is 0 when only it has repeated roots, it turns out $\Delta(1,x)$ is never 0 since the z for which $\wp'(z)$ is 0 yield distinct values of $\wp(z)$ which is not obvious. Moreover, if complex pairs (ω_1,ω_2) and (ω_1',ω_2') form the same lattice, we have that $g_2(\omega_1,\omega_2) = g_2(\omega_1',\omega_2')$ and $g_3(\omega_1,\omega_2) = g_3(\omega_1',\omega_2')$, so $\Delta(\omega_1,\omega_2) = \Delta(\omega_1',\omega_2')$. It follows that $\Delta(x+1) = \Delta(x)$, and

$$\Delta\left(\frac{-1}{x}\right) = \Delta\left(1, \frac{-1}{x}\right) = x^{12}\Delta(x,-1) = x^{12}\Delta(x).$$

More generally, we have that

$$\Delta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} x\right) = (cx+d)^{12}\Delta(x).$$

We now consider the Dedekind η function defined as

$$\eta(x) = e(x/24)\prod_{n=1}^{\infty}(1-e(nx)).$$

It makes sense to consider this function as $\eta(x) = e(x/24)/F(x)$. Notice that $\eta(x+1) = e(x/24)\eta(x)$. We also know that $\eta(-1/x) = \sqrt{-ix}\eta(x)$ which can be derived through polynomial expansions of $\log\eta(x)$. Notice the similarity in how Mobius transformations affect $\Delta(x)$ and how Mobius transformations affect $\eta(x)^{24}$. Thus, we consider $f(x) = \Delta(x)/\eta(x)^{24}$. We know that f is modular as it is meromorphic, it remains the same under composition with Mobius transformations, and it satisfies the Fourier series condition. We also know it has no zeroes as $\Delta(x)$ is never 0. Thus, $\Delta(x)/\eta(x)^{24}$ is a constant. More specifically, it turns out that

$$\Delta(x) = (2\pi)^{12}\eta^{24}(x),$$

which can be seen by matching coefficients in the expansions of both sides of the equation. We know how Mobius transformations affect $\Delta(x)$, so we can get a formula for how they affect $\eta(x)$.

Theorem 4.11. *If* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, *and* $c > 0$, *then*

$$\eta(Ax) = \varepsilon(a,b,c,d)\{-i(cx+d)\}^{1/2}\eta(x)$$

where

$$\varepsilon(a,b,c,d) = e\left(\frac{a+d}{24c} + \frac{s(-d,c)}{2}\right),$$

and

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right).$$

Returning back to $F(x)$ which is the generator function for the partition numbers, notice that $F(i\infty) = 1$. If we could somehow transform $F\left(\frac{h}{k} + i\frac{z}{k^2}\right)$ which is the main term in the integrand used in the circle method into something of the form $F\left(\frac{a}{b} + \frac{ic}{z}\right)$, then as z gets closer to the origin $F\left(\frac{a}{b} + \frac{ic}{z}\right)$ becomes closer to 1. The expression we used to transform $F\left(\frac{h}{k} + i\frac{z}{k^2}\right)$ would then be a good approximation for $F\left(\frac{h}{k} + i\frac{z}{k^2}\right)$, especially since as $N \to \infty$, we have that the chord between $z_1(h,k)$ and $z_2(h,k)$ gets closer to the origin. This motivates the following theorem which translates Theorem 4.11 by replacing $F(x)$ with $\eta(x)$ and using a particular Mobius transformation.

Theorem 4.12. *Let* h *and* k *be defined as above. Let* H *be such that* $Hh \equiv -1 \pmod{k}$. *Then*

$$F\left(\frac{h}{k} + i\frac{z}{k^2}\right) = e(s(h,k)/2)\left(\frac{z}{k}\right)^{1/2} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) F\left(\frac{H}{k} + \frac{i}{z}\right).$$

Proof. Replace $\eta(x)$ with $e(x/24)/F(x)$ in Theorem 4.11 we get

$$F(x) = F(Ax)\varepsilon(a,b,c,d)e\left(\frac{x - Ax}{24}\right)\{-i(cx+d)\}^{1/2}$$

Using $x = h/k + iz/k^2$ and

$$a = H, b = -\frac{hH+1}{k}, c = k, d = -h,$$

we then see

$$F\left(\frac{h}{k} + i\frac{z}{k^2}\right) = e(s(h,k)/2)\left(\frac{z}{k}\right)^{1/2} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) F\left(\frac{H}{k} + \frac{i}{z}\right).$$

