An Elementary Approach to Nevanlinna Theory and Applications

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Abstract

This paper presents an introduction to the elementary foundations of Nevanlinna Theory, assuming only a background in complex analysis and calculus. We define the Nevanlinna characteristic functions and establish the First and Second Main Theorems. These results are then used to derive the Picard Theorems. Beyond the foundational theory, we explore applications of Nevanlinna Theory in analytic number theory and differential equations, and conclude by discussing several significant open problems in the field.

Contents

1 Motivation

For a polynomial of degree d, the Fundamental Theorem of Algebra tells us that for any nonconstant polynomial $p(z) \in \mathbb{C}[z]$, $p(z) = a$ has exactly d solutions counting multiplicities. In particular, a non constant polynomial takes on every value in $\mathbb{C}$ at most d times. Furthermore, the growth of a polynomial is also determined by its degree, and we have

$$\max_{|z|\leq R} |p(z)| = O(R^n) \quad (R \to \infty).$$

Clearly, for a polynomial, the distribution of zeroes, the general distribution of values, and the radial growth of the modulus are all closely linked. However, entire functions behave very differently. For instance, consider the function e^z, which never attains 0 but attains every other value in $\mathbb{C}$ infinitely many times. Additionally, the modulus grows with $O(e^{\Re z})$, which differs from the growth of polynomials. In 1879, Émile Picard proved that any non-polynomial entire function $f : \mathbb{C} \to \mathbb{C}$ assumes every complex value, with at most one exception, infinitely often (Picard's Little Theorem). Although Picard's result is a striking classification, it does not distinguish functions of different order in terms of how "rapidly" or "densely" its values are taken.

To appropriately study the value distributions of meromorphic functions, we need to rely on analog to degree for polynomials. This is the contribution of Nevanlinna's characteristic function, $T(f,r) := N(f,\infty,r)+m(f,\infty,r)$. The results of Nevanlinna theory follow from these more general concepts of growth and counting, and thus provide an elegant "value-distribution theory" for meromorphic functions.

2 The Poisson-Jensen Formulas

Nevanlinna theory is underpinned by the Poisson-Jensen formula, which forms the foundation for the construction and study of the Nevanlinna functions. The first of these, the Poisson formula, arises naturally when considering the solution to the Dirichlet problem on a circle[1].

Theorem 2.1 (Poisson formula). *Let u be harmonic in the open disk $\mathbf{D}(R)$, $R<\infty$ and continuous on the closed disk $\overline{\mathbf{D}(R)}$. For any fixed point z in the disk,*

$$u(z) = \int_0^{2\pi} P(z,Re^{i\theta})u(Re^{i\theta})\,\frac{d\theta}{2\pi},$$

where $P(z,\zeta) = \frac{|\zeta|^2-|z|^2}{|\zeta-z|^2} = \Re\left(\frac{\zeta+z}{\zeta-z}\right)$ denotes the Poisson kernel.

Proof. Define the automorphism $T(w) := \frac{R^2(z-w)}{R^2-\overline{z}w}$. Note that $z=T(0)$. Then, applying the Gauß mean value theorem, we have

$$u(z) = u(T(0)) = \int_0^{2\pi} u(T(Re^{i\theta}))\,\frac{d\theta}{2\pi} = \oint_{|\zeta|=R} u(T(\zeta))\,\frac{d\zeta}{2\pi i\zeta}.$$

Let $w=T(\zeta)$. Then, one can easily show

$$\frac{dw}{w} = \left(\frac{-\zeta}{z-\zeta} + \frac{\overline{z}\zeta}{R^2-\overline{z}\zeta}\right)\frac{d\zeta}{\zeta}.$$

When $|\zeta|=R$, we have

$$\left(\frac{|\zeta|^2-|z|^2}{|\zeta-w|^2}\right)\frac{d\zeta}{\zeta} = P(z,\zeta)\frac{d\zeta}{\zeta}.$$

We can re-parmeterize with $w=Re^{i\theta}$ and recover

$$u(z) = \int_0^{2\pi} P(z,Re^{i\theta})u(Re^{i\theta})\,\frac{d\theta}{2\pi}.$$

□

Remark 2.1. As constructed, we can extend the boundary conditions of u parameterized by $u(Re^{i\theta})$ continuously to the interior of $|z|=R$. This solves the Dirichlet problem for circles.

A straightforward consequence of the Poisson formula is Poisson-Jensen formula.

Theorem 2.2 (Poisson-Jensen Formula). *Let $f \not\equiv 0,\infty$ be a meromorphic function on $\overline{\mathbf{D}(R)}$. Let $a_1,\dots,a_p$ denote the zeroes of f in the open disk $\mathbf{D}(R)$ repeated according to multiplicity, and let $b_1,\dots,b_q$ denote the poles of f in the open disk $\mathbf{D}(R)$ also repeated according to multiplicity. For any $z\in\mathbf{D}(R)$ that is not a zero or pole of f, we have*

$$\begin{aligned}\log|f(z)| &= \int_0^{2\pi} P(z,Re^{i\theta})\log|f(Re^{i\theta})|\frac{d\theta}{2\pi} - \sum_{i=1}^{p}\log\left|\frac{R^2-\overline{a}_i z}{R(z-a_i)}\right| + \sum_{i=1}^{q}\log\left|\frac{R^2-\overline{b}_i z}{R(z-b_i)}\right| \\ &= \int_0^{2\pi} P(z,Re^{i\theta})\log|f(Re^{i\theta})|\frac{d\theta}{2\pi} - \sum_{\zeta\in\mathbf{D}(R)}(\operatorname{ord}_\zeta f)\log\left|\frac{R^2-\overline{\zeta}z}{R(z-\zeta)}\right|.\end{aligned}$$

[1]The Dirichlet problem asks for a function which solves a given partial differential equation, typically taken to be the Laplace equation, in the interior of a region that takes prescribed values on the boundary of the region. For complex functions, this is equivalent to the function being harmonic in the region and its boundary.

Proof. Let

$$F(z) := f(z)\frac{\prod_{i=1}^{p}\left(\frac{R^2-\overline{a}_i z}{R(z-a_i)}\right)}{\prod_{i=1}^{q}\left(\frac{R^2-\overline{b}_i z}{R(z-b_i)}\right)}.$$

In other words, we multiply f by Blaschke products to move all the poles and zeroes of f outside of $\overline{\mathbf{D}(R)}$. Hence, $\log|F(z)|$ is harmonic on any open neighborhood of $\overline{\mathbf{D}(R)}$. The result follows by applying the Poisson formula to $\log|F(z)|$. □

Remark 2.2. The logarithm is an important quantity when studying Nevanlinna theory. One function of the logarithm is to control or smoothen the growth of the modulus of f. However, the more useful property of the logarithm is that it allows us to separate the zeroes and poles of $f(z)$ into summations as in 2.2, which one can interpret as counting zeroes and poles. This hints at a natural connection between the quantity $\log|f(z)|$ and the value distribution of f.

Since the logarithmic derivative arises frequently in the study of Nevanlinna Theory, it is worth restating the Poisson-Jensen Formula in terms of logarithmic derivatives.

Theorem 2.3 (Poisson-Jensen Formula for Logarithmic Derivatives). *Let R, f, a_i, and b_i be as in Theorem 2.2. Then, for all z not a zero or pole of f,*

$$\frac{f'(z)}{f(z)} = \int_0^{2\pi}\frac{2Re^{i\theta}}{(Re^{i\theta}-z)^2}\log|f(Re^{i\theta})|\frac{d\theta}{2\pi} + \sum_{i=1}^{p}\left(\frac{\overline{a_i}}{R^2-\overline{a_i}z}+\frac{1}{z-a_i}\right) - \sum_{i=1}^{q}\left(\frac{\overline{b_i}}{R^2-\overline{b_i}z}+\frac{1}{z-b_i}\right).$$

Proof. We merely take the derivative of Theorem 2.2. Note that

$$\log|f(z)| = \frac{1}{2}[\log f(z) + \log\overline{f(z)}]$$

by anti-holomorphicity [2]. Then, by the Cauchy-Riemann equations,

$$\frac{f'(z)}{f(z)} = \frac{d}{dz}\log f(z) = 2\log\frac{d}{dz}\log|f(z)|.$$

Now taking the derivative of 2.2 and applying this, we can switch the derivative and the integral by the Dominated Convergence Theorem. The result follows. □

3 The Nevanlinna Functions

We start by defining the unintegrated counting function $n(f,a,r)$, which we simply define to be the number of times the function f attains the value a in $\overline{\mathbf{D}(R)}$. We define $n(f,a,r)$ with $\overline{\mathbf{D}(R)}$ so that $n(f,a,0)$ is well defined. However, there are several issues with the n function. For one, it is not a continuous function in r, because it is a discrete function, and for two, it is difficult to integrate into analytic quantities such as those that appear in 2.2. As such, we define the integrated counting function.

Definition 3.1 (Integrated Counting Function). Let $f:\mathbb{CP}^1\to\mathbb{CP}^1$. Let n be the unintegrated counting function as defined above. Then, for any $a\in\mathbb{CP}^1$ and $(r\geq 0)\in\mathbb{R}$,

$$N(f,a,r) := n(f,a,0)\log r + \int_0^r [n(f,a,t)-n(f,a,0)]\,\frac{dt}{t}.$$

It is not at all immediately obvious that $N(f,a,r)$ is a natural counting function. However, we can remark that it is an improvement on $n(f,a,r)$ in that $N(f,a,r)$ is a continuous function in r, and its growth is moderated by the presence of the logarithm. Furthermore, this definition translates well to the language of the Poisson-Jensen formula, as demonstrated below.

[2] This can also be verified elementarily.

Corollary 3.2. *Let f be meromorphic on $\overline{\mathbf{D}(R)}$ with zeros $a_1, \ldots, a_p$ and poles $b_1, \ldots, b_q$ in the open disk $\overline{\mathbf{D}(R)}$, each counted with multiplicity. Then for any z with $|z| = r < R$,*

$$\sum_{i=1}^{p} \log\left|\frac{R^2 - \bar{a}_i\, z}{R(z-a_i)}\right| - \sum_{j=1}^{q} \log\left|\frac{R^2 - \bar{b}_j\, z}{R(z-b_j)}\right| = N(f,\infty,r) - N(f,0,r).$$

Proof. Write $\rho_i = |a_i|$, $\sigma_j = |b_j|$. One checks directly that, for $|z| = r$,

$$\log\left|\frac{R^2 - \bar{a}_i\, z}{R(z-a_i)}\right| = \int_{\rho_i}^{R} \frac{dt}{t} - \int_{\rho_i}^{r} \frac{dt}{t}, \quad \log\left|\frac{R^2 - \bar{b}_j\, z}{R(z-b_j)}\right| = \int_{\sigma_j}^{r} \frac{dt}{t} - \int_{\sigma_j}^{R} \frac{dt}{t}.$$

Hence

$$\sum_{i=1}^{p} \log\left|\frac{R^2 - \bar{a}_i\, z}{R(z-a_i)}\right| - \sum_{j=1}^{q} \log\left|\frac{R^2 - \bar{b}_j\, z}{R(z-b_j)}\right| = \left[\int_0^r \frac{n(f,\infty,t)}{t} dt - \int_0^R \frac{n(f,\infty,t)}{t} dt\right] - \left[\int_0^R \frac{n(f,0,t)}{t} dt - \int_0^r \frac{n(f,0,t)}{t} dt\right].$$

