Extremality in Quasiconformal Maps

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Abstract

This paper explores the geometry of extremal length, a conformally invariant quantity that is used to measure the "thickness" or complexity of curve families on Riemann surfaces. The paper covers essential theorems and definitions that are necessary to understanding extremality and uses classic examples like the rectangular domain and the annulus to show how extremal length can be a useful tool in many types of problems. The paper also includes analytic, topological, and variational techniques that show how extremal length can be used for problems across quasiconformal analysis.

1 Riemann Surfaces

[3] In order to understand extremal length, we first have to define Riemann surfaces.

Definition 1.1. A Riemann surface R is a two-dimensional manifold with oriented conformal structure. R is a connected Hausdorff topological space and there is a covering of R by open sets U_α and there are homeomorphisms z_α from U_α into $\mathbb{C}$ such that the transition maps $f_{\beta\alpha} = z_\alpha \circ z_\beta^{-1}$ from $z_\beta(U_\alpha \cap U_\beta)$ to $z_\alpha(U_\alpha \cap U_\beta)$ are conformal.

The definition above can also be written either to say that a Riemann surface is a one-dimensional, holomorphic manifold or to say a Riemann surface is a one-dimensional, complex analytic manifold.

The pairs (U_α, z_α) are called charts. Two systems of charts (U_α, z_α) and (V_β, ω_β) are compatible if transition maps $f_{\beta\alpha} = \omega_\beta \circ z_\alpha^{-1}$ from $z_\alpha(U_\alpha \cap V_\beta)$ to $\omega_\beta(U_\alpha \cap B_\beta)$ are conformal. The equivalence relation between systems of charts is compatibility because the composition of conformal maps is conformal.

The equivalence class of compatible systems of charts on R is the conformal structure of R. Any system of analytic charts for R can decide an orientation, and the map $z_\alpha : U_\alpha \to \mathbb{C}$ puts orientation on U_α by taking the preimage of the usual orientation for $\mathbb{C}$. On $U_\alpha \cap U_\beta$ the orientation is consistently determined because the Jacobian of $f_{\beta\alpha}$ is equal to $|f'_{\beta\alpha}|^2$ which is positive.

Definition 1.2. A continuous map f from a Riemann surface R to a Riemann surface R_1 is conformal if, given any chart z_α on R and any chart ω_β on R_1, the map $w_\beta \circ f \circ z_\alpha^{-1}$ is conformal on the set where the composition is defined. $f : R \to R_1$ is K-quasiconformal if $K_z(w)(w_\beta \circ f \circ z_\alpha^{-1}) \leq K$ at any point z where the composition is defined.

We should notice that if we choose a different z_{α_1} and ω_{β_1} defined the domains that overall the domains of z_α and ω_β, respectively, then we can say that:

$$w_\beta \circ f \circ z_\alpha^{-1} = g_{\beta\beta_1} \circ \omega_{\beta_1} \circ f \circ z_{\alpha_1}^{-1} \circ \hat{f}_{\alpha\alpha_1}^{-1}$$

The precomposition or postcomposition by conformal transition maps $\hat{f}_{\alpha\alpha_1}^{-1}$ and $g_{\beta\beta_1}$ does not alter dilatation. Because of this, the K-quasiconformal map between Riemann surfaces is well-defined.

Definition 1.3. R and R_1 are conformally equivalent if there is a conformal homomorphism from R to R_1

Definition 1.4. Say that R is a Riemann surface. The quasiconformal moduli space $\mathcal{M}(R)$ is the set of conformal equivalence classes of Riemann surfaces quasiconformally equivalent to R.

2 Defining Extremal Length

2.1 Introduction

[1] Say that F is a family of curves on a Riemann surface. Let us assume that every γ in F is a countable union of open arcs or closed curves. We define the extremal length of F (which we will sometimes write as $\Lambda(F)$) as something akin to the average minimum length of curves in F. We do this because extremal length is a measure of how "thick" a curve family is. Extremal length encodes a conformally invariant way to evaluate the geometry of F, which is especially useful on Riemann surfaces. Let us give a set of metrics for this definition; a metric $\rho(z)|dz|$ is acceptable if:

1) it is invariantly defined for different local parameters z, that is:$\rho_2(z_2)|dz_2|$ where ρ_1, ρ_2 are the representatives of ρ in terms of the parameters z_1, z_2.

