ON THE USE OF COMPLEX METHODS IN ASYMPTOTIC COMBINATORICS

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Abstract. In this paper we explore the power of complex analysis in analyzing the asymptotic behavior of combinatorial sequences through their generating functions. Studying various combinatorial classes such as surjections, unary-binary trees, and Catalan numbers, we will illustrate how we can exploit complex analysis to understand the asymptotic behaviors of these combinatorial structures.

1. Introduction

Asymptotic analysis, at heart, is studying the limiting behavior of functions as their arguments grow large. In combinatorics, we are often interested in determining the growth of coefficients of generating functions. In this paper we refer to ordinary generating functions as OGFs and exponential generating functions as EGFs. Throughout this paper, we will be using the following notation to refer to coefficients: let $f(x)$ be a generating function in x, then $[x^n]f(x)$ denotes the coefficient of x^n in f. For example,

$$[x^n]e^x = \frac{1}{n!} \quad \text{and} \quad [x^n]\frac{1}{1-x} = 1.$$

Where the former is due to the Taylor expansion of $e^x = \sum_{n=0}^{\infty} x^n/n!$ and the latter is due to the geometric series expansion of $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Before we go any further into the paper, we need to introduce a notation:

Definition 1.1. Denote $a_n = [x^n]f(x)$. Then, we say that $a_n \sim g(n)$ as $n \to \infty$ if and only if

$$\lim_{n\to\infty} \frac{a_n}{g(n)} = 1.$$

Traditionally, recurrence relations can sometimes help us to find the asymptotic behavior of sequences in which we can work with. For example, recurrence relations can be used to find the asymptotics of the *Fibonacci Numbers:*

Definition 1.2 (Fibonacci Numbers)**.** The *Fibonacci numbers* F_n, for $n \geq 0$, are defined recursively by: $F_0 = 0, F_1 = 1$, and for all $n \geq 1$,

$$F_{n+1} = F_n + F_{n-1}.$$

We will not provide a full justification of the asymptotic relation as it is a diversion to our primary discussion; however, the interested reader can read the explanation in Section 1.3 of [Wil05] and observe that generating functions offer a powerful way to study sequences. The well known OGF for F_n is

$$F(x) = \frac{x}{1-x-x^2}$$

Date: June 9, 2025.

from which we can derive that $F_n \sim \frac{\varphi^n}{\sqrt{5}}$, where φ is the golden ratio $\frac{1+\sqrt{5}}{2}$.

In the case of the Fibonacci numbers, we benefited from a simple recurrence relation, and hence we were able to find the asymptotic for F_n with ease. However, the *Catalan Numbers* are more complex to analyze asymptotically:

Definition 1.3 (Catalan Numbers)**.** The *Catalan numbers* C_n, where $n \geq 0$, are defined recursively by: $C_0 = 1$ and for $n \geq 1$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}.$$

For example, $C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42$. Beautifully, the Catalan numbers often appear throughout different places in combinatorics such as valid parentheses expressions, binary trees, and triangulations of convex polygons. The well known generating function of the Catalan numbers is

$$C(z) = \frac{1 - \sqrt{1 - 4z}}{2z}.$$

In general, partial fraction decomposition and recursion relations do not work well for Catalan numbers to find the asymptotics. It is at these times that complex methods for studying asymptotics saves the day. In Section 4 we will provide the asymptotic form of $C_n = [z^n]C(z)$.

In this paper, we wish to discuss a starting motivation to understand the beauty of using complex analytic methods to study combinatorial structures and their behaviors. In Section 2 we will discuss some complex analysis terminology and tools as well as prove a fundamental theorem in complex asymptotics. In Section 3 we will dive into analyzing asymptotics via singularities and analyze two combinatorial problems: *surjections* and *unary-binary trees*. Finally, in section 4 we will provide the necessary tools to find the asymptotics for the Catalan numbers. The results mentioned are due to [FS09].

2. Complex Analysis Tools

We will start off by defining the necessary complex analytic terminology that is necessary to understand when the asymptotic theorems are useful:

Definition 2.1 (Analytic Functions)**.** A function $f(z)$ defined over a region Ω is *analytic* at a point $z_0 \in \Omega$ if, for some open disc centered at z_0 and contained in Ω, it can be represented by a convergent power series expansion

$$f(z) = \sum_{n \geq 0} c_n (z - z_0)^n.$$

Definition 2.2 (Singularities)**.** Given a function f defined over the region interior to Γ. A singularity of f is a point z_0 such that f is analytic on a punctured neighborhood around z_0 but not at z_0.

