SHEAF COHOMOLOGY AND THE RIEMANN-ROCH THEOREM

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Abstract. The central goal of this paper is to prove the Riemann-Roch Theorem, which relates the *genus* of a compact Riemann surface to *divisors* of such a surface. To do this, we will of course need to introduce these notions, particularly the formal definitions of Riemann surfaces, *sheaves* and their *cohomology*.

1. Riemann Surfaces

The function $f(z) = \sqrt{z}$ is not a well-defined function on the complex plane; at least, if we want to make it holomorphic, or even continuous, we have to make some sacrifices. Every complex number has two square roots (i.e. $2^2 = (-2)^2 = 4$), and we have to pick one. One standard way to choose which of the two roots to be $\sqrt{z}$ is to always choose the root with positive real part. Namely,

$$z = re^{i\theta}, r > 0, \theta \in (-\pi, \pi) \implies \sqrt{z} = \sqrt{r}e^{i\theta/2}.$$

We run into a problem with the negative reals. What is $\sqrt{-1}$? Is it i, or $-i$? Both have zero real part, and both would be on the boundary of the θ interval, as $e^{i\pi/2}$ and $e^{i(-\pi)/2}$. If we consider $-1 \pm i\varepsilon$, we have

$$\sqrt{-1+i\varepsilon} = \sqrt{\frac{-1+\sqrt{1+\varepsilon^2}}{2}} + i\sqrt{\frac{1+\sqrt{1+\varepsilon^2}}{2}},$$
$$\sqrt{-1-i\varepsilon} = \sqrt{\frac{-1+\sqrt{1+\varepsilon^2}}{2}} - i\sqrt{\frac{1+\sqrt{1+\varepsilon^2}}{2}}.$$

Of course, as $\varepsilon \to 0$, both real parts approach 0, but the top imaginary part approaches 1, while the bottom one approaches -1. Thus, $\sqrt{z}$ is not continuous on the negative real line.

This problem gives rise to the *Riemann surface*, which is a type of surface meant to encode such multi-valued functions.

Definition 1.1 (Complex Chart)**.** Let X be a surface, i.e. a 2-manifold. A *complex chart* $\phi : U \subset X \to V \subset \mathbb{C}$ is a homeomorphism from an open set $U \subset X$ to an open set $V \subset \mathbb{C}$.

Definition 1.2 (Holomorphically Compatible)**.** Let X be a surface and $\phi_1 : U_1 \to V_1, \phi_2 : U_2 \to V_2$ be complex charts. We say that ϕ_1, ϕ_2 are *holomorphically compatible* if

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

is biholomorphic; namely, both $\phi_2 \circ \phi_1^{-1}$ and its inverse, $\phi_1 \circ \phi_2^{-1}$ are holomorphic.

This definition is commutative: ϕ_1, ϕ_2 being holomorphically compatible is equivalent to ϕ_2, ϕ_1 being holomorphically compatible. This notion of holomorphic compatibility gives us

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an idea of "gluing" two subsets together, and the above condition makes sure the gluing works nicely.

Definition 1.3 (Complex Atlas)**.** Let X be a surface and $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X. Then $\mathfrak{U} = \{\phi_\alpha : U_\alpha \to V_\alpha \in \mathbb{C}\}_{\alpha\in A}$ is a *complex atlas* on X if every pair of charts ϕ_α, ϕ_β is holomorphically compatible, when $\alpha, \beta \in A$.

Definition 1.4 (Analytically Equivalent)**.** Let $\mathfrak{U}, \mathfrak{V}$ be complex atlases on a surface X. $\mathfrak{U}$ and $\mathfrak{V}$ are *analytically equivalent* if for any $\phi \in \mathfrak{U}, \psi \in \mathfrak{V}$, ϕ and ψ are holomorphically compatible.

Analytic equivalence is an equivalence relation. We only need to check transitivity; the other two follow from the definition.

Proposition 1.5. *Let $\mathfrak{U}, \mathfrak{V}, \mathfrak{W}$ be complex atlases on a surface X. Suppose $\mathfrak{U}, \mathfrak{V}$ are analytically equivalent and $\mathfrak{V}, \mathfrak{W}$ are as well. Then, $\mathfrak{U}$ and $\mathfrak{W}$ are analytically equivalent.*

Remark 1.6*.* It may be useful to consider sub-atlases and unions of atlases, defined in the natural way. The sub-atlas relation forms a partial order, and thus by an argument using Zorn's lemma, we can say that two atlases are analytically equivalent if and only if they are subsets of the same "maximal atlas."