2) ρ is locally L_2 and greater than 0 everywhere.

3) $A(\rho) = \int\int \rho^2 dxdy \neq 0$ or ∞ the integral is taken over the whole Riemann surface. We should note that $A(\rho)$ is well defined because of the first condition above on ρ; this is because conformal invariance of the metric guarantees that integrating ρ^2 over coordinate patches gives consistent results that are not dependent on local parametrization.

For an allowable ρ like that, we define

$$L_\gamma(\rho) = \int_\gamma \rho|dz|$$

given that ρ is measurable along γ, otherwise we define $L_\gamma(\rho) = +\infty$. Say that $L(\rho) = L(\rho, F) = \inf L_\gamma$ where the infimum is over all curves γ in F. This equation takes the least length over the whole family under a fixed metric ρ. We want to know how short any curve in the family can get, given ρ. This sets up the numerator for a Rayleigh quotient, where the ratio rewards metrics that make the curves long while the area is still small.

The extremal length of the curve family F is

$$\Lambda(F) = \sup_\rho \frac{L(\rho)^2}{A(\rho)}.$$

The extremal length maximizes how efficiently a conformal metric can "stretch" the family F because we are trying to make all curves long while maximizing the area. The above definition perfectly summarizes this.

This identity holds when the supremum is taken over all acceptable metrics. The ratio in this supremum is invariant if ρ is multiplied by a positive scalar, so by trying to evaluate $\Lambda(F)$, we have the option to scale ρ however we want. This allows us to fix one quantity, like length or area, when calculating the supremum. For example, we can scale to make $L(\rho) = 1$ and make $A(\rho)$ as small as possible, or we could scale to make $L(\rho) = A(\rho)$ to make $L(\rho)$ as large as possible.

Now, let us look at a couple examples from Frederick P. Gardiner's and Nikola Lakic's book *Quasiconformal Teichmüller Theory.* [3]

2.2 Examples

Example 1: Say that R is the interior of a rectangle $\{z : 0 \leq x \leq a, 0 \leq y \leq b\}$ and F is the family of arcs in R that join the right vertical side of R to the left vertical side. Given this, we know that $\Lambda(F) = a/b$, so $\Lambda(F)$ is the modulus $m(R)$ of the rectangle R. Here, the modulus a/b refers to the conformal modulus of the rectangle which can measure the aspect ratio in an invariant way under conformal mappings. We are doing this to prove this value as both a lower and upper bound.

For us to see this, let us say that $p \equiv 1$ in R. Given this, $L(\rho) = a$ and $A(\rho) = ab$ and from this we can say that $\Lambda(F) \geq a/b$. We are constructing a test metric $\rho = 1$ because it satisfies the admissibility condition (positive, measurable, finite area). This gives a concrete lower bound on $\Lambda(F)$ since extremal length is the supremum over all metrics.

However, if ρ is any allowable metric on R, then by multiplying ρ by a scalar we can make $L(\rho) = a$. We do this because extremal length is invariant under scaling of the metric, so by normalizing the shortest curve in F to have a unit length a, we can find a lower bound on the area which will give an upper bound on the quotient L^2/A.

Therefore

$$a \leq \int_0^a \rho(x + iy)dx$$

for every y such that $0 \leq y \leq b$. This is because any horizontal line segment from left to right is in the family F so the integral along the curve must be at least a. We use the uniformity of the rectangle and horizontal slices to get an integral conditions hat applies all over y.

Then, by integrating over y and applying Schwarz's inequality, we get

$$ab \leq \int \int_R \rho \ dxdy, \ (ab)^2 \leq ab \int \int_R \rho^2 \ dxdy$$

$$\frac{L(\rho)^2}{A(\rho)} \leq \frac{a^2}{ab} = \frac{a}{b}$$

$$\Lambda(F) \leq \frac{a}{b}$$

This proves that the upper bound on $\Lambda(F)$ by connecting the square of the average of ρ to its L^2-norm.

