MODULAR FORMS AND SUMS OF SQUARES

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Abstract. In this paper, we will introduce modular forms, functions on the upper half plane which satisfy a certain transformation property under the action of the modular group $SL_2(\mathbb{Z})$, as well as a holomorphy condition. We will see important examples of modular forms with number theoretic applications. We then discuss a technical "valence formula," which relates the Laurent series orders of a modular form at points in the orbits of our action, and an analogous formula for *congruence subgroups* of $SL_2(\mathbb{Z})$. These formulas allow us to find the dimension of spaces of modular forms as $\mathbb{C}$-vector spaces. We apply these technical results to the problem of counting the number of ordered pairs of k integers whose squares sum to a certain positive integer n, specifically proving the $k = 4$ case and providing a framework for higher cases.

1. Introduction

It is a classic theorem of Lagrange that every positive integer n can be written as the sum of four nonnegative integer squares. For example, with $n = 26$, we have:

$$26 = 5^2 + 1^2 + 0^2 + 0^2 = 25 + 1 + 0 + 0$$

and also

$$26 = 4^2 + 3^2 + 1^2 + 0^2 = 16 + 9 + 1 + 0.$$

A natural question that arises is the number of distinct ways we can write a given positive integer n in this way. More generally, we ask for the number of ways to write n as a sum of k squares, where k is a positive integer.

Definition 1.1. We define the function $r_k(n)$ by

$$r_k(n) = \#\{(x_1, x_2, \dots, x_k) \in \mathbb{Z}^k | x_1^2 + x_2^2 + \cdots + x_k^2 = n\}.$$

The following theorem of Jacobi answers our question for $k = 4$.

Theorem 1.2. *We have*

$$r_4(n) = 8 \sum_{\substack{d|n \\ 4\nmid d}} d.$$

To prove this theorem, we will introduce the theory of *modular forms*, functions which display a certain invariance property with respect to action by the modular group $SL_2(\mathbb{Z})$. This route will allow us to give a more natural proof of the result than standard elementary methods with the Jacobi Triple Product, and hint towards generalizations to other k.

Date: May 2025.

2. The Modular Group and Upper Half Plane

We begin by reviewing and expanding on some basic background of $\mathrm{SL}_2(\mathbb{Z})$ acting on the upper half plane $\mathbb{H}$.

Definition 2.1. The *modular group* $\mathrm{SL}_2(\mathbb{Z})$ is the group of 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with determinant 1 and entries in $\mathbb{Z}$.

One easily verifies that this construction is in fact a group, and in particular, that all matrices in $\mathrm{SL}_2(\mathbb{Z})$ have inverses with integer entries. It turns out that $\mathrm{SL}_2(\mathbb{Z})$ is finitely generated by two elements. Our proof is due to [3].

Theorem 2.2. *The group $SL_2(\mathbb{Z})$ is generated by the elements $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.*

Proof. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and let G be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by the two elements. Observe that

$$BA^{-1}B^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \implies BA^{-n}B^{-1} = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}.$$

Combined with the fact that $B^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, we get that G contains all elements of $\mathrm{SL}_2(\mathbb{Z})$ of the form $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$. Suppose for the sake of contradiction that G is not all of $\mathrm{SL}_2(\mathbb{Z})$. Let b_0 denote the minimal magnitude of b such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) - G$. It follows that b_0 is nonzero. For some α in $\mathrm{SL}_2(\mathbb{Z}) - G$ with top entries a_0 and b_0, we take some integer n such that $|a_0 - nb_0| < b_0$. Then, the top right entry of $\alpha B^{-1}A^n$ is given by $a_0 - nb_0$, and thus $\alpha B^{-1}A^n \in G$. However, since A and B are invertible, this implies that $\alpha \in G$, contradicting our assumption that $\alpha \in \mathrm{SL}_2(\mathbb{Z}) - G$. ■

We recall the standard group action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane $\mathbb{H}$ by mobius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}.$$

The proof that this map respects the group operation of $\mathrm{SL}_2(\mathbb{Z})$ can be done, by example, with direct computation. We can check that this action maps elements of $\mathbb{H}$ back to $\mathbb{H}$ via the formula

$$\Im(\alpha z) = \frac{\Im(z)}{|cz+d|^2}.$$

Note that the non-identity matrix $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is in the kernel of this action, and is the only such nonidentity matrix. Thus, restricting this action to the quotient group $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{I, -I\}$ makes the action faithful (Some authors define the modular group as $\mathrm{PSL}_2(\mathbb{Z})$ rather than $\mathrm{SL}_2(\mathbb{Z})$ for this reason).