Proof. Let $\phi \in \mathfrak{U}, \psi \in \mathfrak{V}, \xi \in \mathfrak{W}$. Then, $\phi \circ \psi^{-1}$ and $\psi \circ \xi^{-1}$ are holomorphic. Thus, $(\phi \circ \psi^{-1}) \circ (\psi \circ \xi^{-1}) = \phi \circ \xi^{-1}$ is holomorphic. Similar holds for the inverse, $\xi \circ \phi^{-1}$. Thus, $\forall \phi \in \mathfrak{U}, \psi \in \mathfrak{W}$, $\phi \circ \xi^{-1}$ is biholomorphic. ∎

Thus, under analytic equivalence, complex atlases on a surface X form equivalence classes.

Definition 1.7 (Riemann surface)**.** A *Riemann surface* is defined as a connected surface X combined with a *complex structure*, or an equivalence class of complex atlases on X.

Example (The complex plane is a Riemann surface)*.* Let $X = \mathbb{C}$, $\mathfrak{U} = \{\phi\}$, with $\phi : X = \mathbb{C} \to \mathbb{C}$ satisfying $\phi(z) = z$.

Example (The Riemann sphere)*.* Let $X = \mathbb{C} \cup \{\infty\}$, with the topology of a 2-sphere. If $U \subset \mathbb{C}$ is open in $\mathbb{C}$, then U is open in X, and if K is compact in $\mathbb{C}$, then $\mathbb{C} - K \cup \{\infty\}$ is open in X. We give X a complex structure with the following atlas.

$$\phi_1 : U_1 = \mathbb{C} \in X \to \mathbb{C} \qquad \forall z \in \mathbb{C},\ \phi_1(z) = z$$
$$\phi_2 : U_2 = X - \{0\} \to \mathbb{C} \qquad \forall z \neq 0, \phi_2(z) = 1/z$$

where $1/\infty = 0$.

Many of our notions from analysis on the complex plane extend to analysis on Riemann surfaces. One such example is holomorphicity.

Definition 1.8 (Holomorphic)**.** Let Y be an open subset of a Riemann surface X. Let $f : Y \to \mathbb{C}$. We define f to be *holomorphic*, if, for any chart $\phi : U \to V$ on X,

$$f \circ \phi^{-1} : \phi(U \cap Y) \to \mathbb{C}$$

is holomorphic as a complex function.

Example ($1/z$ on the Riemann sphere)*.* Let X be the Riemann sphere as defined previously. Let $f : Y = X - \{0\} \to \mathbb{C}$ be defined by $f(z) = 1/z$, with $1/\infty = 0$. We will verify that f is holomorphic on Y. Consider ϕ_1. We have that $Y \cap U_1 = \mathbb{C} - \{0\}$, and $1/z$ is holomorphic on $\mathbb{C} - \{0\}$. Similarly, with ϕ_2, ϕ^{-1} is just $1/z$, so $f \circ \phi^{-1}$ is the identity, which is holomorphic.

2. Sheaves

The concept of a sheaf is quite an abstract one. The general motivation behind the definition is to take a topological space, and encode some sort of information about each open set. We will begin with some definitions.

Definition 2.1 (Presheaf). Let X be a topological space, $\mathfrak{O}$ be the collection of open sets in X. A *presheaf* $\mathcal{F}$ of abelian groups (note: we can define a sheaf of any category) is a collection of abelian groups $\{\mathcal{F}(U) : U \in \mathcal{O}\}$. For every $U \supset V, U, V \in \mathfrak{O}$, we have a restriction homomorphism $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ with the following two properties:

(1) $\rho_U^U = \mathrm{Id}_{\mathcal{F}(U)}$
(2) If $W \subset V \subset U$ are open sets, then $\rho_W^U = \rho_W^V \circ \rho_V^U$.

We often use the notation $s|V = \rho_V^U(s)$, where $s \in \mathcal{F}(U)$.

A presheaf is generally not specific enough for our needs: instead, we like to consider the gluing of open sets, in order to make sure that our information is compatible (i.e. glues uniquely) when taking unions.

Definition 2.2 (Sheaf). A presheaf $\mathcal{F}$ on a topological space X is called a *sheaf* if for every open set $U \in \mathfrak{O}$, open cover $\{U_\alpha\}_{\alpha \in A}$ of U, the following axioms (the "sheaf axioms") are satisfied:

(1) If $s, t \in \mathcal{F}(U)$ with $s|U_\alpha = t|U_\alpha$ for any $\alpha \in A$, then $s = t$.
(2) If $s_\alpha \in \mathcal{F}(U_\alpha), \alpha \in A$ satisfy

$$s_\alpha|(U_\alpha \cap U_\beta) = s_\beta|(U_\alpha \cap U_\beta)$$

for any $\alpha, \beta \in A$, then there exists $s \in \mathcal{F}(U)$ with $s|U_\alpha = s_\alpha$ for all $\alpha \in A$.