Applying the definition of N, we can write

$$\int_0^r \frac{n(f,\infty,t)}{t} dt - \int_0^R \frac{n(f,\infty,t)}{t} dt = N(f,\infty,r) - N(f,\infty,R),$$

$$\int_0^R \frac{n(f,0,t)}{t} dt - \int_0^r \frac{n(f,0,t)}{t} dt = N(f,0,R) - N(f,0,r).$$

Since f has no zeros or poles on $|z| = R$, one checks $N(f,\infty,R) = N(f,0,R)$, and hence their difference cancels. Thus,

$$\sum_{i=1}^{p} \log\left|\frac{R^2 - \bar{a}_i\, z}{R(z-a_i)}\right| - \sum_{j=1}^{q} \log\left|\frac{R^2 - \bar{b}_j\, z}{R(z-b_j)}\right| = N(f,\infty,r) - N(f,0,r).$$

□

Corollary 3.3. *Let $f \not\equiv 0, \infty$ be meromorphic on $\overline{\mathbf{D}(R)}$. Then,*

$$\log|\operatorname{ilc}(f,0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta + N(f,\infty,r) - N(f,0,r),$$

where $\operatorname{ilc}(\mathrm{f},0)$ *denotes the initial Laurent coefficent of f about $z = 0$.*

Proof. Apply the Poisson-Jensen Formula to the function $F(w) := f(w)w^{-(\operatorname{ord}_0 f)}$, and expand $\log|F(0)|$. The result follows from 3.2. □

To define the mean proximity function, we must first introduce the Weil functions. Given a point $a \in \mathbb{CP}^1$, a Weil function is a continuous map $\lambda_a : \mathbb{CP}^1 \setminus \{a\} \to \mathbb{R}$, which has the property that on every open neighborhood of a, there is a continuous function α such that

$$\lambda_a(z) = -\log|z-a| + \alpha(z).$$

At $z = \infty$, we take the local holomorphic coordinate $\frac{1}{z}$. Weil functions have two important properties that we can exploit. The first is that the difference between any two Weil functions is a continuous function, and on the compact space $\mathbb{CP}^1$, this forces the difference to be bounded. Secondly, the Weil functions grow large exactly when z is close to a, and thus, we can use the size of the Weil function to estimate the proximity of z to a. For a meromorphic function $f(z)$ on $\mathbf{D}(R)$, $\lambda_a(f(z))$ will tell us the proximity of $f(z)$ to a. Thus, we can define a mean proximity function by

$$m(f,\lambda_a,r) := \int_0^{2\pi} \lambda_a(f(re^{i\theta}))\,\frac{d\theta}{2\pi}.$$

As it turns out, most results in Nevanlinna theory do not depend on the choice Weil function, instead, we tend to think of m as depending only on a and writing $m(f, a, r)$. However, we wish to choose a Weil function to act as the default in the definition of $m(f, a, r)$. One of the simplest functional choices is the one by R. Nevanlinna himself, who defined

$$\lambda_a(z) := \begin{cases} \log^+ \frac{1}{|z-a|} & \text{if} \quad a, z \neq \infty, \\ \log^+ |z| & \text{if} \quad a = \infty, \end{cases} \quad \text{and} \quad \lambda_a(\infty) = 0 \quad \text{if} \quad a \neq \infty.$$

This leads to the definition of the analytic proximity function.

Definition 3.4 (Analytic Mean Proximity Function). Let f be meromorphic on $\overline{\mathbf{D}(R)}$, $R \leq \infty$. Then,

$$m(f, a, r) := \int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\theta}) - a} \right| \frac{d\theta}{2\pi} \quad \text{if} \quad a \neq \infty \quad \text{and}$$
$$m(f, \infty, r) := \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| \frac{d\theta}{2\pi},$$

where $\log^+ x = \max(\log x, 0)$.

When looking at things from an analytic perspective, we find it convenient to use Nevanlinna's choice of Weil function as above. However, this function does not lend itself naturally to a geometric perspective, so we construct another Weil function based on the geometric structure of $\mathbb{CP}^1$.

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ be two points in $\mathbb{C}$. We use stereographic projection to identify the complex plane with the sphere of radius $\frac{1}{2}$ centered at the origin in $\mathbb{R}^3$. Let $(\rho_j, \theta_j, \zeta_j)$ be the cylindrical coordinate representations of the images of the z_j, then we have by projection

$$\rho_j = \frac{r_j}{1 + r_j^2} \quad \zeta_j = \frac{r_j^2 - 1}{4(1 + r_j^2)}.$$

The square of the standard Euclidean distance in $\mathbb{R}^3$ is given by

$$\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_1 - \theta_2) + \zeta_1^2 + \zeta_2^2 - 2\zeta_1\zeta_2.$$

Rewriting with the appropriate transformations allows us to define the chordal distance between two points in $\mathbb{C}$ to be

$$||z_1, z_2||^2 := \frac{|z_1 - z_2|^2}{(1 + |z_1|^2)(1 + |z_2|^2)}.$$

One can check that the chordal distance satisfies the triangle inequality and is a valid metric. We can continuously extend the definition of the chordal distance to $\mathbb{CP}^1$ by

$$||z, \infty||^2 = \frac{1}{1 + |z|^2}.$$

Hence, we define the geometric Weil function by $\mathring{\lambda} := -\log ||z, a||$.

Definition 3.5 (Geometric Mean Proximity Function). Let f be meromorphic on $\overline{\mathbf{D}(R)}$, $R \leq \infty$. Then for all $r \leq R$,

$$\mathring{m}(f, a, r) := \int_0^{2\pi} -\log ||f(re^{i\theta}), a|| \frac{d\theta}{2\pi}.$$

The final of the Nevanlinna functions is the characteristic function, which we simply define by combining the defintions of the counting function and proximity function.

Definition 3.6 (Nevanlinna Characteristic Function). Let f be meromorphic on $\overline{\mathbf{D}(R)}$, $R \leq \infty$. Then, for all $r \leq R$ the analytic characteristic function is defined by

$$T(f, a, r) := N(f, a, r) + m(f, a, r),$$

and the geometric characteristic function is defined by

$$\mathring{T}(f,a,r) := N(f,a,r) + \mathring{m}(f,a,r) + c_{\text{fmt}}(f,a),$$

where c_{fmt} is a constant not depending on r defined in section four.

4 First Main Theorem

The first truly remarkable result of Nevanlinna Theory is the First Main Theorem. In its essence, the First Main Theorem states that the Nevanlinna characteristic is (essentially) invariant of the value of a. Thus, the characteristic is the generalization of the degree of a polynomial for meromorphic functions that we require to properly study the value distribution of meromorphic functions.

The adept reader might remark that the Poisson-Jensen formula is in many ways already invariant of the value of a, at least in structure. In fact, as we will see, the First Main Theorem is nothing more than a formal repackaging of the Poisson-Jensen formula.

Theorem 4.1 (First Main Theorem). *Let $a \in \mathbb{CP}^1$, and let $f \not\equiv a, \infty$ be a meromorphic function in $\mathbf{D}(R)$, $R \le \infty$. Then,*

$$N(f,a,r) + m(f,a,r) = T(f,r) + O(1),$$

where $T(f,r) := T(f,\infty,r)$. Alternatively, for all $a, b \in \mathbb{CP}^1$,

$$T(f,a,r) = T(f,b,r) + O(1).$$

Proof. Apply 3.3 to $h = \frac{1}{f-a}$. Note that h has zeroes at the poles of f and poles at the a-points of f. We have

$$\log|\operatorname{ilc}(h,0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\theta})|\,d\theta + N(h,\infty,r) - N(h,0,r) = -\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta}) - a|\,d\theta + N(f,a,r) - N(f,\infty,r).$$

Note that $\log x = \log^+ x - \log^+ \frac{1}{x}$. Rearranging and expanding, we have

$$N(f,a,r) - N(f,\infty,r) = \frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta}) - a|\,d\theta - \frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|f(re^{i\theta}) - a|}\,d\theta.$$

The first integral on the right hand side is $m(f,\infty,r) + O(1)$ [3], and the second is simply $m(f,a,r)$. Hence,

$$N(f,a,r) - N(f,\infty,r) = m(f,\infty,r) - m(f,a,r) + O(1),$$

from which we recover

$$N(f,a,r) + m(f,a,r) = T(f,r) + O(1).$$

□

Remark 4.1. The choice to define $T(f,r)$ with $a = \infty$ is merely convention. Similarly, we define $N(f,r) := N(f,\infty,r)$ and $m(f,r) := m(f,\infty,r)$, for notational convenience. We also note that the First Main Theorem holds for general Weil functions, but these are typically with poor error terms.

A natural question is whether the $O(1)$ error term can be made explicit. In fact, one can achieve equality using the geometric functions. This is a significant advantage over the analytic functions, as in those cases, we can at most say the difference is bounded.

Theorem 4.2 (Geometric First Main Theorem). *Let $a \in \mathbb{CP}^1$, and let $f \not\equiv a, \infty$ be a meromorphic function in $\mathbf{D}(R)$, $R \le \infty$. Let*

[3] This follows from elementary inequalities. In particular, one can use

$$\log^+|f-a| \le \log^+|f| + \log^+|a| + \log 2.$$

$c_{\text{fmt}}(f,a)$ be defined by

$$c_{\text{fmt}}(f,a) := \begin{cases} \log ||f(0), a|| & \text{if} \quad f(0) \neq a \\ \log|\operatorname{ilc}(f-a,0)| + 2\log||a,\infty|| & \text{if} \quad f(0) = a, a \neq \infty \\ -\log|\operatorname{ilc}(f,0)| & \text{if} \quad f(0) = a = \infty. \end{cases}$$

Then, for any $a, b \in \mathbb{CP}^1$,

$$\mathring{T}(f,a,r) = \mathring{T}(f,b,r).$$

Proof. The proof is a routine application of definitions. By the definition of $\mathring{m}$ and 3.3,

$$\mathring{m}(f,a,r) - \mathring{m}(f,\infty,r) = N(f,\infty,r) - N(f,a,r) - \log|\operatorname{ilc}(f-a,0)| - \log||a,\infty||.$$

The result follows by choosing the appropriate constant to force cancellation. □

The First Main Theorem is in essence a generalization of the Fundamental Theorem of Algebra. If it were true that $N(f,a,r)$ were essentially independent of a, we would have our generalization, however, as we have seen, this cannot possibly be true. But the First Main Theorem does tell us that the quantity $m(f,a,r) + N(f,a,r)$ does not really depend on a. Our generalization thus measures the growth of the set of points where f is equal to a along with the set of points where f is "close" (in proximity to) a. Also note that $m(f,a,r)$ is always non positive. Thus, the First Main Theorem gives an upper bound on the number of times f attains a. The more difficult lower bound relies on the Second Main Theorem, and requires further machinery.

Those familiar with Picard's Great Theorem will know that any non-constant meromorphic function on the complex plane can omit at most two values in $\mathbb{CP}^1$. One might suspect that for most a, the counting function dominates the sum $N(f,a,r) + m(f,a,r)$ [4]. This is indeed the case, and we can prove deeper results along this line using the Second Main Theorem.

4.1 Some Elementary Computations With the First Main Theorem

Consider

$$f(z) = c\frac{z^p + a_{p-1}z^{p-1} + \cdots + a_0}{z^q + b_{q-1}z^{q-1} + \cdots + b_0},$$

where the numerator and denominator share no factors, and $c \neq 0$. We have multiple cases depending on p and q. Let $p > q$, then $\lim_{z\to\infty} f(z) = \infty$, so $m(f,a,r) = O(1)$ for all finite a. The equation $f(z) = a$ has p roots for all finite a, making $N(f,a,r) = p\log r + O(1)$. Hence,

$$T(f,r) = m(f,a,r) + N(f,a,r) = p\log r + O(1). \tag{1}$$

Similarly, $f(z) = \infty$ has q solutions, so $N(f,\infty,r) = q\log r + O(1)$. And by the First Main Theorem, we have

$$m(f,\infty,r) = (p-q)\log r + O(1).$$

The cases $p = q$ and $p < q$ are left as exercises to the reader.