3. Modular Forms

In this section, we define and motivate *modular forms*, our main objects of study.

Definition 3.1. Let k be an integer and f be a meromorphic function on the upper half plane $\mathbb{H}$. We say that f is *weakly modular* of weight k if

$$f(\alpha z) = (cz+d)^k f(z) \text{ for all } z \in \mathbb{H} \text{ and } \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{3.1}$$

Note that using $\alpha = -I$ (with I denoting the 2×2 identity) in the equation (3.1) gives $f(z) = (-1)^k f(z)$, implying that no nonzero weakly modular functions of odd weight exist. Thus, in what follows we assume k is even. At first glance, this definition seems strange and unnatural. We observe the following property of the definition.

Proposition 3.2. *If equation (3.1) is satisfied for two elements α, β of $SL_2(\mathbb{Z})$, it is satisfied for their product $\alpha\beta$.*

Proof. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$, we define

$$j(\gamma, z) = cz + d.$$

Then, our modularity condition becomes $f(\gamma, z) = j(\gamma, z)^k f(z)$. By the fact that $\mathrm{SL}_2(\mathbb{Z})$ operates through fractional linear transformations as a group action on $\mathbb{H}$, we have:

$$j(\alpha\beta, z) = j(\alpha, \beta z) j(\beta, z).$$

Thus,

$$f((\alpha\beta)z) = f(\alpha(\beta z)) = j(\alpha, \beta z) f(\beta z) = j(\alpha, \beta z) j(\beta, z) f(z) = j(\alpha\beta, z) f(z)$$

by consectively applying (3.1) for α and β. Comparing the left and right hand sides gives equation (3.1) for $\alpha\beta$, as desired. ■

Remark 3.3. The function $j(\gamma, z)$ is often known as the *factor of automorphy.*

Corollary 3.4. *Let f be meromorphic on $\mathbb{H}$. Then, f is weakly modular of weight k if and only if it satisfies the following relations:*

$$f(z+1) = f(z) \tag{3.2}$$

$$f\left(\frac{-1}{z}\right) = z^k f(z) \tag{3.3}$$

for all $z \in \mathbb{H}$.

Proof. Let A and B denote the same generators of $\mathrm{SL}_2(\mathbb{Z})$ as in Theorem 2.2. If f is weakly modular of weight k, using the matrices A and B respectively as α in equation (3.1) gives the two relations above. Conversely, since A and B generate $\mathrm{SL}_2(\mathbb{Z})$, it suffices by Proposition 3.2 to verify the weak modularity equation for just these two matrices and their inverses. Checking A^{-1} gives $f(z-1) = f(z)$, which is equivalent to (3.2). Checking B^{-1} gives $f\left(\frac{1}{-z}\right) = (-z)^k f(z) = z^k f(z)$, which is equivalent to (3.3). ■

Corollary 3.3 gives a much more natural condition which is equivalent to weak modularity. Note that weak modularity of weight 0 is the same as invariance under $\mathrm{SL}_2(\mathbb{Z})$ by our proof of proposition 3.2.

Further motivation comes from a connection to functions that behave well with complex lattices, which we leave out in this paper. More details can be found in [1]. While on the topic, we state a result about lattices which will be useful later. We do not prove this result here, instead referring to [2].