■

Letting $\omega(h,k) = e(s(h,k)/2)$ and $\Psi_k(z) = z^{1/2}\exp(\pi/(12z) - \pi z/(12k^2))$. Plugging this into the circle method integral, we now have

$$p(n) = \sum_{h,k} \frac{i}{k^2} e\left(\frac{-nh}{k}\right) \int_{z_1(h,k)}^{z_2(h,k)} F\left(\frac{h}{k} + i\frac{z}{k^2}\right) e\left(\frac{-inz}{k^2}\right) dz.$$
$$= \sum_{h,k} ik^{-5/2} e\left(\frac{-nh}{k}\right) \int_{z_1(h,k)}^{z_2(h,k)} \omega(h,k)\Psi_k(z) F\left(\frac{H}{k} + \frac{i}{z}\right) \exp\left(\frac{2n\pi z}{k^2}\right) dz.$$

4.3. **Splitting and Bounding the Integral.** Split the integral into

$$I_1(h,k) = \int_{z_1(h,k)}^{z_2(h,k)} \Psi_k(z) \exp\left(\frac{2n\pi z}{k^2}\right) dz,$$
$$I_2(h,k) = \int_{z_1(h,k)}^{z_2(h,k)} \Psi_k(z) \left(F\left(\frac{H}{k} + \frac{i}{z}\right) - 1\right) \exp\left(\frac{2n\pi z}{k^2}\right) dz.$$

Let us show that $I_2 \to 0$ as $N \to \infty$. We will do this through the ML inequality. Note that $|e^z| = e^{\mathrm{Re}(z)}$ and recall that for all z on or in K, we have $\mathrm{Re}(z) \le 1$ and $\mathrm{Re}(z^{-1}) \ge 1$.

$$\left|\Psi_k(z)\left(F\left(\frac{H}{k}+\frac{i}{z}\right) - 1\right)\exp\left(\frac{2n\pi z}{k^2}\right)\right|,$$
$$= |z|^{1/2}\exp\left(\frac{\pi}{12}\,\mathrm{Re}\left(\frac{1}{z}\right) - \frac{\pi}{12k^2}\,\mathrm{Re}(z)\right)$$
$$\times \exp\left(\frac{2n\pi\,\mathrm{Re}(z)}{k^2}\right)\left|\sum_{m=1}^{\infty} p(m) e\left(\frac{Hm}{k} + i\frac{2\pi m}{z}\right)\right|,$$
$$\le |z|^{1/2}\exp\left(\frac{\pi}{12}\,\mathrm{Re}\left(\frac{1}{z} + \frac{2n\pi}{k^2}\right)\right)\sum_{m=1}^{\infty}\left|p(m) e\left(\frac{Hm}{k} + i\frac{2\pi m}{z}\right)\right|,$$
$$\le |z|^{1/2}\exp(2n\pi)\sum_{m=1}^{\infty} p(m) e\left(\frac{Hm}{k} + 2i\pi\left(m - \frac{1}{24}\right)\mathrm{Re}\left(\frac{1}{z}\right)\right),$$
$$\le |z|^{1/2}\exp(2n\pi)\sum_{m=1}^{\infty} p(m) e\left(2i\pi\left(m - \frac{1}{24}\right)\right).$$

Because n is constant in this scenario, the magnitude integrand of I_2 is less than $c|z|^{1/2}$ for some c. On the path from $z_1(h,k)$ to $z_2(h,k)$, we have that $|z|$ is at most $\sqrt{2}k/N$, so the integrand is less than $c2^{1/4}(k/N)^{1/2}$. The length of the path is less than $2\sqrt{2}k/N$, so multiplying the two, we have

$$|I_2(h,k)| < Ck^{3/2}N^{-3/2},$$

for some constant C not dependent on N. Plugging this into the sum over Ford circles, we have

$$\left|\sum_{h,k} ik^{-5/2}\omega(h,k) e\left(\frac{-nh}{k}\right) I_2(h,k)\right| \le \sum_{h,k} Ck^{-1}N^{-3/2} = CN^{-1/2}.$$

Therefore,

$$p(n) = \sum_{h,k} ik^{-5/2}\omega(h,k)e\left(\frac{-nh}{k}\right) I_1(h,k) + O(N^{-1/2}).$$

We just need to find I_1 now. We provide the main steps and tools needed as described in [1]. Through an analogous process of bounding, we can find that

$$p(n) = \sum_{h,k} ik^{-5/2}\omega(h,k)e\left(\frac{-nh}{k}\right) \int_K \Psi_k(z)\exp\left(\frac{2n\pi z}{k^2}\right) + O(N^{-1/2}).$$

Where the integral is done around K in the clockwise direction. Using the theory of Bessel Functions and taking $N \to \infty$, we then have

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right),$$

where

$$A_k(n) = \sum_{\substack{0\le h\le k\\(h,k)=1}} e\left(\frac{s(h,k)}{2} - \frac{h}{k}\right).$$

Thus, we now have an infinite series for the partition function.

References

[1] T. M. Apostol, *Modular functions and dirichlet series in number theory*, (1976).
[2] S. J. Miller and R. Takloo-Bighash, *The circle method*, **155** (2004).