Now consider $f(z) = e^z$. We can easily compute $m(f,\infty,r)$. Note that

$$m(f,\infty,r) = \int_0^{2\pi} \log^+ \left|e^{re^{i\theta}}\right| \frac{d\theta}{2\pi} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r\cos\theta \,\frac{d\theta}{2\pi} = \frac{r}{\pi}.$$

It is clear that $N(f,\infty,r) = 0$, thus $T(f,r) = r/\pi$. By the First Main Theorem, $T(f,a,r) = r/\pi + O(1)$.

In general, note that if $a = 0, \infty$, then $N(f,a,r) = 0$. We can also compute

$$m(f,0,r) = \int_0^{2\pi} \log^+ \left|\frac{1}{e^{re^{i\theta}}}\right| \frac{d\theta}{2\pi} = \int_{\pi/2}^{3\pi/2} (-r\cos\theta)\,\frac{d\theta}{2\pi} = \frac{r}{\pi}.$$

[4] A good example of this is the function e^z. For this function, the mean proximity function is only significant near at 0 and ∞. In all other cases, the counting function dominates.

Thus, for $a = 0, \infty$, we have $N(f, a, r) = 0$ and $m(f, a, r) = r/\pi$; for all other $a \in \mathbb{C}$, we have $m(f, a, r) = O(1)$ and $N(f, a, r) = r/\pi + O(1)$. To see this, we can uniformly bound m. Note that

$$m(f,a,r) = \int_0^{2\pi} \log^+ \left| \frac{1}{e^{re^{i\theta}} - a} \right| \frac{d\theta}{2\pi} \leq \int_0^{2\pi} \log^+ \left(\frac{1}{|e^{re^{i\theta}}| - |a|} \right) \frac{d\theta}{2\pi} = \int_0^{2\pi} \log^+ \left(\frac{1}{e^{r\cos\theta} - |a|} \right) \frac{d\theta}{2\pi}.$$

Now fix $\varepsilon \in (0, \pi/2)$ and split the integral as

$$\int_0^{2\pi} \log^+ \left(\frac{1}{e^{r\cos\theta} - |a|} \right) \frac{d\theta}{2\pi} = \int_{|\theta-\pi|<\varepsilon} \log^+ \left(\frac{1}{e^{r\cos\theta} - |a|} \right) \frac{d\theta}{2\pi} + \int_{|\theta-\pi|\geq\varepsilon} \log^+ \left(\frac{1}{e^{r\cos\theta} - |a|} \right) \frac{d\theta}{2\pi}.$$

In the second region, $\cos\theta \leq \cos(\pi - \varepsilon) = -\cos\varepsilon < 0$, so $e^{r\cos\theta} \leq e^{-r\cos\varepsilon} \to 0$ as $r \to \infty$, and hence $1/(e^{r\cos\theta} - |a|)$ is bounded. In the first region, the length of the arc is 2ε, so we bound

$$\int_{|\theta-\pi|<\varepsilon} \log^+ \left(\frac{1}{e^{r\cos\theta} - |a|} \right) \frac{d\theta}{2\pi} \leq \frac{2\varepsilon}{2\pi} \cdot \log^+ \left(\frac{1}{e^{-r} - |a|} \right).$$

This is $O(\varepsilon \cdot r)$, but since ε is arbitrary, we can choose $\varepsilon = 1/r$ so the whole integral is bounded by $O(1)$. Thus, $m(f, a, r) = O(1)$ as $r \to \infty$.

Ironically, we can also use the First Main Theorem to provide an alternate proof of the Fundamental Theorem of Algebra.

Theorem 4.3 (Fundamental Theorem of Algebra). *Let $p(z) \in \mathbb{C}[z]$ be a non constant polynomial. Then, p has at least one root in $\mathbb{C}$.*

Proof. Assume for the sake of contradiction that p has no roots [5]. By equation 1, $T(p, a, r) = d \log r + O(1)$, where d is the degree of the polynomial. Take $a = 0$, so $T(p, 0, r) = d \log r + O(1)$. Now, note that $N(p, 0, r) = 0$ by hypothesis. Further note that $|p(z)|$ attains a minimum M for some finite r by the Extreme Value Theorem. Thus,

$$m(p, 0, r) = \int_0^{2\pi} \log^+ \left| \frac{1}{p(re^{i\theta})} \right| \frac{d\theta}{2\pi} \leq \log^+ \frac{1}{M}.$$

Hence, $m(p, 0, r) = O(1)$. By definition, $T(p, 0, r) = m(p, 0, r) + N(p, 0, r) = O(1)$, which contradicts the growth of T. □

4.2 Growth Order

In order to study these deficient values, we need to define the *order* of a function f:

Definition 4.4. The *order* of an entire function f is defined as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(f,r)}{\log r}.$$

5 Logarithmic Derivatives

A meromorphic function g is a logarithmic derivative of some meromorphic function f if

$$g = \frac{d}{dx} \log(f) = \frac{f'}{f}.$$

Logarithmic derivatives are a significant point of interest in value distribution theory. Growth of logarithmic derivatives is elementary when f is a polynomial; f' has a smaller degree than f, so $|f'/f| \to 0$ as $|z| \to \infty$. This behavior cannot be naively generalized to all meromorphic functions. If $f = e^{z^2}$, then $|f'/f| = 2z \to \infty$ as $|z| \to \infty$. However, the growth rates of f and f'/f are still wildly different. In particular, the rate at which $|f'/f|$ approaches infinity is slow compared to that of $T(f, r)$. This result can be generalized to all meromorphic functions in what is known as the Lemma on the Logarithmic Derivative, which states that if f is a meromorphic

[5] Note that this forces $p(0) \neq 0$.

function, then $m(f'/f, \infty, r)$ cannot approach infinity quickly compared to the rate at which $T(f, r)$ tends to infinity. While the result of the Lemma on the Logarithmic Derivative is fairly intuitive, the proof is very technical. The aim of this section is to prove the Lemma on the Logarithmic Derivative. Next section, we will prove the Nevanlinna Second Main Theorem using using the lemma.

Lemma 5.1 (Smirnov's inequality). *Let $R < \infty$, and let F be analytic in $\mathbf{D}(R)$. For $0 \leq \theta \leq 2\pi$, define $u(\theta)$ by*

$$u(\theta) = \liminf_{z \to Re^{i\theta}, |z| < R} |F(z)|.$$

If either $\Re(F(z))$ or $\Im(F(z))$ has constant sign, then for any α with $0 < \alpha < 1$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} u(\theta)^\alpha \, d\theta \leq \sec(\frac{\pi\alpha}{2}) |F(0)|^\alpha$$

Proof. Without loss of generality, assume that $\Re(F(z)) > 0$ for all $|z| < R$. Since F is non zero in $\mathbf{D}(R)$, we can fix a choice of $\arg(F(z))$ such that $|\arg(F(z))| < \frac{\pi}{2}$ for all $|z| < R$. Thus, the function

$$F(z)^\alpha = |F(z)|^\alpha e^{i\alpha \arg(F(z))}$$

is analytic in $\mathbf{D}(R)$, and it follows that $\Re(F(z)^\alpha)$ is harmonic. Since

$$\Re(F(z)^\alpha) = |F(z)|^\alpha \cos(\alpha \arg(F(z))) \geq |F(z)|^\alpha \cos(\frac{\alpha\pi}{2}),$$

we have

$$\frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^\alpha \, d\theta \leq \frac{\sec(\frac{\alpha\pi}{2})}{2\pi} \int_0^{2\pi} \Re(F(re^{i\theta})) \, d\theta = \sec(\frac{\alpha\pi}{2}) \Re(F(0)^\alpha),$$

where the last line follows from the Gauß mean value theorem. Therefore,

$$\frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^\alpha \, d\theta \leq \sec(\frac{\alpha\pi}{2}) |F(0)|^\alpha$$

The final step follows from a lemma in measure theory which we shall present here without proof.

Lemma 5.2 (Fatou's Lemma). *Let $\{f_n\}$ be a sequence of nonnegative measurable functions on an integrable measure space. Then,*

$$\int \liminf_{n \to \infty} f_n \, d\mu \leq \liminf_{n \to \infty} \int f_n \, d\mu$$

Fatou's lemma gives us

$$\frac{1}{2\pi} \int_0^{2\pi} u(\theta)^\alpha \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left| \liminf_{r \to R} |F(re^{i\theta})| \right|^\alpha d\theta \leq \liminf_{r \to R} \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^\alpha \, d\theta \leq \sec(\frac{\alpha\pi}{2}) |F(0)|^\alpha$$

as desired.

□

Lemma 5.3 (Kolokolnikov's inequality). *Let $R < \infty$, and let $\{c_k\}$ be a finite sequence of complex numbers in $\mathbf{D}(R)$. For $\delta_k = \pm 1$, define*

$$H(z) = \sum_k \frac{\delta_k}{z - c_k}$$

Then, for any $\alpha \in (0, 1)$ and for $0 < r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |H(re^{i\theta})|^\alpha \, d\theta \leq (2 + 2^{2-\alpha}) \sec(\frac{\alpha\pi}{2}) \left(\frac{n(H, \infty, R)}{r} \right)^\alpha$$

Proof. Let $\varphi_k = \arg c_k$, $p_k = \delta_k \cos\varphi_k$, and $q_k = \delta_k \sin\varphi_k$. We write

$$\begin{aligned} H(z) &= \sum_{|c_k|>r, p_k>0} \frac{e^{i\varphi_k} p_k}{z-c_k} + \sum_{|c_k|>r, p_k<0} \frac{e^{i\varphi_k} p_k}{z-c_k} \\ &\quad + i \sum_{|c_k|>r, p_k>0} \frac{-e^{i\varphi_k} p_k}{z-c_k} + i \sum_{|c_k|>r, p_k<0} \frac{-e^{i\varphi_k} p_k}{z-c_k} \\ &\quad + \sum_{|c_k|\le r, \delta_k=1} \frac{\delta_k}{z-c_k} + \sum_{|c_k|\le r, \delta_k=-1} \frac{\delta_k}{z-c_k} \\ &= F_1(z) + F_2(z) + F_3(z) + F_4(z) + F_5(z) + F_6(z). \end{aligned}$$

Since $|c_k| > r$ and $|z| < r$, $\Re(\frac{e^{i\varphi_k}}{z-c_k}) < 0$, and thus F_1, F_2, F_3, F_4 all satisfy the assumptions of 5.1. Therefore, for $j = 1, 2, 3, 4$, we have

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} |F_j(re^{i\theta})|^\alpha \, d\theta &\le \sec\left(\frac{\alpha\pi}{2}\right) |F_j(0)|^\alpha \\ &\le \sec\left(\frac{\alpha\pi}{2}\right) |F_j(0)|^\alpha \left(\sum_{|c_k|>r} \frac{1}{|c_k|}\right)^\alpha \\ &\le \sec\left(\frac{\alpha\pi}{2}\right) |F_j(0)|^\alpha \left(\frac{n(H,\infty,R) - n(H,\infty,r)}{r}\right)^\alpha. \end{aligned}$$