Example (Constant Sheaf). We can see that if we have a topological space X and an abelian group A, we can simply define a sheaf $\mathcal{F}$ on X as $\mathcal{F}(X) = A$, where each restriction homomorphism is the identity.

Example (Sheaf of Holomorphic Functions). An example of a sheaf of abelian groups (and in fact a sheaf of rings) is defined for any Riemann surface X. Let $\mathcal{O}(U)$ be the ring of holomorphic functions defined on $U \subset X$. We define ρ_V^U as the map that restricts each function $f \in \mathcal{O}(U)$ to V. The sheaf axioms are satisfied: (1) if two holomorphic functions agree on an open cover of a set, they agree at every point, and are thus identical. The proof of (2) is left as an exercise.

Example (A Presheaf that is not a Sheaf). Let X be a topological space and S be a set. Consider the sheaf of sets $\mathcal{F}$ on X such that $\mathcal{F}(U) = S$ for any $U \in \mathfrak{O}$, and let ρ_V^U be the identity. We can see that $\mathcal{F}$ is a presheaf. However, if we let $U = \emptyset$ and consider the empty covering, then $s|U_\alpha = t|U_\alpha$ for any α in our cover, vacuously, where $s, t \in A$. Thus, $s = t$. But we can just set $s \neq t$, so $\mathcal{F}$ is not a sheaf.

We will provide one more definition, which is useful both because the style of its definition is iterated later on, and because the concept is used in the proof of the Riemann-Roch theorem.

Definition 2.3 (Stalk of a Presheaf). Let $\sqcup$ denote the disjoint union. Let $\mathcal{F}$ be a presheaf on a topological space X with a chosen point a. We define an equivalence relation $\sim_a$ on the disjoint union

$$\bigsqcup_{U \ni a} \mathcal{F}(U)$$

where if $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$, then if $a \in W \subset U \cap V$ and $f|W = g|W$, then $f \sim_a g$. We define the *stalk* of $\mathcal{F}$ at a as

$$\mathcal{F}_a = \left(\bigsqcup_{U \ni a} \mathcal{F}(U) \right) \Big/ \sim_a .$$

Namely, the stalk of $\mathcal{F}$ at a is the set of equivalence classes of functions such that they are identical "around" a.

3. Sheaf Cohomology

We now proceed with some highly technical definitions meant to introduce the method of sheaf cohomology.

Definition 3.1 (Cochain Group)**.** Let q be a nonnegative integer, and $\mathcal{F}$ be a sheaf of abelian groups on a topological space X. Moreover, let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of X. Then we define the *qth cochain group* $C^q(\mathcal{U}, \mathcal{F})$ as the direct product

$$\prod_{(\alpha_0, \ldots, \alpha_q) \in A^{q+1}} \mathcal{F} \left(\bigcap_{i=0}^{q} U_{\alpha_i} \right).$$

Each element of such a group is called a q-cochain.

Definition 3.2 (Coboundary Operator)**.** We will define the *coboundary operator*

$$\delta : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$$

which we sometimes denote as

$$C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F}),$$

as, for $(f_\alpha)_{\alpha \in A}$,

$$\delta((f_\alpha)_{\alpha \in A}) = ((f_\beta - f_\alpha)|(U_\alpha \cap U_\beta))_{\alpha, \beta \in A}.$$

Similarly, we define the coboundary operator $\delta : C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F})$ by mapping a 1-cochain $(f_{\alpha\beta})$ to a 2-cochain $((f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})|(U_\alpha \cap U_\beta \cap U_\gamma))$.

Each of these coboundary operators is a group homomorphism.

Definition 3.3 (Coboundary Group)**.** We let the 1-*coboundary* group $B^1(\mathcal{U}, \mathcal{F})$ be defined as

$$\operatorname{Im}(C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F})).$$

Definition 3.4 (Cocycle Group)**.** We let the 1-*cocyle* group $Z^1(\mathcal{U}, \mathcal{F})$ be defined as

$$\ker(C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^2(\mathcal{U}, \mathcal{F})).$$

We can see that a 1-cochain is a cocycle if for all $\alpha, \beta, \gamma \in A$, $f_{\alpha\gamma} = f_{\alpha\beta} + f_{\beta\gamma}$ when restricted to $U_\alpha \cap U_\beta \cap U_\gamma$. We also get $f_{\alpha\alpha} = 0$ and $f_{\alpha\beta} = -f_{\beta\alpha}$. Finally, every coboundary is a cocycle. Namely, $B_1(\mathcal{U}, \mathcal{F}) \subset Z^1(\mathcal{U}, \mathcal{F})$.

This will allow us to define the 1*st cohomology group*.