Proposition 3.5. *For* $\omega_1', \omega_2' \in \mathbb{C}$*, we have* $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ *and* $\frac{\omega_1'}{\omega_2'} \in \mathbb{H}$ *if and only if*

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \alpha \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

for some $\alpha \in SL_2(\mathbb{Z})$*.*

Our definition of modular forms comes from an addition of useful holomorphy conditions to the transformation properties of Definition 3.1. Let f denote a weakly modular function of weight k. By the additive periodicity in equation (3.2), we can write

$$f(z) = g\left(e^{2\pi i z}\right)$$

for some meromorphic function g.

Definition 3.6. A weakly modular function f is *holomorphic at* ∞ if g extends to a holomorphic function at 0.

The reference to ∞ comes from the fact that as $e^{2\pi i z} \to 0$, writing $z = x + yi$ gives: $e^{-2\pi y} \cdot e^{2\pi i x}$ approaches 0, meaning y goes to ∞. If this condition is satisfied, we write $f(\infty) = g(0)$. Definition 3.4 allows us to define modular forms.

Definition 3.7. Let k be an integer and f be a holomorphic function on $\mathbb{H}$. If f is weakly modular of weight k and holomorphic at ∞, we call f a *modular form* for the group $SL_2(\mathbb{Z})$

It is easy to see from the definition that the set of modular forms of weight k form a $\mathbb{C}$-vector space. With some computation, we can also verify that modular forms form a graded ring with their weights. In the next sections, we will see two important examples of modular forms. One special case we would like to distinguish is when $f(\infty) = 0$.

Definition 3.8. A cusp form of weight k (for the group $\mathrm{SL}_2(\mathbb{Z})$) is a modular form f of weight k such that $f(\infty) = 0$.

We specificy "for $\mathrm{SL}_2(\mathbb{Z})$" in Definitions 3.5 and 3.6 because it is possible to define modular forms for finite-index subgroups Γ of $\mathrm{SL}_2(\mathbb{Z})$ by restricting α to come from Γ in equation (3.1). We will see an important example of a class of subgroups we would like to consider in section 5.

4. Eisenstein Series

Our first example of nonzero modular forms is a class of functions we are already familiar with from the coefficients of the Laurent series of the Weiererstrass $\wp$ function.

Definition 4.1. Let $k \geq 4$ be a positive even integer. Let Λ be a lattice in the complex plane. Then the series

$$G_k(\Lambda) = \sum_{\gamma \in \Lambda, \gamma \neq 0} \frac{1}{\gamma^k}$$

is called the Eisenstein series of index k.

By considering lattices Λ_z generated by 1 and z, we can view Eisenstein series as functions on $\mathbb{H}$, which we denote $G_k(z)$. With comparison to a double integral, we can prove that these functions are absolutely convergent (the restriction $k \geq 4$ arises by failure of this convergence for $k = 2$). By Proposition 3.5, for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we get that the lattice generated by $az + b$ and $cz + d$ is the same as Λ_z. Thus,

$$\begin{aligned} G_k(z) &= \sum_{(m,n)\neq 0} \frac{1}{m(az+b) + n(cz+d)} \\ &= (cz+d)^{-k} \sum_{(m,n)\neq 0} \frac{1}{m(\alpha z) + n} \\ &= (cz+d)^{-k} G_k(\alpha z) \end{aligned}$$

proving weak modularity of weight k for Eisenstein series. For a fixed $z \in \mathbb{H}$, let $q = e^{2\pi i z}$. We recall the following result about Eisenstein series:

Lemma 4.2. *For even $k \geq 4$ and $z \in \mathbb{H}$, we have*

$$G_k = 2\zeta(k) + 2\frac{(-1)^{\frac{k}{2}}(2\pi)^k}{(k-1)!} \sum_{r=1}^{\infty} \sigma_{k-1}(r) q^r. \tag{4.1}$$

where $\sigma_n(r)$ is the sum of the n^{th} powers of the divisors of r.