Now that we have stated the main definitions, we can discuss a fundamental theorem that kickstarts using complex asymptotic methods:

Theorem 2.3. *(Cauchy Coefficient Formula) Let $f(z)$ be analytic in a region containing the origin, and let Γ be a positively oriented simple closed contour enclosing the origin. Then* *the coefficient of z^n in the power series expansion of $f(z)$ is given by:*

$$[z^n]f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z^{n+1}}\,dz.$$

Proof. We will use the Cauchy Residue Theorem. Hence, we have that

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z^{n+1}}\,dz = \operatorname{Res}\left(\frac{f(z)}{z^{n+1}};0\right).$$

Since $f(z)$ can be expressed as $\sum_{k=0}^{\infty} a_k z^k$, we have that the Laurent expansion of $f(z)/z^{n+1}$ is

$$\frac{f(z)}{z^{n+1}} = \sum_{k=0}^{\infty} a_k \cdot z^{k-n-1}.$$

We have the coefficient of z^{-1} is when $k = n$, hence we have that

$$\operatorname{Res}\left(\frac{f(z)}{z^{n+1}};0\right) = a_n = [z^n]f(z).$$

Bringing both parts together, we have that

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z^{n+1}}\,dz = \operatorname{Res}\left(\frac{f(z)}{z^{n+1}};0\right) = [z^n]f(z),$$

as needed. ∎

As a simple example, we can use Theorem 2.3 for the function $f(z) = \frac{1}{1-z}$. We already know that f represents a geometric series. In particular, from Section 1 we already saw that $[z^n]f(z) = 1$. Nonetheless, let us verify this with the Cauchy Coefficient Formula: as f is analytic only for $|z| < 1$, denote Γ to be positively oriented circle around the origin, $|z| = r$ for some $0 < r < 1$. Then, we have that

$$[z^n]f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z^{n+1}}\,dz = \frac{1}{2\pi i}\int_\Gamma \frac{1}{(1-z)z^{n+1}}\,dz = 1.$$

Both methods result in the same answer, validating our result. While the function we used was a simple example, Theorem 2.3 is limited to nice generating functions. However, at times generating functions have nontrivial singularities. Hence, we must talk about how to handle these singularities.

3. Singularity Analysis

In this section, we will introduce a general theorem that allows us to find the asymptotic behavior of coefficients from only the dominant singularity of a generating function.

Theorem 3.1. *Let $f(z)$ be an analytic function at the origin and denote $R := R_{sing}(f;0)$ as the distance from the origin of the singularity of $f(z)$ that is closest to the origin (dominant singularity). Then the coefficients $f_n = [z^n]f(z)$ satisfy that*

$$f_n = \left(\frac{1}{R}\right)^n \cdot \theta(n),$$

where $\limsup_{n\to\infty} |\theta(n)|^{1/n} = 1$.

The proof of this theorem is a result of the Boundary Singularity Theorem and Pringsheim's theorem, which we will not discuss here.

To illustrate Theorem 3.1 we will consider two examples: *surjections* and *unary-binary trees*. We will start with applying the above result for surjections:

Definition 3.2 (Surjection)**.** A *surjection* from a finite set $\mathcal{A}$ to a set $\mathcal{B}$ is a function $f : \mathcal{A} \to \mathcal{B}$ such that for every $y \in \mathcal{B}$, there exists at least one $x \in \mathcal{A}$, such that $f(x) = y$.

A well-known EGF counting the number of surjections from $|\mathcal{A}| = n$ to $|\mathcal{B}| = 2$ is given by

$$R(z) = \frac{1}{2 - e^z}.$$

Note that the denominator of R is an entire function and hence its singularities only come from its zeros at $\chi_k = \log 2 + 2ik\pi$, with $k \in \mathbb{Z}$. We have that the singularity closest to the origin is hence $\log 2$. Hence, from Theorem 3.1, we have that if $r_n = [z^n]R(z)$, then

$$r_n \sim \left(\frac{1}{\log 2}\right)^n \theta(n),$$

where $\limsup_{n\to\infty} |\theta(n)|^{1/n} = 1$. Hence, we have the coefficients of the EGF for these types of surjections grow approximately like 1.4427^n.

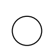
This application illustrates how the location of a single singularity in the complex plane can directly determine the exponential behavior of a generating function's coefficients.

Our next application are *unary-binary trees.* The definition is motivated by [Sed22]:

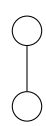
Definition 3.3. A *unary-binary tree* is an unlabeled and unordered rooted tree in which every internal node has either one child (unary) or two children (binary).