Note that if $F^\star$ is the family of arcs in R that can join the top of rectangle R to the top, then $\Lambda(F^\star) = b/a$ and $\Lambda(F^\star)\Lambda(F) = 1$. This relationship is a general identity that says that if F and $F^\star$ are dual families, then their extremal lengths multiply to 1. This shows the duality of vertical and horizontal moduli in conformal rectangles.

Example 2: Say that $R = z : r_1 < |z| < r_2$ and say that F be the family of closed curves in R that are homotopic to the curve γ_1 where $\gamma_1(\theta) = \frac{e^{i\theta}(r_1+r_2)}{2}$ where $0 \leq \theta < 2\pi$. Let us have $\rho_0(z) = (2\pi|z|)^{-1}$. We choose this ρ_0 because its analytical structure is known and produces a constant unit length for each circular path $|z| = r$. This makes sense for the extremal metric because it distributed the metric equally along angular curves and it turns out to be optimal.

We can say that, for any curve γ homotopic to γ_1:

$$(2\pi i)^{-1} \int_\gamma \frac{dz}{z}.$$

This integral is the winding number of γ around 0. Since all $\gamma \in F$ are homotopic to γ_1, they enclose the origin once and the integral gives a topological invariant. It gives a lower bound on the weighted length under ρ_0.

Therefore:

$$\int_\gamma \rho_0(z)|dz| \geq 1 \text{ and } L(\rho_0) \geq 1,$$

So now we have a lower bound on length under this specific metric. This shows that ρ_0 is admissible for F. Now we can compute the area:

$$A(\rho_0) = \int\int_R \frac{r dr d\theta}{(2\pi)^2 r^2} = \frac{1}{2\pi}\log(\frac{r_2}{r_1}).$$

This means that:

$$\Lambda(F) \geq 2\pi(\log(\frac{r_2}{r_1}))^{-1}$$

We have constructed a specific admissible metric that gives thee above equation as the lower bound. So now we pivot to proving that it is the maximum by bounding from above.

However, for any allowable metric $\rho|dz|$:

$$L(\rho) \leq \int_0^{2\pi} \rho(re^{i\theta}) r d\theta,$$

$$\frac{L(\rho)}{r} \leq \int_0^{2\pi} \rho(re^{i\theta}) d\theta,$$

We use the above inequality to control $L(\rho)$ by an average value of ρ on each circle. This makes the problem one-dimensional integrals over θ which we can then integrate in r.

$$L(\rho)\log(\frac{r_2}{r_1}) \leq \int\int_R \rho dr d\theta,$$

Apply Cauchy-Schwarz:

$$(L(\rho))^2(\log(\frac{r_2}{r_1}))^2 \leq \int\int \frac{1}{r} dr d\theta \int\int \rho^2 r dr d\theta,$$

So:

$$\frac{L(\rho)^2}{A(\rho)} \leq 2\pi(\log(\frac{r_2}{r_1}))^{-1}.$$

The above calculations prove the upper bound using a general metric ρ. These steps are similar to those in the rectangle case: normalize or control $L(\rho)$ and bound $A(\rho)$ from below. The rotational symmetry of the annulus makes this argument manageable.

The same kind of argument above shows that if $F^\star$ is the family of curves joining the two horizontal sides of R, then $\Lambda(F^\star) = (2\pi)^{-1}\log(\frac{r_2}{r_1})$. Again, we see duality because circular and radial families are orthogonal and mutually constraining. This duality holds across all annuli and shows that extremal length has the geometry of the annulus in both directions.

2.3 Theorems and Lemmas

[1] The definitions mentioned above give us many useful and foundational properties.