Note that when $|z| = r$, we have

$$\left|\sum_{|c_k|\le r} \frac{1}{z-c_k}\right| = \left|\sum_{|c_k|\le r} \frac{r}{r^2 - \overline{c}_k z}\right|.$$

Define

$$\tilde{F}_5(z) = \sum_{|c_k|\le r, \delta_k=1} \frac{r}{r^2 - \overline{c}_k z}.$$

Since $|c_k| \le r$ and $|z| < r$, we have $\Re(r^2 - \overline{c}_k z) > 0$, we can apply 5.1 to

$$|F_5(re^{i\theta})| = |\tilde{F}_5(z)| = \left|\sum_{|c_k|\le r, \delta_k=1} \frac{r}{r^2 - \overline{c}_k re^{i\theta}}\right|,$$

which gives

$$\frac{1}{2\pi}\int_0^{2\pi} |F_5(re^{i\theta})|^\alpha \, d\theta \le \sec\left(\frac{\alpha\pi}{2}\right) |\tilde{F}_5(0)|^\alpha \le \sec\left(\frac{\alpha\pi}{2}\right) \left(\frac{n(H,\infty,r)}{r}\right)^\alpha.$$

Similarly, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |F_6(re^{i\theta})|^\alpha \, d\theta \le \sec\left(\frac{\alpha\pi}{2}\right) \left(\frac{n(H,\infty,r)}{r}\right)^\alpha.$$

Now, note that if d_j are non-negative real number and $0 < \alpha < 1$, then

$$\left(\sum_j d_j\right)^\alpha \leq \sum_j d_j^\alpha,$$

and thus we conclude that

$$\frac{1}{2\pi}\int_0^{2\pi}|H(re^{i\theta})|^\alpha\,d\theta \leq \frac{1}{2\pi}\sum_{j=1}^{6}\int_0^{2\pi}|F_j(re^{i\theta})|^\alpha\,d\theta \leq 2\sec\left(\frac{\alpha\pi}{2}\right)\left(\frac{n(H,\infty,R)}{r}\right)^\alpha\left(2\left(1-\frac{n(H,\infty,r)}{n(H,\infty,R)}\right)^\alpha+\left(\frac{n(H,\infty,r)}{n(H,\infty,R)}\right)^\alpha\right).$$

To finish, note that if $0 < d \leq 1$, then

$$(1-d)^\alpha + d^\alpha \leq 2^{1-\alpha},$$

so

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}|H(re^{i\theta})|^\alpha\,d\theta &\leq 2\sec\left(\frac{\alpha\pi}{2}\right)\left(\frac{n(H,\infty,R)}{r}\right)^\alpha\left(2\left(1-\frac{n(H,\infty,r)}{n(H,\infty,R)}\right)^\alpha+\left(\frac{n(H,\infty,r)}{n(H,\infty,R)}\right)^\alpha\right)\\ &\leq (2+2^{2-\alpha})\sec\left(\frac{\alpha\pi}{2}\right)\left(\frac{n(H,\infty,R)}{r}\right)^\alpha,\end{aligned}$$

as desired. □

5.1 The Gol'dberg-Grinshtein Inequality

Now that we have the necessary tools, our aim is to bound the integral of $|f'/f|^\alpha$ over the circle of radius r. This is done by the Gol'dberg-Grinshtein estimate, which bounds the integral in terms of $m(f,[0]+[\infty],s)$ and $n(f,[0]+[\infty],s)$, where $s > r$ and

$$\begin{aligned}m(f,[0]+[\infty],s) &:= m(f,0,s)+m(f,\infty,s),\\ n(f,[0]+[\infty],s) &:= n(f,0,s)+n(f,\infty,s),\end{aligned}$$

and similarly

$$N(f,[0]+[\infty],s) := N(f,0,s)+N(f,\infty,s).$$

Lemma 5.4 (Gol'dberg-Grinshtein). *Let f be a meromorphic function on $\mathbf{D}(R)$, $R \neq 0,\infty$, and let $0<\alpha<1$. Then, for $r<s<R$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^\alpha \leq \left(\frac{s}{r(s-r)}m(f,[0]+[\infty],s)\right)^\alpha + (2+2^{3-\alpha})\sec\left(\frac{\alpha\pi}{2}\right)\left(\frac{n(f,[0]+[\infty],s)}{r}\right)^\alpha.$$

Proof. Let $a_1,\dots,a_p$ and $b_1,\dots,b_q$ denote the zeroes and poles of f in $\mathbf{D}(S)$ counted with multiplicity. From 2.3 we have that for $z \neq a_i, b_j$,

$$\frac{f'(z)}{f(z)} = \frac{1}{2\pi}\int_0^{2\pi}\frac{2se^{i\varphi}}{(se^{i\varphi}-z)^2}\log|f(se^{i\varphi})|\,d\varphi + \sum_{j=1}^{p}\left(\frac{\overline{a}_j}{s^2-\overline{a}_j z}+\frac{1}{z-a_j}\right) - \sum_{j=1}^{q}\left(\frac{\overline{b}_j}{s^2-\overline{b}_j z}+\frac{1}{z-b_j}\right).$$

As a result,

$$\left|\frac{f'(z)}{f(z)}\right| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{2s}{|se^{i\varphi}-z|^2}\log|f(se^{i\varphi})|\,d\varphi + \left|\sum_{j=1}^{p}\frac{\overline{a}_j}{s^2-\overline{a}_j z} - \sum_{j=1}^{q}\frac{\overline{b}_j}{s^2-\overline{b}_j z}\right| + \left|\sum_{j=1}^{p}\frac{1}{z-a_j} - \sum_{j=1}^{q}\frac{1}{z-b_j}\right|.$$

For $0 < \alpha < 1$ and for positive real d_j, we know that

$$\left(\sum_j d_j\right)^\alpha \leq \sum_j d_j^\alpha.$$

Applying this to our current inequality gives us

$$\left|\frac{f'(z)}{f(z)}\right|^\alpha \leq \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{2s}{|se^{i\varphi}-z|^2}\log|f(se^{i\varphi})|\,d\varphi\right)^\alpha + \left|\sum_{j=1}^{p}\frac{\overline{a}_j}{s^2-\overline{a}_j z} - \sum_{j=1}^{q}\frac{\overline{b}_j}{s^2-\overline{b}_j z}\right|^\alpha + \left|\sum_{j=1}^{p}\frac{1}{z-a_j} - \sum_{j=1}^{q}\frac{1}{z-b_j}\right|^\alpha.$$

Now, we let $z = re^{i\theta}$ and integrate with respect to θ, giving us

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f'(z)}{f(z)}\right|^\alpha d\theta \leq{}& \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{2s}{|se^{i\varphi}-z|^2}\log|f(se^{i\varphi})|\,d\varphi\right)^\alpha d\theta \\
&+ \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{j=1}^{p}\frac{\overline{a}_j}{s^2-\overline{a}_j z} - \sum_{j=1}^{q}\frac{\overline{b}_j}{s^2-\overline{b}_j z}\right|^\alpha d\theta \\
&+ \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{j=1}^{p}\frac{1}{z-a_j} - \sum_{j=1}^{q}\frac{1}{z-b_j}\right|^\alpha d\theta \\
={}& I_1 + I_2 + I_3.
\end{aligned}$$

We can bound I_1 by swapping the order of integration.

$$\begin{aligned}
I_1^{1/\alpha} &\leq \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{2s}{|se^{i\varphi}-z|^2}\log|f(se^{i\varphi})|\,d\varphi\right) d\theta \\
&= \frac{2s}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|se^{i\varphi}-re^{i\theta}|^2}\,d\theta\right)\left|\log|f(se^{i\varphi})|\right|\,d\varphi.
\end{aligned}$$

Now, we have the following as a result of the Poisson formula, 2.1, applied where $u \equiv 1$.

$$1 = \frac{1}{2\pi}\int_0^{2\pi}\frac{|se^{i\varphi}|^2 - |re^{i\theta}|^2}{|se^{i\varphi}-re^{i\theta}|^2}\,d\theta$$

Applying this to the above inner integral gives us

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|se^{i\varphi}-re^{i\theta}|^2}\,d\theta = \frac{1}{s^2-r^2}.$$

Furthermore,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left|\log|f(se^{i\varphi})|\right|\,d\varphi &= \frac{1}{2\pi}\int_0^{2\pi}\left|\log^+|f(se^{i\varphi})|\right|\,d\varphi + \frac{1}{2\pi}\int_0^{2\pi}\left|\log^+\left|f\left(\frac{1}{se^{i\varphi}}\right)\right|\right|\,d\varphi \\ &= m(f,\infty,s)+m(f,0,s) \\ &= m(f,[0]+[\infty],s).\end{aligned}$$

Thus, since $s+r\geq 2r$,

$$I_1 \leq \left(\frac{2s}{s^2-r^2}m(f,[0]+[\infty],s)\right)^\alpha \leq \left(\frac{s}{r(s-r)}m(f,[0]+[\infty],s)\right)^\alpha$$

Now, we estimate I_2. For each zero a_j and each pole b_j, let $\zeta_{1j}=\Re(a_j)$ and $\eta_{1j}=\Re(b_j)$. Similarly, let $\zeta_{2j}=\Im(a_j)$ and $\eta_{2j}=\Im(b_j)$. Now we have

$$\begin{aligned}\sum_{j=1}^{p}\frac{\overline{a}_j}{s^2-\overline{a}_jz}-\sum_{j=1}^{q}\frac{\overline{b}_j}{s^2-\overline{b}_jz} &= \left(\sum_{\zeta_{1j}>0}\frac{\zeta_{1j}}{s^2-\overline{a}_jz}-\sum_{\eta_{1j}<0}\frac{\eta_{1j}}{s^2-\overline{b}_jz}\right) \\ &\quad -\left(\sum_{\zeta_{1j}\leq 0}\frac{-\zeta_{1j}}{s^2-\overline{a}_jz}-\sum_{\eta_{1j}\geq 0}\frac{-\eta_{1j}}{s^2-\overline{b}_jz}\right) \\ &\quad -i\left(\sum_{\zeta_{2j}>0}\frac{\zeta_{2j}}{s^2-\overline{a}_jz}-\sum_{\eta_{2j}<0}\frac{\eta_{2j}}{s^2-\overline{b}_jz}\right) \\ &\quad +i\left(\sum_{\zeta_{2j}\leq 0}\frac{-\zeta_{2j}}{s^2-\overline{a}_jz}-\sum_{\eta_{2j}\geq 0}\frac{-\eta_{2j}}{s^2-\overline{b}_jz}\right) \\ &= h_1(z)-h_2(z)-ih_3(z)+ih_4(z).\end{aligned}$$

As a result,

$$I_2 \leq \sum_{j=1}^{4}\frac{1}{2\pi}\int_0^{2\pi}|h_j(re^{i\theta})|^\alpha\,d\theta$$

Since both $\Re(s^2-\overline{a}_jre^{i\theta})$ and $\Re(s^2-\overline{b}_jre^{i\theta})$ are positive, we apply 5.1 to get

$$\begin{aligned}\frac{s^{2\alpha}}{\sec\left(\frac{\alpha\pi}{2}\right)}I_2 &\leq s^{2\alpha}\sum_{j=1}^{4}|h_j(0)|^\alpha \\ &\leq \sum_{k=1}^{2}\left(\left(\sum_{\zeta_{kj}>0}|\zeta_{kj}|+\sum_{\eta_{kj}<0}|\eta_{kj}|\right)^\alpha+\left(\sum_{\zeta_{kj}\leq 0}|\zeta_{kj}|+\sum_{\eta_{kj}\geq 0}|\eta_{kj}|\right)^\alpha\right) \\ &\leq s^\alpha\sum_{k=1}^{2}\left(\left(\sum_{\zeta_{kj}>0}1+\sum_{\eta_{kj}<0}1\right)^\alpha+\left(\sum_{\zeta_{kj}\leq 0}1+\sum_{\eta_{kj}\geq 0}1\right)^\alpha\right),\end{aligned}$$

where the last line follows because $|\zeta_{kj}|,|\eta_{kj}|\leq s$. Now, define a new function γ_k as the following.