Theorem 4.3. *The Eisenstein series $G_k(z)$ is a modular form of weight k.*

Proof. Viewing (4.1) as a convergent power series in q, we see by analytic and holomorphic being equivalent that E_k is holomorphic (making the appropriate change of variables through composition back to z). Furthermore, we see that we see that we can analytically extend the right hand side as a function of q to a holomorphic function at 0, with $E_k(\infty) = 2\zeta(k)$. Since we have already shown weak modularity of weight k, $E_k(z)$ being a modular form of weight k follows. ■

5. Congruence Subgroups

We define an important class of subgroups of $\mathrm{SL}_2(\mathbb{Z})$, especially relevant to number theoretic applications of modular forms. We construct two important examples of modular forms for these subgroups, which will be important in our proof of Theorem 1.2.

Definition 5.1. Let N be a positive integer. The *principal congruence subgroup of level N* is the group

$$\Gamma(N) = \left\{ \alpha \in \mathrm{SL}_2(\mathbb{Z}) | \ \alpha \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}, \right\}$$

where matrix congruence is taken entry-wise.

Definition 5.2. A *congruence subgroup* is a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ containing $\Gamma(N)$ for some N, which we call the *level* of Γ.

For our purposes, the most important example of a congruence subgroup is given by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | \ c \equiv 0 \pmod{N} \right\}.$$

Note that for $N|M$, we have the inclusions

$$\Gamma(M) \subset \Gamma_0(M) \subset \Gamma_0(N).$$

We see two important examples of modular forms for congruence subgroups, first finding a way to reconcile the notion of the Eisenstein series of index 2.

Definition 5.3. We define the normalized Eisenstein series E_k by:

$$E_k(z) = \frac{G_k(z)}{2\zeta(k)}.$$

Despite $G_2(z)$ not converging absolutely, we can still meaningfully talk about $E_2(z)$ as a holomorphic function. It turns out that $E_2(z)$ fails to satisfy weak modularity, and thus is not a modular form. In particular, $E_2(z)$ satisfies the transformation property

$$z^{-2}E_2\left(\frac{-1}{z}\right) = E_2(z) - \frac{1}{4\pi i z} \tag{5.1}$$

which contradicts relation (3.3). We can, however, show that

$$E_2^n(z) = E_2(z) - nE_2(nz)$$

is a modular form of weight 2 for the congruence subgroup $\Gamma_0(n)$. The proof of weak modularity is a simple computation based on (5.1) and the generation of $\mathrm{SL}_2(\mathbb{Z})$ by A and B from earlier, while holomorphy requires some more technical work. We now look at another example, which is familiar as a special case of the more general θ function:

Definition 5.4. We define the *theta function* $\theta(z)$ by

$$\theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

with q defined as in earlier sections.

We know that θ is a holomorphic function on $\mathbb{H}$. Most of the reason we care here about the θ function is due to its powers.

Proposition 5.5. *Let k be a positive integer. The θ function satisfies:*

$$\theta(z)^k = \sum_{n=0}^{\infty} r_k(n) q^n.$$

This result is easy to see by considering the direct multiplication and regrouping terms. We get one q^n term with coefficient 1 for each combination of squares summing to n. Moreover, θ^k is a modular form over a congruence subgroup for even k, though we will not discuss the proof here.

Theorem 5.6. *Let k be an even positive integer. Then the function θ^k is a modular form of weight $\frac{k}{2}$ for the congruence subgroup $\Gamma_0(4)$.*

6. Valence Formula and Dimension

In this section we discuss a technical result and its analog over congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. These results are both powerful and important in their own right, but also give us information about the dimension of the $\mathbb{C}$-vector space of modular forms of weight k.

Let f be a meromorphic function on $\mathbb{H}$, and p a point in H. Then, we can write f as a Laurent series around p:

$$f(z) = a_n(z-p)^n + a_{n+1}(z-p)^{n+1} + \dots$$

for some $n \in \mathbb{Z}$ and $a_i \in \mathbb{C}$. We call n the *order of valuation* or *order* of f at p, and denote it $\nu_p(f)$. If f is weakly modular, equation (3.1) along with the fact that $\nu_p(cz+d) = 0$ implies that $\nu_{\alpha p}(f) = \nu_p(f)$ for any $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ (the name and notation derive from ν being a discrete valuation on the field of Laurent series). Thus, order only depends on p's orbit under the action by $\mathrm{SL}_2(\mathbb{Z})$. We write $\nu_\infty(f)$ to denote the order at 0 of the function g defined in section 3. The following miraculous formula, known as the *valence formula* relates the orders of valuation of a weakly modular function f at each of these orbits.