Lemma 2.1. *Conformal Invariance: If Γ is a family of curves with the domain Ω and f is an injective holomorphic mapping from Ω to Ω' then we can say that $M(\Gamma) = M(f(\Gamma))$. In other words, the modulus is invariant under conformal maps, or more generally, it is invariant under injective holomorphic graphs.*

We can prove this by using the change of variables formulas:

$$\int_\gamma p \circ f|f'| \cdot ds = \int_{f(\gamma)} p \cdot ds$$

$$\int_\Omega (p \circ f)^2|f'|^2 \cdot dxdy = \int_{f(\Omega)} p \cdot dxdy$$

The above two equations represent the arc length and the area change of variable formulas respectively. This step is important because it tells us how to take integrals from the image domain back to the preimage under a conformal map. The first integral comes from the fact that for a conformal map f, the differential of the arc length is written as $ds_{f(\gamma)} = |f'(z)|ds_\gamma$. The second integral comes from the multidimensional change of variables theorem where the Jacobian determinant of a holomorphic function $f : \Omega \to \Omega'$ is $|f'|^2$.

The integrals above lead to: $p \in \mathscr{A}(f(\Gamma))$ then $|f'| \cdot p \circ f \in \mathscr{A}(f(\Gamma))$. By taking the infimum over these metrics we get $M(f(\Gamma)) \leq M(\Gamma)$. $\square$

Lemma 2.2. *Monotonicity: If Γ_0 and Γ_1 are path families with every $\gamma \in \Gamma_0$ containing a curve in Γ_1, then $M(\Gamma_0) \leq M(\Gamma_1)$ and $\lambda(\Gamma_0) \geq \lambda(\Gamma_1)$*

The proof is apparent because $\mathscr{A}(\Gamma_0) \supset \mathscr{A}(\Gamma_1)$. Since the modulus is an infimum over $\mathscr{A}(\Gamma)$, making the admissible set larger can only decrease the infimum. This same reasoning inverts the inequality for $\lambda = \frac{1}{M}$. $\square$

Lemma 2.3. *Grötsch Principle: If Γ_0 and Γ_1 are curve families in disjoint domains then $M(\Gamma_0 \cup \Gamma_1) = M(\Gamma_0) + M(\Gamma_1)$*

Say that ρ_0 and ρ_1 are admissible for Γ_0 and Γ_1. Take $\rho = \rho_0$ and $\rho = \rho_1$ in their respective domains. This step helps construct a global admissible function by stitching together two local ones by using the fact that the domains are disjoint, so that the pieces don't interfere.

From here, it is simple to see that ρ is admissible for $\Gamma_0 \cup \Gamma_1$ and since the domains are disjoint then we can say:

$$\int \rho^2 = \int \rho_1^2 + \int \rho_2^2$$

Since the supports are disjoint, the area functional $A(\rho)$ separates as a sum of integrals over each domain. This proves that the combined metric has a total energy equal to the sum of the individual energies.

Therefore, $M(\Gamma \cup \Gamma_1) \leq M(\Gamma_0) + M(\Gamma_1)$, and by restricting an admissible metric ρ to each domain, a similar argument proves the opposite direction. This completes our argument by symmetry because any admissible metric for the union can be restricted back to each subdomain.

By combining monotonicity and the Grötsch principle, we get:

Lemma 2.4. *Parallel Rule: Say that Γ_0 and Γ_1 are path families in disjoint domains $\Omega_0, \Omega_1 \subset \Omega$ that connect the disjoint sets E, F in $\partial\Omega$. If Γ is the path family that connects E and F in Ω then:*

$$M(\Gamma) \geq M(\Gamma_0) + M(\Gamma_1).$$

Lemma 2.5. *Series Rule: Say that Γ_0 and Γ_1 are curve families in disjoint domains and every curve of $\mathscr{F}$ contains a curve from both Γ_0 and Γ_1. Therefore*

$$\lambda(\Gamma) \geq \lambda(\Gamma_0) + \lambda(\Gamma_1).$$

We can prove this because if $\rho_j \in \mathscr{A}(\Gamma_j)$ for $j = 0, 1$, then

$$\rho_t = \rho_0(1-t) + t\rho_1$$

is admissible for Γ. This convex combination interpolates between the two admissible metrics. The goal here is to construct a single metric that is valid for all of Γ by blending those valid for each of the subfamilies. This is valid because admissibility preserved under convex combinations when the domains are disjoint.