$$\gamma_k=\frac{\sum_{\zeta_{kj}>0}1+\sum_{\eta_{kj}<0}1}{n(f,[0]+[\infty],s)}\leq 1$$

Thus, we have

$$s^\alpha \sum_{k=1}^{2}\left(\left(\sum_{\zeta_{kj}>0} 1+\sum_{\eta_{kj}<0} 1\right)^\alpha+\left(\sum_{\zeta_{kj}\le 0} 1+\sum_{\eta_{kj}\ge 0} 1\right)^\alpha\right)=s^\alpha \sum_{k=1}^{2}\left((\gamma_k(f,[0]+[\infty],s))^\alpha+((1-\gamma_k)n(f,[0]+[\infty],s)^\alpha\right),$$

and therefore,

$$\frac{s^{2\alpha}}{\sec\left(\frac{\alpha\pi}{2}\right)} I_2 \le (n(f,[0]+[\infty],s))^\alpha \sum_{k=1}^{2}(\gamma_k^\alpha+(1-\gamma_k)^\alpha) \le 2^{2-\alpha}.$$

Thus,

$$\begin{aligned}
I_2 &\le \sec\left(\frac{\alpha\pi}{2}\right) s^{-2\alpha} s^\alpha n(f,[0]]+[\infty],s)^\alpha 2^{2-\alpha}\\
&= 2^{2-\alpha}\sec\left(\frac{\alpha\pi}{2}\right)\left(\frac{n(f,[0]+[\infty],s)}{s}\right)^\alpha\\
&\le 2^{2-\alpha}\sec\left(\frac{\alpha\pi}{2}\right)\left(\frac{n(f,[0]+[\infty],s)}{r}\right)^\alpha
\end{aligned}$$

since $s>r$. Finally, it remains to estimate I_3. We apply 5.3 to $\mathbf{D}(S)$, giving us

$$I_3 \le (2+2^{2-\alpha})\sec\left(\frac{\alpha\pi}{2}\right)\left(\frac{n(f,[0]+[\infty],s)}{r}\right)^\alpha.$$

Combining the estimates for I_1, I_2, and I_3 gives us

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f'(z)}{f(z)}\right|^\alpha d\theta &\le I_1+I_2+I_3\\
&\le \left(\frac{s}{r(s-r)} m(f,[0]+[\infty],s)\right)^\alpha\\
&\quad + 2^{2-\alpha}\sec\left(\frac{\alpha\pi}{2}\right)\left(\frac{n(f,[0]+[\infty],s)}{r}\right)^\alpha\\
&\quad + (2+2^{2-\alpha})\sec\left(\frac{\alpha\pi}{2}\right)\left(\frac{n(f,[0]+[\infty],s)}{r}\right)^\alpha\\
&= \left(\frac{s}{r(s-r)} m(f,[0]+[\infty],s)\right)^\alpha\\
&\quad + (2+2^{3-\alpha})\sec\left(\frac{\alpha\pi}{2}\right)\left(\frac{n(f,[0]+[\infty],s)}{r}\right)^\alpha.
\end{aligned}$$

as desired. □

Remark 5.1. The treatment of Gol'dberg Grinshtein below leaves the constants on the right hand side independent of f. In this case, we want to bound the left hand side using the Nevanlinna characteristic function T. Unfortunately, due to the use of the first main theorem, it is thus necessary to give the assumption $F(0)=1$ or to allow the right hand side to depend on f. In this case, we choose to let the right hand side depend on f to alleviate restrictions on f.

Theorem 5.5. *(Gol'dberg-Grinshtein) Let f be a meromorphic function in $(0<R\le\infty)$, and let $0<\alpha<1$. Then, for all $r_0<r<p<R$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta \le C_{gg}(\alpha)\left(\frac{p}{r(p-r)}\right)^{\alpha}(2T(f,p)+\beta_1)^{\alpha},$$

where

$$\beta_1 = \beta_1(f,r_0) = |\operatorname{ord}_0 f|\log^+\frac{1}{r_0} + |\log|\operatorname{ilc}(f,0)|| + \log 2$$

and

$$C_{gg}(\alpha) = 2^{\alpha} + (8+2\cdot 2^{\alpha})\sec\left(\frac{\alpha\pi}{2}\right)$$

Proof. Define $s := (r+\rho)/2$. Then, 5.4 gives us

$$\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^{\alpha}\frac{d\theta}{2\pi} \le \left(\frac{s}{r(s-r)}m(f,[0]+[\infty],s)\right)^{\alpha} + (2+2^{3-\alpha})\sec(\alpha\pi/2)\left(\frac{n(f,[0]+[\infty],s)}{r}\right)^{\alpha}. \tag{2}$$

Note that since $s-r=\rho-s=(\rho-r)/2$, we can write the following via elementary manipulation:

$$\frac{s}{r(s-r)} = \frac{r+\rho}{r(\rho-r)} \le \frac{2\rho}{r(\rho-r)}.$$

Using the definitions of the Nevanlinna functions together with the First Main Theorem, we see that

$$\begin{aligned} m(f,0,s) &= T(f,0,s) - N(f,0,s) \\ &\le T(f,0,s) + \max\{0,\operatorname{ord}_0 f\}\log^+\frac{1}{r_0} \\ &\le T(f,\infty,s) + \max\{0,\operatorname{ord}_0 f\}\log^+\frac{1}{r_0} + |\log|\operatorname{lc}(f,0)|| + \log 2. \end{aligned}$$

Similarly, we can bound

$$m(f,\infty,s) \le T(f,\infty,s) + \max\{0,-\operatorname{ord}_0 f\}\log^+(1/r_0),$$

and thus

$$m(f,[0]+[\infty],s) = m(f,0,s) + m(f,\infty,s) \le 2T(f,s) + \beta_1.$$

Furthermore, because $\rho > s$, we know that $T(f,s) < T(f,\rho)$. Thus, the first term on the RHS of 2 is bounded by

$$2^{\alpha}\left(\frac{\rho}{r(\rho-r)}\right)^{\alpha}(2T(f,\rho)+\beta_1)^{\alpha}.$$

In a similar manner, we can estimate the second term on the RHS of 2 in terms of $T(f,\rho)$. We have the following:

$$\begin{aligned} &\frac{\rho}{\rho-s}\left(N(f,[0]+[\infty],\rho) + |\operatorname{ord}_0 f|\log^+\frac{1}{r_0}\right) \\ &\ge \frac{\rho}{\rho-s}\int_s^{\rho} n(f,[0]+[\infty],t)\frac{dt}{t} \\ &\ge \frac{\rho}{\rho-s}\int_s^{\rho} n(f,[0]+[\infty],s)\frac{dt}{\rho} \\ &= n(f,[0]+[\infty],s). \end{aligned}$$

As a result of the First Main Theorem, we have the following.

$$\begin{aligned} n(f,[0]+[\infty],s) &\leq \frac{\rho}{\rho-s}\left(N(f,[0]+[\infty],\rho)+|\operatorname{ord}_0 f|\log^+\frac{1}{r_0}\right) \\ &\leq \frac{2\rho}{\rho-r}(2T(f,\rho)+\beta_1), \end{aligned}$$

Thus, the second term on the right in 2 is bounded by

$$(8+2\cdot 2^{\alpha})\sec(\alpha\pi/2)\left(\frac{\rho}{r(\rho-r)}\right)^{\alpha}(2T(f,\rho)+\beta_1)^{\alpha}.$$

as desired. Fixing a value for α lets us bound $m(f'/f,\infty,r)$.

□

Corollary 5.6. *Let f be a meromorphic function on $\mathbf{D}(R)$ where $0<R\leq\infty$. Then, for all r and ρ such that $r_0<r<\rho<R$,*

$$m(f'/f,\infty,r)\leq\log^+\left(\frac{\rho(2T(f,\rho)+\beta_1)}{r(\rho-r)}\right)+c_{gg}$$

$$\leq\log^+T(f,\rho)+\log^+\frac{\rho}{r(\rho-r)}+\log^+\beta_1+c_{gg}+\log 2,$$

where

$$\beta_1=\beta_1(f,r_0)=|\operatorname{ord}_0 f|\log^+\frac{1}{r_0}+|\log|\operatorname{ilc}(f,0)||+\log 2,$$

and $c_{gg}:=\frac{1}{\alpha}\log C_{gg}(\alpha)$

Proof. As a result of the concavity of $\log^+$, we can remove it from the integral and then apply 5.5 for

$$\begin{aligned} m(f'/f,\infty,r) &= \frac{1}{\alpha}\int_0^{2\pi}\log^+\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^{\alpha}\frac{d\theta}{2\pi}\leq\frac{1}{\alpha}\log^+\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^{\alpha}\frac{d\theta}{2\pi} \\ &\leq\frac{1}{\alpha}\log^+C_{gg}(\alpha)+\log^+\frac{\rho(2T(f,\rho)+\beta_1)}{r(\rho-r)}. \end{aligned}$$

We now apply the following identity.

$$\log^+(x+y)\leq\log^+x+\log^+y+\log 2,$$

This gives us

$$\log^+\frac{\rho(2T(f,\rho)+\beta_1)}{r(\rho-r)}\leq\log^+T(f,\rho)+\log^+\frac{\rho}{r(\rho-r)}+\log^+\beta_1+2\log 2.$$

The proof is completed by swapping $\frac{1}{\alpha}\log^+C_{gg}(\alpha)$ with c_{ss}.

□

5.2 Proof of the Lemma on the Logarithmic Derivative

We state the following form of the Lemma on the Logarithmic Derivative from [CY01].

Theorem 5.7. *Let c_{gg} be the constant from 5.6, and let f be a non-constant meromorphic function on $D(R)$, $R \leq \infty$. Assume that $T(f, r_0) \geq e$ for some r_0, and that in the case $R = \infty$ that $r_0 \geq e$. Let*

$$\beta_1 = \beta_1(f, r_0) = |\operatorname{ord}_0 f| \log^+ \frac{1}{r_0} + |\log|\operatorname{ilc}(f, 0)|| + \log 2,$$

and let ψ be a Khinchin function.

Let $\phi(r)$ be a positive, nondecreasing, continuous function defined for $r_0 \leq r < \infty$. Then, there exists a closed set E of radii such that

$$\int_E \frac{dr}{\phi(r)} \leq 2k_0(\psi) + 1,$$

and such that for all $r \geq r_0$ outside of E, we have

$$m(f'/f, \infty, r) \leq \log T(f, r) + \log^+ \frac{\psi(T(f, r))}{\phi(r)} + c_{gg} + \log^+ \beta_1 + 3\log 2 + 1.$$

Moreover, if $\phi(r) \leq r$, then

$$\int_E \frac{dr}{\phi(r)} \leq k_0(\psi) + 1,$$

and for all $r \geq r_0$ outside of E, we have

$$m(f'/f, \infty, r) \leq \log T(f, r) + \log \frac{\psi(T(f, r))}{\phi(r)} + c_{gg} + \log^+ \beta_1 + 3\log 2 + 1.$$

Note that this deviates significantly from the original description of the Lemma on the Logarithmic Derivative, which stated that the growth of $m(f'/f, r)$ was bounded by the growth of $T(f, r)$. But this theorem statement is equivalent to this description, as the RHS of both inequalities involving $m(f'/f, r)$ are dependent on $T(f, r)$ up to a constant.