Theorem 6.1. *Let f be a (nonzero) meromorphic function on $\mathbb{H}$, which is weakly modular of weight k and meromorphic at ∞ (defined analogously to holomorphic). Then, we have:*

$$\nu_\infty(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_\rho(f) + \sum_{p \in S} \nu_p(f) = \frac{k}{12}, \tag{6.1}$$

where S is a set with one representative for each orbit of $SL_2(\mathbb{Z})$ acting on $\mathbb{H}$ besides those for i and $\rho = e^{\frac{2\pi i}{3}}$.

The proof is long and not included here. See [4] for the complete proof. The idea is to evaluate a certain contour integral, where the sum of the orders naturally appears as a result of using the argument principle.

Definition 6.2. We define M_k as the $\mathbb{C}$-vector space of modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$.

Theorem 6.1 allows us to derive a powerful and surprising about the dimension of M_k. We require one additional lemma before we prove the main theorem.

Lemma 6.3. *Let S_k denote the vector space of cusp forms of weight k. Then, the vector spaces S_{k+12} and M_k are isomorphic, implying*

$$\dim S_{k+12} = \dim M_k.$$

Proof. Define the modular form Δ by

$$\Delta = \frac{(240E_4)^3 - (504E_6)^2}{1728}.$$

Here, the seemingly random coefficients are chosen so that the constant term vanishes in the q-expansion of Δ, which we can see by computation. We get that Δ is a cusp form of weight 12. Applying Theorem 6.1, we get that all orders must be nonnegative by Δ being holomorphic, and thus $\nu_\infty(\Delta) = 1$, with all other orders equal to 0. Equivalently, Δ has a simple zero at ∞ as its only zero. Thus, it follows that left multiplation by Δ:

$$f \to \Delta \cdot f$$

defines a vector space isomorphism between M_k and S_{k+12}. We can easily check the linear map properties, and bijectivity follows by invertibility. From there, the result $\dim S_{k+12} = \dim M_k$ follows. ■

Theorem 6.4. *Let k be an even positive integer. Then we have:*

$$\dim M_k = \begin{cases} \lfloor \frac{k}{12} \rfloor & k \equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor + 1 & k \not\equiv 2 \pmod{12}. \end{cases}$$

Proof. Equation (6.1), together with the fact that all orders are nonnegative, implies that there are no nonzero modular forms of negative weight and that $M_0 = \mathbb{C}$, giving $\dim M_0 = 1$. Considering denominators, we also get that there are no nonzero modular forms of weight 2 (providing another verification that E_2 is not a modular form). Let $k \geq 4$ be an even positive integer. From our proof of Theorem 4.3, we know that Eisenstein series do not vanish at ∞. Thus, we can write any modular form of weight k as a linear combination of a cusp form of weight k and E_k, giving

$$M_k = S_k \oplus \mathbb{C} \cdot E_k$$

Taking dimensions, we get

$$\dim M_k = \dim S_k + 1.$$

By Lemma 6.3, we get $\dim M_k = \dim M_{k-12} + 1$. Using induction with our cases above as base cases, we get the desired form for $\dim M_k$. ■

An analogous valence formula exists for modular forms for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. It's statement is much longer and more technical, and it's proof is similarly laborious, using Theorem 6.1 and some additional results from group theory. We refer to [1] for its statement and proof.

Definition 6.5. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. We define $M_k(\Gamma)$ as the the $\mathbb{C}$-vector space of modular forms of weight k for the subgroup Γ.