The domains are disjoint, so we can assume that $\rho_0\rho_1 = 0$. This allows us to simplify ρ_t^2 into $\rho_0^2(1-t)^2 + t^2\rho_1^2$ without cross terms. By integrating ρ^2, we see that:

$$M(\Gamma) \leq M(\Gamma_0)(1-t)^2 + t^2 M(\Gamma_1),$$

for each t. This computes the area functional of the metric. Since it is admissible for Γ, this expression provides an upper bound on $M(\Gamma)$.. Our goal now is minimizing the right-hand side with respect to t to find the tightest bound.

We differentiate the right hand side above and set it equal to 0 so that we can find the optimal t set $a = M(\Gamma_1), b = M(\Gamma_0)$. This is an optimization step because we find the value of t that minimizes the area for our admissible family, which will give the best upper boundon $M(\Gamma)$.

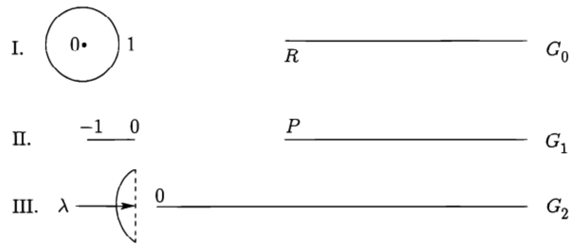

Figure 1. Three symmetric cases to find the maximum of $M(G)$

When we do that, we get:

$$2at - 2b(1-t) = 0$$

Solving for t, we get $t = \frac{b}{a+b}$.By plugging this t into the inequality above, we get:

$$M(\mathscr{F}) \leq at^2 + b(1-t^2) = \frac{b^2aa^2b}{(a+b)^2} = \frac{ab(a+b)}{(a+b)^2} = \frac{ab}{a+b} = \frac{1}{\frac{1}{a}+\frac{1}{b}}$$

This simplifies the upper bound to the harmonic mean of a and b, which is the exact formula for combining resistances in series, which shows that extremal length behaves analogously.

Which we can rewrite as

$$\lambda(\Lambda) \geq \lambda(\Gamma_0) + \lambda(\Gamma_1)$$

Taking the recipricol completes the proof and switches from modulus to extremal length. This confirms that the total "resistance" of the path family Γ is at least the sum of its component "resistances". □

3 Geometric Extremal Problems

[2] Say that we have G, a doubly connected region in the finite plane, and let C_1 be the bounded and C_2 be the unbounded component of its complement. Our goal is to find the largest value of the module $M(G)$ given one of the following conditions. The goal is to maximize the modulus, which geometrically corresponds to making the domain as "thin" as possible conformally:

1. C_1 is the unit disk, meaning that $|z| \leq 1$ and C_2 contains the point $R > 1$.
2. C_1 contains 0 and -1 and C_2 contains a point that is the distance P from the origin.
3. If $\text{diam}(C_1 \cap \{|z| \leq 1\}) \geq \lambda,$ then C_2 contains the origin.

We can say that the maximum of $M(G)$ can be found in the three symmetric cases seen in Figure 1. Let us consider each of the three cases separately.

Case 1: Say that Γ is the family of closed curves that separates C_1 and C_2 and we know that $\lambda(\Gamma) = M(G)^{-1})$. We can say this because in double connected regions, Γ is usually regarded as the family of closed curves that circle the inner boundary. The recipricol relationship above comes from the definition of extrmal length in an annular context.

When we compare Γ with the family $\tilde{\Gamma}$ of closed curves that lie in the complement of $C_1 \cup \{R\}$, we can see that we have 0 winding number about R and nonzero winding number about the origin. We do this to distinguish the curves since Γ must enclose the origin but $\tilde{\Gamma}$ does not enclose R. This way, we can embed Γ into a larger, more symmetric family $\tilde{\Gamma}$ and compare their extremal lengths.

We know that $\Gamma \subset \tilde{\Gamma}$, meaning that $\lambda(\Gamma) \geq \lambda(\tilde{\Gamma})$. We know this because of the monotonicity lemma, since $\Gamma \subset \tilde{\Gamma}$ the extremal length of Γ must be greater than or equal to the extremal length of $\tilde{\Gamma}$. This gives a lower bound on $\lambda(\Gamma)$ and therefore an upper bound on $M(G)$.