Denote $\mathcal{M}_n$ to denote the number of unique unary-binary trees with n nodes. In particular, $\mathcal{M}_n = [z^n]M(z)$, where $M(z)$ is the OGF for unary-binary trees. Before we go on to find the asymptotic for $\mathcal{M}_n$, let us first do an initial experimentation for finding the first few values of $\mathcal{M}_n$:

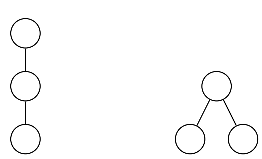
We have that $\mathcal{M}_1 = 1$:



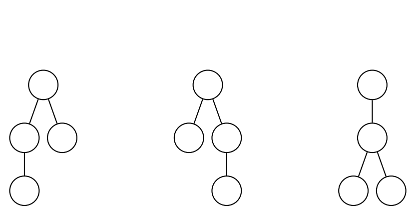
We have that $\mathcal{M}_2 = 1$:



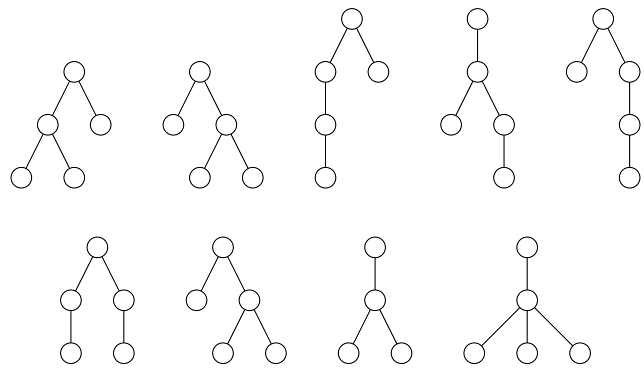
We have that $\mathcal{M}_3 = 2$:



We have that $\mathcal{M}_4 = 4$:



Finally, we have that $\mathcal{M}_5 = 9$:



Now that we are clearer with what unary-binary trees are, we can find the generating function. By definition, the generating function equation for unary-binary trees is $M(z) = z + zM(z) + zM(z)^2$, with the explicit form as an OGF in the form

$$M(z) = \frac{1 - z - \sqrt{(1-3z)(1+z)}}{2z}.$$

Notice that $M(z)$ is analytic in the complex plane in a disk around the origin of radius $\frac{1}{3}$ with singularities at $z = -1$ and $z = \frac{1}{3}$. Hence, from Theorem 3.1, we have that

$$\mathcal{M}_n \sim 3^n.$$

We can actually find a stronger result expressing the coefficients of $M(z)$ more exactly as

$$\mathcal{M}_n = \frac{1}{\sqrt{4\pi n^3/3}} 3^n,$$

but the reason for the multiplication of $1/\sqrt{4\pi n^3/3}$ is out of the scope of this paper and requires stronger results. For completion, we have that $\mathcal{M}_n$ are referred to as the *Motzkin Numbers*.

4. The Standard Transfer Theorem and Catalan Numbers

For our final theorem, we will discuss a powerful result known as the *Standard Transfer Theorem* which enables us to find the asymptotic behavior of C_n as promised in Section 1.

Theorem 4.1. *Let $f(z) = (1-z)^{-\alpha}$ with $\alpha \notin \{0, -1, -2, \cdots, \}$. Then the coefficient of z^n satisfies:*

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \frac{e_1(\alpha)}{n} + \frac{e_2(\alpha)}{n^2} + \cdots\right)$$

where each $e_k(\alpha)$ is a polynomial of degree $2k$ in α.

Now, we can find the asymptotics for $C(z)$. It is well known that

$$C(z) = \frac{1 - \sqrt{1-4z}}{2z}.$$

We have a singularity at $z = 1/4$ hence we have that $C(z) \sim (1-4z)^{1/2}$. Therefore, we apply the above Theorem 4.1 to $(1-4z)^{-1/2}$. Let $u = 4z$. Then,

$$(1-4z)^{-1/2} = (1-u)^{-1/2}.$$

Since this is in the form we need, we have that

$$[z^n](1-4z)^{-1/2} = 4^n[u^n](1-u)^{-1/2} = \frac{4^n n^{-3/2}}{\sqrt{\pi}}.$$

Hence,

$$C_n \sim \frac{4^n n^{-3/2}}{\sqrt{\pi}}.$$

Hence, we are able to find the asymptotic behavior of the Catalan numbers instead of using more complicated results such as *Stirling's Approximation*.

5. Conclusion

The usage of complex analysis to understand the behavior of combinatorial structures is a powerful tool as we have seen throughout this paper. However, there are many more advanced theorems that give us more accurate approximations for a wider range of generating functions, such as Darboux's method and other Transfer Theorems apart from Theorem 4.1. Finally, such approximations and the understanding of asymptotic behavior is beneficial in fields such as statistical physics.

Acknowledgments. The author would like to thank Dr. Simon-Rubinstein Salzedo for the suggestion of the topic and Mark Takken for the feedback making this paper possible.

References

[FS09] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics.* Cambridge University Press, Cambridge, UK, 2009.

[Sed22] Robert Sedgewick. Analytic combinatorics lecture slides — lecture 5 (aaqa5). Online lecture slides, Princeton University, 2022. Part of the *Analytic Combinatorics* course.

[Wil05] Herbert S. Wilf. *generatingfunctionology.* A K Peters, Ltd., Wellesley, MA, third edition, 2005.

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