In order to prove the Lemma on the Logarithmic Derivative, we will prove a different lemma on the growth of $\int \frac{\mathrm{dr}}{\phi(r)}$, which can be applied in tandem with 5.6.

6 The Second Main Theorem

The most important consequence of the lemma on the logarithmic derivative is the Nevanlinna second main theorem, which is arguably the central result of Nevanlinna theory as a whole. The second main theorem can be seen as a generalization of Picard's little theorem; it bounds the growth of a meromorphic function f by a measure of the amount of times f takes on a finite collection of complex values. This section aims to prove the second main theorem via the lemma on logarithmic derivatives, vaguely following the original proof by Rolf Nevanlinna.

First, we restate the lemma on the logarithmic derivative and prove a corollary.

Theorem 6.1 (Lemma on the logarithmic derivative). *Let f be a nonconstant meromorphic function. Then*

$$m(f'/f, r) = o(T(f, r))$$

outside of a possible exceptional set of finite linear measure. Furthermore, if f has finite order, then

$$m(f'/f, r) = O(\log(r)).$$

Corollary 6.2. *Let f be a nonconstant meromorphic function and k be a positive integer. Then*

$$m(f^{(k)}/f, r) = o(T(f, r)),$$
$$T(f^{(k)}, r) \leq (k+1)T(r, f) + o(T(f, r)).$$

Proof. We proceed via induction. When $k = 1$, the first claim is just the the lemma on the logarithmic derivative. So, our inductive assumption is that

$$m(f^k/f, r) \leq m(f, r) + m(f^k/f, r) = m(f, r) + o(T(f, r)).$$

The meromorphic function f has a pole of order $p \geq 1$ at z_0 if and only if f^k has a pole of order $m + p \geq m + 1$ at z_0, so it follows that $n(f^k, r) \leq (k+1)n(f, r)$. Therefore,

$$N(f^k, r) \leq (k+1)N(f, r),$$

and thus,

$$T(f^k, r) \leq (k+1)T(f, r) + o(T(f, r)),$$

which is equivalent to the second claim in the corollary. It follows that

$$m(f^{k+1}/f^k, r) = o(T(f^l, r)) = o(T(f, r)),$$

and therefore,

$$m(f^{k+1}/f, r) \leq m(f^{k+1}/f^k, r) + m(f^k/f, r) = o(T(f, r)) + o(T(f, r)) = o(T(f, r)).$$

□

We are now able to state and prove the second main theorem. First, note that

$$n(f, r) = \sum_{\zeta \in \mathbf{D}(R)} \max\{\text{ord}_\zeta(1/f), 0\},$$

as both sides give the number of poles f has within the disk of radius r. Now, consider the function

$$n_{ram}(f, r) := 2n(f, r) - n(f', r) + n(1/f', r) = \sum_{\zeta \in \mathbf{D}(R)} \nu_\zeta(f),$$

where this

$$\nu_\zeta(f) = 2\max\{\text{ord}_\zeta(1/f), 0\} - \max\{\text{ord}_\zeta(1/f'), 0\} + \max\{\text{ord}_\zeta(f'), 0\}.$$

The purpose of the function $\nu_\zeta(f)$ is to count high-multiplicity (ramified) terms. If f has a pole of order p at $z = \zeta$, then $\nu_\zeta(f) = 2p - (p+1) + 0 = p - 1$. Now, suppose that f takes the value a at ζ with multiplicity p. Again, $\nu_\zeta(f) = 0 + 0 + p - 1 = p - 1$. So, $n_{ram}(f, r)$ counts values of high multiplicity. The function $N_{ram}(f, r)$ is defined similarly and is known as the *ramification term.*

Theorem 6.3 (Nevanlinna's second main theorem). *Let f be a transcendental meromorphic function. For $q \geq 2$, let $a_1 \dots a_q \in \mathbb{C}$ be q distinct points. Then*

$$(q-1)T(f, r) \leq N(f, r) + \sum_{j=1}^{q} N\left(\frac{1}{f - a_j}, r\right) - N_{ram}(f, r) + o(T(f, r)).$$

Proof. Let $l = \min_{1 \leq i < j \leq q} |a_i - a_j|$. Fix a point $z \in \mathbb{C}$ such that $f'(z) = 0, \infty, a_j$ for all $j \in [1, q]$. Then, choose $k \in [1, q]$ such that $|f(z) - a_k|$ is minimal. Thus, $|f(z) - a_k| \leq |f(z) - a_j|$ for all $j \in [1, q]$. Finally, define the following

$$C(a_1, a_2, \dots, a_q) = \log^+ \sum_{j=1, j \neq k}^{q} |a_j| + (q-1)\left(\log^+ \frac{2}{l} + \log 2\right).$$

For all $j \in [1, q] \setminus \{k\}$, $|f(z) - a_j| \geq \frac{l}{2}$. Therefore,

$$\log^+|f(z)-a_j| = \log|f(z)-a_j| + \log^+\frac{1}{|f(z)-a_j|} \le \log|f(z)-a_j| + \log^+\frac{2}{l}.$$

So

$$\log^+|f(z)| \le \log^+|f(z)-a_j| + \log^+|a_j| + \log 2 \le \log|f(z)-a_j| + \log^+\frac{2}{l} + \log^+|a_j| + \log 2,$$

and this gives

$$(q-1)\log^+|f(z)| \le \sum_{j=1,j\neq k}^{q} \log|f(z)-a_j| + C(a_1,a_2,\dots,a_q). \tag{3}$$

Now,

$$\begin{aligned}\sum_{j=1,j\neq k}^{q} \log|f(z)-a_j| &= \sum_{j=1}^{q}\log|f(z)-a_j| + \log\frac{1}{|f(z)-a_k|}\\ &= \sum_{j=1}^{q}\log|f(z)-a_j| - \log|f'(z)| + \log\frac{|f'(z)|}{|f(z)-a_k|}\\ &\le \sum_{j=1}^{q}\log|f(z)-a_j| - \log|f'(z)| + \log\left(\sum_{j=1}^{q}\frac{|f'(z)|}{|f(z)-a_j|}\right). \end{aligned}\tag{4}$$

The point of this manipulation is to eliminate k from the inequality, so that none of the a_j are treated any differently than the others. Using 3 and 4, we have

$$\begin{aligned}(q-1)m(f,r) &= \frac{1}{2\pi}\int_0^{2\pi}(q-1)\log^+|f(re^{i\theta})|\,d\theta\\ &\le \frac{1}{2\pi}\sum_{j=1}^{q}\int_0^{2\pi}\log|f(re^{i\theta})-a_j| - \frac{1}{2\pi}\int_0^{2\pi}\log|f'(re^{i\theta})|\,d\theta\\ &\quad+\frac{1}{2\pi}\int_0^{2\pi}\log\left(\sum_{j=1}^{q}\frac{|f'(re^{i\theta}|}{|f(re^{i\theta})-a_j|}\right)d\theta + C(a_1,a_2,\dots,a_q). \end{aligned}\tag{5}$$

Recall 3.3. Applying this to $f'(z)$ and $f(z)-a_j$ allows us to eliminate most of the integrals in 5. We have

$$\begin{aligned}(q-1)m(f,r) - \sum_{j=1}^{q} N\left(\frac{1}{f-a_j},r\right) + qN(f,r) + N\left(\frac{1}{f'},r\right) - N(f',r) &\le \frac{1}{2\pi}\int_0^{2\pi}\log\left(\sum_{j=1}^{q}\frac{|f'(re^{i\theta})|}{|f(re^{i\theta})-a_j|}\right)d\theta - \log|\operatorname{ilc}(f',0)|\\ &\quad+\sum_{j=1}^{q}\log|\operatorname{ilc}(f-a_j,0)| + C(a_1,a_2,\dots,a_q).\end{aligned}$$

The left hand side is

$$(q-1)T(f,r) - \left(N(f,r) + \sum_{j=1}^{q} N\left(\frac{1}{f-a_n},r\right)\right) + N_{ram}(f,r).$$

To finish the proof, we note that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\log\left(\sum_{j=1}^{q}\frac{|f'(re^{i\theta})|}{|f(re^{i\theta})-a_j|}\right)d\theta &\leq \frac{1}{2\pi}\int_0^{2\pi}\log^+\left(\sum_{j=1}^{q}\frac{|f'(re^{i\theta})|}{|f(re^{i\theta})-a_j|}\right)d\theta\\ &\leq m\left(\sum_{j=1}^{q}\frac{f'(z)}{f(z)-a_j},r\right)\\ &\leq \sum_{j=1}^{q}m\left(\frac{f'(z)}{f(z)-a_j},r\right)+\log q\\ &= o(T(f,r)).\end{aligned}$$

□

6.1 Deficient Values

In this section, we will be studying something called *deficient values*. The idea is that we can predict how often f will hit a given value c based on how fast f grows. However, sometimes f hits c less often than expected. To measure this discrepancy, we define a function called the *Nevanlinna deficiency function*:

Definition 6.4 (Nevanlinna deficiency function). Let f be an entire function and let $c \in \mathbb{C}$. The *Nevanlinna deficiency function* $\delta_N(c,f)$ is defined as

$$\delta_N(c,f) = \liminf_{r\to\infty}\frac{m(f,c,r)}{T(f,r)} = 1 - \limsup_{r\to\infty}\frac{N(f,c,r)}{T(f,r)}.$$

Now, since $m(f,c,r) \leq T(f,r)$ (asymptotically) with both nonnegative, we know that $0 \leq \delta_N(c,f) \leq 1$. If $\delta_N(c,f) = 0$, then $N(f,c,r) \sim T(f,r)$ as $r \to \infty$ so the frequency at which f hits c is on the level of f's growth. This is what we would usually expect, and is indeed what happens most of the time. However, there are some rare cases where $\delta_N(c,f) > 0$, and these c are what we call *deficient values*:

Definition 6.5 (Deficient Value). A *deficient value* of an entire function f is a complex number c such that

$$\delta_N(c,f) > 0.$$

Example. Take $f(z) = e^z$, and $c = 0$. Then

$$\delta_N(0,f) = 1 - \limsup_{r\to\infty}\frac{N(f,0,r)}{T(f,r)} = 1 - \limsup_{r\to\infty}\frac{0}{T(f,r)} = 1 - 0 = 1,$$

since e^z never hits 0.

6.2 Picard's Theorem

One of the principle motivations behind the development of Nevanlinna Theory was to provide a deeper understanding of *why* Picard's theorems hold true. Although Picard and subsequent mathematicians offered rigorous analytical proofs of these theorems, these arguments were often seen as ad hoc, and did not illuminate the underlying mechanisms that made them natural. Fortunately, with the machinery of Nevanlinna theory, the proofs of both the Little and Great Picard Theorems become not only streamlined but also philosophically satisfying.

Theorem 6.6 (Picard's Little Theorem). *Let $f : \mathbb{C} \to \mathbb{CP}^1$ be a non-constant meromorphic function. Then, f can omit at most two values in $\mathbb{CP}^1$.*

Proof. Assume for the sake of contradiction, there exists a non-constant meromorphic function that omits $a_1, a_2, a_3 \in \mathbb{CP}^1$. Note that

$\delta(f, a_i) = 1$ directly by definition. By the defect version of the Second Main Theorem, we have

$$\sum_{a \in \mathbb{CP}^1} \delta(a, f) \leq 2.$$

But $\delta(f, a_1) + \delta(f, a_2) + \delta(f, a_3) = 3 > 2$, leading to a contradiction. □

Remark 6.1. Picard's Little Theorem can be generalized far beyond $\mathbb{CP}^1$ with the Second Main Theorem, see for instance [Don24].