However, since $\tilde{\Gamma}$ is a symmetric family we know that $\lambda(\tilde{\Gamma}) = \frac{1}{2}\lambda(\tilde{\Gamma}^+)$. This comes from the symmetry principle, which states that if the family $\tilde{\Gamma}$ is symmetric under reflection, the each curve can be split into two mirror halves. The extremal length of the family would then be half the extremal length of the "positive half" $\tilde{\Gamma}^+$. Likewise, if Γ_0 is the family Γ in the extremal case, then we can say that $\lambda(\Gamma_0) = \frac{1}{2}\lambda(\Gamma_0^+)$.

We can see that $\tilde{\Gamma}^+ = \Gamma_0^+$. We know that every curve $\tilde{\gamma}$ in $\tilde{\Gamma}$ has points P_1 and P_2 on $(-\infty, -1)$ and $(1, R)$ respectively. If we divide $\tilde{\gamma}$ into arcs labeled $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ such that $\tilde{\gamma} = \tilde{\gamma}_1 + \tilde{\gamma}_2$, then we can say that $\tilde{\gamma}^+ = \tilde{\gamma}_1^+ + \tilde{\gamma}_2^+ = (\tilde{\gamma}_1 + \tilde{\gamma}_2^{+-})^+$.

In the above equations, $\tilde{\gamma}_1^+ + \tilde{\gamma}_2^{+-}$ belong to Γ_0 so we can say that $\tilde{\gamma}^+ \in \Gamma_0^+$, and therefore $\tilde{\Gamma}^+ \subset \tilde{\Gamma}_0^+$. This containment allows us to apply the monotonicity of extremal length again to the "positive halves" of the families. So, we have shown that $\lambda(\Gamma) \geq \lambda(\tilde{\Gamma}) = \lambda(\Gamma_0)$ so $M(G) \leq M(G_0)$. $\square$

Case 2: Say that $z = f(\zeta)$ maps $|\zeta| < 1$ conformally onto $C_1 \cup G$ where $f(0) = 0$. According to Koebe's one-quarter theorem $|f'(0)| \leq 4P$ where $G = G_1$. If $f(a) = -1$, the distortion theorem shows that:

$$1 = |f(a)| \leq \frac{|a||f'(0)|}{(1-|a|)^2} \leq \frac{4P|a|}{(1-|a|)^2} \text{ with equality at } G = G_1$$

The module $M(G)$ is the same as the module between the unit circle and the image of C_1. Then, using inversion and applying Case 1, if $|a|$ is given, then the module is largest for a line segment and increases when $|a|$ increases.

Case 3: Let us open up the plane by $\zeta = \sqrt{z}$. The result is a figure that is symmetric with respect to the origin with two component images of C_1 and two component images of C_2. We do this because, by construction, the preimage of a doubly connected region under $\zeta = \sqrt{z}$ becomes a symmetric region with double boundary components. This process allowed the geometry to be "doubled" and create a split between the upper and lower halves. The laws of composition show that $M(G) \leq \frac{1}{2}M(\hat{G})$ where $\hat{G}$ is the region in between C_1^+ and C_1^-. We can see that equality holds in the symmetric situation (refer to Figure 2).

Let's assume that C_1 contains the points z_1 and x_2 with $|z_1| \leq 1$, $|z_2| \leq 1$, $|z_1 - z_2| \geq \lambda$. This constraint makes sure that there is minimal separation in the unit disk, which prevents degeneration. This is important because the lower bound on this diameter will grow with this transformation. Say that $\zeta_1, \zeta_2 \in C_1^+$ and $-\zeta_1, -\zeta_2 \in C_1^-$ are the corresponding points in the ζ-plane. Here, we are lifting the original points under $\zeta = \sqrt{z}$, mapping them to symmetric points in the ζ-plane. We use the linear transformation:

$$\omega = \frac{\zeta + \zeta_1}{\zeta - \zeta_1} \cdot \frac{\zeta_1 + \zeta_2}{\zeta_1 - \zeta_2}$$