Using a similar line of reasoning to Theorem 6.3, we obtain a *bound* on the dimension of $M_k(\Gamma)$:

Theorem 6.6. *Let $[SL_2(\mathbb{Z}) : \Gamma]$ denote the index of Γ in $SL_2(\mathbb{Z})$. We have*

$$\dim M_k(\Gamma) \leq 1 + \left\lfloor \frac{k}{24} [SL_2(\mathbb{Z}) : \Gamma] \right\rfloor .$$

Most commonly, these indices are calculated by viewing congruence subgroups as kernels of certain group homomorphisms. For a much deeper discussion of analyzing dimension of modular form vector spaces, including an explicit (complicated) formula for $\dim M_k(\Gamma)$, see chapter 3 of [2].

7. Sums of Squares

Using all our previously developed theory, we are ready to tackle the proof of Theorem 1.2, which we restate below. In particular, we combine our results about Eisenstein series and the θ function with the dimension bounds from the previous section.

Theorem 1.2. *We have*

$$r_4(n) = 8 \sum_{\substack{d \mid n \\ 4 \nmid d}} d.$$

Proof. Let $\Gamma_0(4)$ be as defined in section 5. We can compute that $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4)] = 12$ by considering a sequence of canonical group homomorphisms whose kernels form a chain of inclusions to $\Gamma_0(4)$. Thus, by Theorem 6.5, $\dim M_2(\Gamma_0(4)) \leq 1 + \lfloor 1 \rfloor = 2$. Since we can exhibit the two linearly independent elements, E_2^2 and E_2^4, this dimension is exactly two, and moreover E_2^2 and E_2^4 are a basis for the vector space $M_2(\Gamma_0(4))$. Furthermore, by Theorem 5.6, the exponentiated θ function θ^4 is a modular form of weight 2 for $\Gamma_0(4)$, and thus $\theta^4 \in M_2(\Gamma_0(4))$. Therefore, we can write θ^4 as a linear combination:

$$\theta^4 = c_1 E_2^2 + c_2 E_2^4$$

for some $c_1, c_2 \in \mathbb{C}$. Using the q-expansions determined by Lemma 4.2 (normalized as in Definiton 5.3) and Proposition 5.5, we can write:

$$\begin{aligned} \theta^4 &= 1 + 8q + 24q^2 + O(q^3) \\ E_2^2 &= -1 - 24q - 24q^2 + O(q^3) \\ E_2^4 &= -3 - 24q - 72q^2 + O(q^3) \end{aligned}$$

Comparing early coefficients gives $c_1 = 0$ and $c_2 = -\frac{1}{3}$. Thus,

$$\theta^4 = -\frac{1}{3}E_2^4 = 1 + 8\sum_{n\geq 1}\left(\sigma(n) - 4\sigma\left(\frac{n}{4}\right)\right).$$

where we use the q-expansion of E_2^4. By proposition 5.5, we compare coefficients with θ^4 to get

$$r_4(n) = 8\left(\sigma(n) - 4\sigma\left(\frac{n}{4}\right)\right) = 8\sum_{\substack{d|n \\ 4\nmid d}} d.$$

■

We note that our proof method generalizes to get analogous identities for some even $k > 4$. In general, we compute the dimension of $M_{\frac{k}{2}}(\Gamma_0(4))$ using the bounds in section 6 and demonstration of linearly independent elements. Then, the remainder of our work is computation of linear combinations, as above. In particular, the proof of an analogous result for $k = 8$ proceeds nearly exactly as above. Using the dimension formula found in [2] makes our method possible for k in which we don't have equality with the bound of Theorem 6.5.

References

[1] Peter Bruin and S Dahmen. "Modular forms". In: *Course notes online at https://www.few.vu.nl/ sdn249/modularforms16/notes.pdf* (2016).

[2] Fred Diamond and Jerry Michael Shurman. *A first course in modular forms*. Vol. 228. Springer, 2005.

[3] Toshitsune Miyake. *Modular forms*. Springer Science & Business Media, 2006.

[4] Jean-Pierre Serre. *A course in arithmetic*. Vol. 7. Springer Science & Business Media, 2012.