The more difficult Picard's Great Theorem requires more work.

7 Applications to Differential equations

Nevanlinna Theory is particularly powerful in the study of second-order linear differential equations with variable coefficients. These have the form

$$f'' + A(z)f' + B(z)f = 0 \tag{6}$$

where $A(z)$ and $B(z) \not\equiv 0$ are entire.

Much of the machinery for differential equations relies on deficiencies. The first essential question is to determine when a function of order ρ has deficiencies. In particular, we can show that such an f exists for all $\rho > \frac{1}{2}$.

Theorem 7.1. *Let $\rho > \frac{1}{2}$. Then, there exists an equation of the form 6 with a solution f such that*

$$\rho(B) \leq \rho(A) \leq \rho(f) = \rho;$$
$$\delta_N(0, f) > 0.$$

See ([GHW21], p.17) for the proof.

It turns out that there are no functions of order $\leq \frac{1}{2}$ that have deficient values. Thus we call Theorem 7.1 *sharp* with respect to ρ, because it gives the sharpest possible bound.

You are probably wondering why functions of order $\leq \frac{1}{2}$ cannot have deficient values. The proof is a bit involved, so we only give a broad explanation of how it works. (See [GO08], p.207 for more details).

Take any function f of order $\rho \leq 1/2$. We want to show that f cannot have any Nevanlinna deficient values. It suffices to show $\delta(0, f) = 0$, and we may assume that $f(0) = 1$. There is a certain inequality that can be proven holds for all σ with $\rho < \sigma < 1$. Then, we manipulate the equation to get that $\delta(0, f) \leq$ an expression of σ. Since $\rho \leq \frac{1}{2} < 1$, we are allowed to make σ approach $\frac{1}{2}$ in the inequality, and that yeilds $\delta(0, f) \leq 0$. Since deficient values can't be negative, we have $\delta(0, f) = 0$, as desired.

We can explicitly construct deficient functions f with order $\rho > \frac{1}{2}$. One such construction is the class of *Lindelöf functions*.

Definition 7.2 (Lindelöf Function). A Lindelöf function L_p is defined by

$$L_\rho(z) := \prod_{n=1}^{\infty} \left(1 + \frac{z}{n^{1/\rho}}\right) e_{\lfloor \rho \rfloor} \left(-\frac{z}{n^{1/\rho}}\right),$$

where $e_k(z)$ is defined as

$$e_k(z) := \begin{cases} 1, & \text{if} \quad k = 0 \\ e^{\sum_{j=1}^{k} \frac{z^j}{j}} & \text{if} \quad k \in \mathbb{Z}^+. \end{cases}$$

For the motivation of this definition, see [GHW21].

For slowly growing functions f, we can consider a function analogous to the Nevanlinna deficiency function, called the *Valiron deficiency function*:

Definition 7.3. Let f be an entire function, and let $c \in \mathbb{C}$. The *Valiron deficiency function* is defined as

$$\delta_V(c, f) = \limsup_{r\to\infty} \frac{m(f,c,r)}{T(f,r)} = 1 - \liminf_{r\to\infty} \frac{N(f,c,r)}{T(f,r)}.$$

This may appear to be the same as the definition for $\delta_N(c, f)$. However, notice that in the definition of δ_N we have

$$\liminf_{r\to\infty} \frac{m(f,c,r)}{T(f,r)},$$

while in the case of δ_V we have $\limsup$ instead of $\liminf$. In particular, this implies that

$$0 \le \delta_N(c, f) \le \delta_V(c, f) \le 1.$$

Just as with δ_N, the vast majority of times it happens that $\delta_V(c, f) = 0$. When this is false, we call c a *Valiron deficient value* of f:

Definition 7.4. We say that c is a *Valiron deficient value* of f if

$$\delta_V(c, f) > 0.$$

For slowly growing entire functions, Nevanlinna deficient values are not possible, but Valiron deficient values are. The growth of such functions f can be measured in terms of the *logarithmic order* $\rho_{\log}(f)$:

Definition 7.5. We define the *logarithmic order* $\rho_{\log}(f)$ to be

$$\rho_{\log}(f) = \limsup_{r\to\infty} \frac{\log T(f,r)}{\log\log r}.$$

There is an analogous result to Theorem 7.1 using Valiron deficiency and logarithmic order:

Theorem 7.6. *Let $\rho > 2$. Then there exists an equation of the form 6 with a solution f satisfying $\delta_V(0, f) = 1$ and*

$$\rho_{\log}(B) \le \rho_{\log}(A) \le \rho_{\log}(f) = \rho,$$

where $A(z)$ is transcendental.

Remark. Here, transcendental means that $A(z)$ is not a polynomial of z.

As with Theorem 7.1, the result fails for all $\rho < 2$. But unlike before, the case $\rho = 2$ is unsolved. Thus we call Theorem 7.6 *essentially sharp* with respect to ρ.

Now, here is a useful theorem for finding differential equations for entire functions f:

Theorem 7.7 (Gol'dberg's Theorem). *Let $f \not\equiv 0$ be an entire function whose zeros all have multiplicity at most $n-1$, for $n \in \mathbb{N}$. Then f is a solution of some differential equation of the form*

$$f^{(n)}(z) + A_{n-1}(z)f^{(n-1)}(z) + \cdots + A_1(z)f'(z) + A_0(z)f(z) = 0, \tag{7}$$

where the A_i's are entire and $A_0 \not\equiv 0$.

Proof. In the case $n = 1$, f has no zeros and the proof is trivial: Take $A_1(z) = 1$ and $A_0(z) = -\frac{f'(z)}{f(z)}$. For convinience sake, we will only prove the case $n = 2$. The general case is quite similar.

We assume that f has zeros, because if f has no zeros then the statement is trivial. Next, let's write

$$f''(z) + A(z)f'(z) + B(z)f(z) = 0$$

for A, B entire and $B \not\equiv 0$. Since f has zeros, we cannot simply make $B(z)$ cancel with everything else. If we want to satisfy the differential equation for the zeros z_k of f, we need

$$f''(z_k) + A(z_k)f'(z_k) = 0,$$

or

$$A(z_k) = -\frac{f''(z_k)}{f'(z_k)} =: \sigma_k$$

for all k. (Recall that if f has a simple zero at z then $f'(z) \neq 0$.) If $\{z_k\}$ is a finite sequence, then $A(z)$ can be chosen as the Lagrange interpolation polynomial. If it is an infinite sequence, then $A(z)$ can be constructed using theorems of Weirstraß and Mittag-Leffler (see 9.1). Also, fix $\zeta \neq z_k$. In addition to our constraints on z_k, we require that

$$A(\zeta) \neq -\frac{f''(\zeta)}{f'(\zeta)}.$$

(To meet this requirement, just pick a different value for $A(\zeta)$.) This guarantees $f''(z) + A(z)f'(z) \not\equiv 0$. After finding an entire function $A(z)$ that satisfies all these properties, we define $B(z)$ by

$$-\frac{f''(z) + A(z)f'(z)}{f(z)}.$$

The numerator has zeros at the simple zeros of f, thus turning those into removable singularities when we divide out. Also, since the numerator is not congruent to 0, we must have that B is an entire function such that $B(z) \not\equiv 0$. You can check that these A and B satisfy 6, and thus we are done. □

Remark. If f is an entire function, there exists a constant c for which $g := f - c$ has all simple zeros (see 9.2). A consequence of this observation is that many properties of f, such as the number of deficient values, remain valid for g, and by Gol'dberg's Theorem, g solves some equation of the form 6.

Example.

1) For any finite number $q \geq 2$, there exists a solution of 6 with q Nevanlinna-deficient values. Indeed, set

$$f(z) = \int_0^z e^{-s^q} ds$$

and

$$a_k = e^{2\pi ki/q} \int_0^\infty e^{-s^q} ds,$$

where $1 \leq k \leq q$. Then f is entire, $\delta_N(a_k, f) = 1/q$, and $\delta_N(c, f) = 0$ whenever $c \neq a_k$ for all $1 \leq k \leq q$ (see [Wri65] p.46-47). We know that $f'(z) = e^{-z^q}$ has no zeros. Thus we find that for any constant c, the function $g = f - c$ has all simple zeros and exactly q Nevanlinna deficient values (since a simple zero is a zero z such that $f'(z) \neq 0$). Therefore by Gol'dberg's Theorem, g is a solution of some equation of the form 6.

2) Eremenko proved ([GO08], p.132) that for any countable set $E \subseteq \mathbb{C}$ and any $\rho > 1/2$, there exists an entire function f of order ρ such that $E_N(f) = E$. If f is any such function, then for a suitable $c \in \mathbb{C}$, the function $g = f - c$ has only simple zeros and countably many Nevanlinna deficient values, and g solves an equation of the form 6.

3) Let f be an entire function with uncountably many Valiron deficient values. For a suitable $c \in \mathbb{C}$, $g = f - c$ has only simple zeros and uncountably many Valiron deficient values, and g solves an equation of the form 6.

Gol'dberg's Theorem does not give information about the orders of the coefficients in 7. Thus, although Gol'dberg's Theorem is useful in the above discussions, it cannot be used to prove the respective inequalities

$$\rho(B) \leq \rho(A) \leq \rho(f)$$

and
$$\rho_{\log}(B) \leq \rho_{\log}(A) \leq \rho_{\log}(f)$$
in Theorems 7.1 and 7.6.

7.1 Malmquist's Theorem

A significant result in the area of differential equations is that of Malmquist's Theorem, a useful tool for solving rational ordinary nonhomogenous differential equations.

Theorem 7.8 (Malmquist's Theorem). *Denote by $\mathbb{C}(z)$ the field of rational functions in the variable z with complex coefficients. Denote by $\mathbb{C}(z)[X]$ the one-variable polynomial ring with coefficients in $\mathbb{C}(z)$. Let $P(X)$ and $Q(X)$ be relatively prime elements of $\mathbb{C}(z)[X]$. If the differential equation*
$$f' = \frac{P(f)}{Q(f)}$$
has a transcendental meromorphic solution f, then Q has degree zero in X and P has degree at most 2 in X.

Again, recall that a meromorphic function f is *transcendental* if it is not a quotient of polynomials.

We break the proof of this theorem into pieces. First we prove a proposition relating the quotient $P(f)/Q(f)$ to the characteristic function of f:

Lemma 7.9. *Let P and Q be as in the above theorem, and let f be an arbitrary transcendental meromorphic function. Write $g = P(f)/Q(f)$. Then*
$$T(g, r) = \max\{\deg P, \deg Q\} T(f, r) + o(T(f, r)).$$

The proof is a bit involved, and we will not go into it (see [CY01], p.130). We now state a weaker version of Malmquist's Theorem as a lemma:

Lemma 7.10. *Let $P(X)$ and $Q(X)$ be relatively prime polynomials in $\mathbb{C}(z)[X]$ and let f be a transcendental meromorphic solution to*
$$f' = \frac{P(f)}{Q(f)}.$$
Then, $\deg P \leq 2$ *and* $\deg Q \leq 2$.