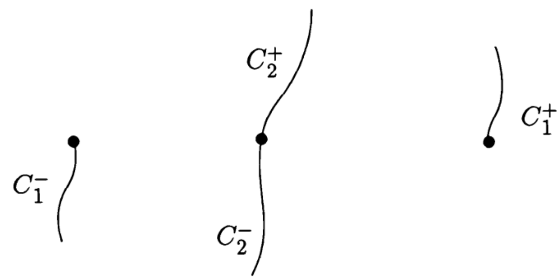

Figure 2. Symmetric Situations

which carries $(-\zeta_1, -\zeta_2)$ into $(0, 1)$ and $\zeta \to \infty, \zeta_2 \to \omega_0$. The purpose of this transformation is to normalize the domain into a more standard configuration where a triple set of points $(\infty, \zeta_2, -\zeta_1)$ is sent to $(\omega_0, 1, 0)$. This change of variables makes the extremely length calculation easier since the modulus of an annulus can now be written as cross ratios.

These conditions apply where:

$$\omega_0 = -(\frac{\zeta_2 + \zeta_1}{\zeta_2 - \zeta_1})^2.$$

Let us set:

$$u = \frac{\zeta_2 + \zeta_1}{\zeta_2 - \zeta_1}$$

We do this because writing $\omega_0 = -u^2$ allows to to write the modulus in terms of u which will then be bounded using the constraints on z_1 and z_2. This step turns the modulus into an explicit inequality. When we do this, we have:

$$u + \frac{1}{u} = \frac{2(\zeta_2^2 + \zeta_1^2)}{\zeta_2^2 - \zeta_1^2} = \frac{2(z_1 + z_2)}{z_2 - z_1}$$

We can do this because $\zeta^2 = z$ which means that we can write everything in terms of our original coordinates. This is vital because the constraints we placed on the geometry we applied on z_1, z_2 and now we can pull that structure through the transformation. Since we know that

$$|z_2 + z_1|^2 = 2(|z_1|^2 + |z_2|^2) - |z_2 - z_1|^2 \leq 4 - \lambda^2$$

This inequality tells us how "close" the midpoint of z_1 and z_2 is to the origin. This allows us to control $|u|$ which also bounds the modulus through $|\omega_0| = |u|^2$. Therefore, we can say that

$$|u| - \frac{1}{|u|} \leq \frac{2}{\lambda}\sqrt{4 - \lambda^2}$$

$$|u| \leq \frac{2 + \sqrt{4 - \lambda^2}}{\lambda}$$

$$|\omega_0| \leq (\frac{2 + \sqrt{4 - \lambda^2}}{\lambda})^2$$

This gives us an upper bound on the cross ratio defining the modulus. Since the modulus of the domain is a monotonic function of this cross ratio, bounding $|\omega_0|$ above gives us an explicit upper bound on $M(G)$. We can confirm that this equality holds for the symmetric case, and by using Case 2 we can see that $M(G)$ is a maximum in the case of Figure 1.

Using these three cases, we can explore the implications. We will use the notation from Hans P. Künzi's book *Quasikonforme Abbildungen* to write the extremal modules:

$$\text{I)}\ \frac{1}{2\pi}\log\Phi(R)$$

$$\text{II)}\ \frac{1}{2\pi}\log\Psi(P)$$

$$\text{III)}\ \frac{1}{2\pi}\log X(\lambda)$$

Relationships: There are many relationships between these functions. For example, since the reflection of G_0 gives a ring that is twice as wide as a ring of the type G_1, we can say that

$$\Phi(R)^2 = \Psi(R^2 - 1).$$

We can find another relationship by mapping the outside of the unit circle on the outside of the segment $(-1, 0)$:

$$\Phi(R) = \Psi(\frac{1}{4}(\sqrt{R} - \frac{1}{\sqrt{R}})^2).$$

This equation comes from the Schwarz-Christoffel-type maps from circular domains to slit domains. The square root transformation maps the annulus into a slit plane, as seen by the identity above. By using this latest relationship in tandem with the first relationship we found, we can say that

$$\Phi(R) = \Phi[\frac{1}{2}(\sqrt{R} + \frac{1}{\sqrt{R}})]^2$$

. Using the calculations in Case 3, we can state

$$X(\lambda) = \Phi(\frac{\sqrt{4+2\lambda} + \sqrt{4-2\lambda}}{\lambda}).$$

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