Proof. Let $d = \max\{\deg P, \deg Q\}$. Since $f' = P(f)/Q(f)$, by the previous proposition we have
$$\begin{aligned} dT(f, r) &= T(P(f)/Q(f)) + o(T(f, r)) \\ &= T(f', r) + o(T, r) \\ &= m(f', \infty, r) + N(f', \infty, r) + o(T(f, r)). \end{aligned}$$
Now, $N(f', \infty, r) \leq 2N(f, \infty, r)$ for $r \geq 1$, and by Proposition 1.5.1 ([CY01]),
$$m(f', \infty, r) \leq m\left(\frac{f'}{f}, \infty, r\right) + m(f, \infty, r).$$
Clearly, $m(f, \infty, r) + 2N(f, \infty, r) \leq 2T(f, r)$, and so we conclude from the Logarithmic Derivative Lemma (Theorem 3.4.1 of [CY01]) that
$$dT(f, r) \leq 2T(f, r) + o(T(f, r))$$
for a sequence of $r \to \infty$. This means that $d \leq 2$, thus $\deg(P) \leq 2$ and $\deg(Q) \leq 2$. ∎

The above lemma reduces the proof of Malmquist's Theorem to the case when P and Q are quadratic. The proof is essentially completed by the following elementary lemma about quadratic polynomials:

Lemma 7.11. *Let*

$$P(X) = a_2X^2 + a_1X + a_0 \text{ and } Q(X) = b_2X^2 + b_1X + b_0,$$

where a_0, a_1, a_2, b_0, b_1, and b_2 are all in $\mathbb{C}(z)$. Let

$$\tilde{P}(X) = a_0X^2 + a_1X + a_2 \text{ and } \tilde{Q}(X) = b_0X^2 + b_1X + b_2.$$

Then, P and Q are relatively prime in $\mathbb{C}(z)[X]$ if and only if $\tilde{P}$ and $\tilde{Q}$ are relatively prime in $\mathbb{C}(z)[X]$.

Proof. Since P, Q are quadratics, the only nontrivial common factor they could have is a linear one, i.e. a factor of the form $c_1X + c_0$. But, by the symmetry of polynomial multiplication, this is equivalent to the statement that $c_0X + c_1$ is a common factor of $\tilde{P}$ and $\tilde{Q}$. □

With all of our ingredients, we may now prove Malmquist's Theorem:

Proof of Malmquist's Theorem. Let f be a transcendental meromorphic solution to the differential equation

$$f' = \frac{Q(f)}{P(f)}.$$

By Lemma 7.10, we have

$$P(X) = a_2X^2 + a_1X + a_0 \text{ and } Q(X) = b_2X^2 + b_1X + b_0,$$

where a_0, a_1, a_2, b_0, b_1, and b_2 are in $\mathbb{C}(z)$. We need to show that

$$b_1(z) \equiv b_2(z) \equiv 0.$$

Let $g = 1/f$. Then,

$$\begin{aligned} g' = -\frac{f'}{f^2} &= -\frac{a_0 + a_1f + a_2f^2}{f^2[b_0 + b_1f + b_2f^2]} \\ &= -\frac{g^2[a_0 + a_1/g + a_2/g^2]}{b_0 + b_1/g + b_2/g^2} \\ &= -\frac{g^2[a_0g^2 + a_1g + a_2]}{b_0g^2 + b_1g + b_2}. \end{aligned}$$

Thus, g is a transcendental meromorphic solution to

$$g' = \frac{g^2\tilde{P}(g)}{\tilde{Q}(g)},$$

where $\tilde{P}$ and $\tilde{Q}$ are defined as in Lemma 7.11. Now, the above fraction must reduce to a fraction of quadratics by Lemma 7.10. Since $X^2\tilde{P}(X)$ has degree 4, this means that it and $\tilde{Q}(X)$ must have a quadratic factor in common. But since $\tilde{P}$ and $\tilde{Q}$ are relatively prime by Lemma 7.11, this means that X^2 divides $\tilde{Q}(X)$, and by definition of $\tilde{Q}$ this implies that $b_1(z) \equiv b_2(z) \equiv 0$.□

8 Research Trends and Open Problems

8.1 Generalizations

Nevanlinna Theory has been applied in a surprising number of seemingly unrelated fields. We have already briefly discussed the applications to Number Theory and Differential Equations, though we note that there is a great abundance of literature on this topic that we have not begun to comprehensively cover.

Of course, a major question is whether the ideas and results of Nevanlinna Theory can be generalized to settings beyond $\mathbb{CP}^1$. The development of non-Archimedean analogs of Nevanlinna Theory have recently emerged as an active area of research, particularly in the context of p-adic analysis. Analogues of the First and Second Main Theorem have been developed for p-adic meromorphic functions on $B_\rho(0)$.

Theorem 8.1 (p-adic First Main Theorem). *Let f be a non-constant meromorphic function in $B(\rho)$. Then, for every $a \in \mathbb{C}_p$,*

$$m\left(\frac{1}{f-a}, r\right) + N\left(\frac{1}{f-a}, r\right) = T(f,r) + O(1) \qquad (r \to \rho).$$

Theorem 8.2 (p-adic Second Main Theorem). *Let f be a non-constant meromorphic function in $B(\rho)$ and let $a_1, \dots, a_q$ be distinct numbers in $\mathbb{C}_p$. Then,*

$$(q-1)T(f,r) \leq N(f,r) + \sum_{j=1}^{q} N\left(\frac{1}{f-a}, r\right) - 2N(f,r) + N(f',r) - N\left(\frac{1}{f'}, r\right) - \log r + O(1).$$

The interested reader may refer to [YH99] for proofs of these theorems.

Extending these results to higher dimensional p-adic varieties remains unresolved, as the connections between algebraic geometry and value distribution remain poorly understood. Furthermore, technical results regarding ramification and error terms also demand further investigation.

Some research has also been done extending Nevanlinna theory to difference operators. While Nevanlinna theory provides estimates involving the derivative $f \mapsto f'$, estimates on the exact difference $f \mapsto \Delta f = f(z+c) - f(z)$ are less understood. Some progress have been made for the c- paired points of meromorphic functions f, where a point a is said to be c-paired if $f(z) = f(z+c) = a$. [HK05] proved an analog of the Second Main Theorem in this case, which we state below.

Theorem 8.3 (Second Main Theorem for Difference Operators). *Let $c \in \mathbb{C}$, and let f be a meromorphic function of finite order such that $\Delta_c f \not\equiv 0$. Let $q \geq 2$, and let $a_1(z), \dots, a_q(z)$ be distinct meromorphic periodic functions with period c such that $a_k \in \mathcal{I}(f)$ for all $k = 1, \dots, q$. Then*

$$m(r,f) + \sum_{k=1}^{q} m\left(r, \frac{1}{f-a_k}\right) \leq 2T(r,f) - N_{pair}(r,f) + S(r,f),$$

where

$$N_{pair}(r,f) := 2N(r,f) - N(r, \Delta_c f) + N\left(r, \frac{1}{\Delta_c f}\right),$$

and the exceptional set associated with $S(r,f)$ is of at most finite logarithmic measure.

Other results surrounding this topic can be found in [HK05].

Extending Nevanlinna theory to several variables has proven notoriously difficult, as the naive generalization of counting functions fails to capture the complexity of higher dimensional mappings.

8.2 The Inverse Problem

The so-called direct problem of Nevanlinna Theory focuses on analyzing the distribution of values taken on by a fixed meromorphic function. In contrast, the inverse problem poses a more delicate question. In its essence, the inverse problem asks to what extent the value distribution of a meromorphic function be prescribed under given hypothesis.

The Problem can formally be stated as follows: Given a finite or countable set of values $a_{j_{j\in J}} \subset \mathbb{CP}^1$ and corresponding deficiencies $\delta_{j_{j\in J}} \in [0,1]$ satisfying

$$\sum_{j\in J} \delta_j \leq 2,$$

does there exists a non-constant meromorphic function f on $\mathbb{C}$ such that $\delta(a_j, f) = \delta_j$ for all $j \in J$?

As of current, this problem remains unsolved in its general case, though constructions in specific cases have been found. We present part of a famous construction due to [Wri65] for entire functions under the conditions above.

9 Appendix

9.1 Mittag-Leffler and Weirstraß

Consider two sequences $\{a_k\}$ and $\{b_k\}$, where $\{a_k\}$ are isolated points and all terms in $\{a_k\}$ are distinct. As in Gol'dberg's Theorem, we want to find an entire function $f(z)$ such that $f(a_k) = b_k$ for all k.

Theorem 9.1 (Weirstraß Factorization Theorem). *Let*

$$0, \dots, 0, a_1, a_2, \dots$$

(with m zeros) be a sequence of complex numbers, sorted by absolute value, so that $a_n \to \infty$ as $n \to \infty$. Then

$$f(z) = z^m \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e_n\left(\frac{z}{a_n}\right)$$

is an entire function that vanishes at 0 (if $m > 0$) and the a_k's and nowhere else, so that the order of the zero at a is equal to the number of occurences of a in the sequence.

In other words, $f(z)$ has zeros at each of the terms in the sequence, counting multiplicity.

Theorem 9.2 (Mittag-Leffler Theorem). *Let U be an open set in $\mathbb{C}$ and let $E \subset U$ be a subset whose limit points occur only on the boundary of U. For each $a \in E$, let $p_a(z)$ be a polynomial in $1/(z-a)$ without constant coefficient, i.e. of the form*

$$p_a(z) = \sum_{n=1}^{N_a} \frac{c_{a,n}}{(z-a)^n}.$$

Then there exists a meromorphic function f on U whose poles are precisely the elements of E and so that for each such pole $a \in E$, the function $f(z) - p_a(z)$ has a removable singularity at a.

Now, plugging the sequence $\{a_k\}$ into the Wierstraß Factorization Theorem yields a function $f_1(z)$ that has simple zeros at the a_k's. Plugging in $E = \{a_k\}$ and

$$p_a(z) = \frac{b_k}{f_1'(a_k)(z - a_k)},$$

we get a function $f_2(z)$ with poles at the a_k's such that

$$f_2(z) - \frac{b_k}{f_1'(a_k)(z - a_k)} =: h_k(z)$$

has a removable singularity at a_k. (Note that $f_1'(a_k) \neq 0$ since f_1 has a simple zero there.) Then, we define $f(z) = f_1(z)f_2(z)$. Since f_2 has simple poles where f_1 has zeros, we have that f is an entire function. For all k, we compute that

$$\begin{aligned}
f(a_k) &= \lim_{z \to a_k} f_1(z)f_2(z) \\
&= \lim_{z \to a_k} f_1(z)h_k(z) + \frac{b_k f_1(z)}{f_1'(a_k)(z - a_k)} \\
&= \lim_{z \to a_k} \frac{b_k f_1(z)}{f_1'(a_k)(z - a_k)} \\
&= \lim_{z \to a_k} \frac{b_k f_1'(z)}{f_1'(a_k)} \\
&= \frac{b_k f_1'(a_k)}{f_1'(a_k)} \\
&= b_k,
\end{aligned}$$

by L'hospital's rule. Thus we have found our f.

9.2 Functions with All Simple Zeros

If $f(z)$ is an entire function, we want to prove that there exists a constant c such that $g(z) := f(z) - c$ has all (if any) of its zeros simple. Let E be the set of zeros of $f'(z)$. In order for our function $g(z)$ to have all simple zeros, there must not exist z such that $g(z) = 0$ and $g'(z) = f'(z) = 0$. In other words, we need a constant c such that whenever $z \in E$, $f(z) \neq c$. But E must be countable since it's the set of zeros of some function. If E is countable, then the image $f(E)$ is also countable, so there exist complex numbers $c \notin f(E)$. Such a c satisfies the desired properties.

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