Definition 3.5 (Cohomology Group)**.** We define the 1*st cohomology group* as

$$H^1(\mathcal{U}, \mathcal{F}) = Z^1(\mathcal{U}, \mathcal{F})/B^1(\mathcal{U}, \mathcal{F}).$$

All of these definitions seem rather abstract. However, the general idea is that we are looking for a way to measure how well our sheaf axioms are working - specifically those regarding the gluing of sets in an open cover. That said, we currently only have a definition of a cohomology group for a specific open cover. We would like to generalize this notion to cohomology groups of a sheaf over a topological space, without regard to a specific open cover.

We define a cover to be finer than another cover if every covering set of the first cover is entirely contained within a covering set of the second cover. This definition gives us a natural mapping for two such covers. Suppose $\mathcal{V} = (V_\beta)_{\beta \in B}$ is finer than $\mathcal{U} = (U_\alpha)_{\alpha \in A}$. We define $\tau : B \to A$ as

$$V_\beta = U_{\tau(\beta)} \forall \beta \in B.$$

Here is where our sheaf structure becomes valuable. We define

$$t : Z^1(\mathcal{U}, \mathcal{F}) \to Z^1(\mathcal{V}, \mathcal{F})$$

such that for any $f = (f_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{F})$,

$$g_{\alpha\beta} = f_{\tau(\alpha)\tau(\beta)} | V_\alpha \cap V_\beta$$

when $\alpha, \beta \in A$. We can see that t induces a homomorphism of the cohomology groups $H^1(\mathcal{U}, \mathcal{F}) \to H^1(\mathcal{V}, \mathcal{F})$. For ease, we call this t as well.

Lemma 3.6. *We will now show that t is well-defined (i.e. independent of τ) and one-to-one.*

Proof. Let τ and σ be two such mappings as previously defined. Observe that for any β, $V_\beta \in U_{\tau(\beta)}, U_{\sigma(\beta)}, U_{\tau(\beta)} \cap U_{\sigma(\beta)}$. Now suppose we define $t(f) = g$ and $t'(f) = g'$ under τ and σ respectively. Then, we have

$$\begin{aligned} g_{\alpha\beta} - g'_{\alpha\beta} &= f_{\tau(\alpha)\tau(\beta)} - f_{\sigma(\alpha)\sigma(\beta)} \\ &= f_{\tau(\alpha)\tau(\beta)} + f_{\tau(\beta)\sigma(\alpha)} - f_{\tau(\beta)\sigma(\alpha)} - f_{\sigma(\alpha)(\sigma(\beta)} \\ &= f_{\tau(\alpha)\sigma(\alpha)} - f_{\tau(\beta)\sigma(\beta)}, \end{aligned}$$

which when restricted to $V_\alpha \cap V_\beta$ gives that $g_{\alpha\beta} - g'_{\alpha\beta}$ is a coboundary by definition. Thus $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ are in the same equivalence class, or what we call *cohomologous.*

To show t is one-to-one, we need to show that if $t((f_{\alpha\beta}))$ is a cocycle whose image is a coboundary, then $(f_{\alpha\beta})$ is itself a coboundary by linearity (t is a homomorphism). We write

$$f_{\tau(\alpha)\tau(\beta)} = g_\alpha - g_\beta | V_\alpha \cap V_\beta, \qquad g_\alpha \in \mathcal{F}(V_\alpha).$$

We do a similar manipulation as above to get that

$$g_\alpha - g_\beta = f_{\gamma\tau(\alpha)} - f_{\gamma\tau(\beta)},$$

or $f_{\gamma\tau(\alpha)} + g_\alpha$ is fixed no matter the choice of α. Thus by gluing we have some

$$h_\gamma = f_{\gamma\tau(\alpha)} + g_\alpha | U_\gamma \cap V_\alpha$$

where $h_\gamma \in \mathcal{F}(U_\gamma)$. Applying this, we get for any γ and restricted to $U_\alpha \cap U_\beta \cap V_\gamma$

$$f_{\alpha\beta} = f_{\alpha\tau(\gamma)} + f_{\tau(\gamma)\beta} = f_{\alpha\tau(\gamma)} + g_\gamma - f_{\beta\tau(\gamma)} - g_\gamma = h_\alpha - h_\beta,$$

and once again we glue over $U_\alpha \cap U_\beta$, giving us our desired result. ■

We are now able to proceed to working with cohomology groups over a sheaf. We will say $\mathcal{V} < \mathcal{U}$ if $\mathcal{V}$ is finer than $\mathcal{U}$. We define an equivalence relation on

$$\bigsqcup_{\bigcup \mathcal{U} \supset X} H^1(\mathcal{U}, \mathcal{F})$$

where if $\zeta \in H^1(\mathcal{U}_2, \mathcal{F}), \eta \in H^1(\mathcal{U}_2, \mathcal{F})$, and there exists $\mathcal{V} < \mathcal{U}_1, \mathcal{U}_2$ such that $t_1(\zeta) = t_2(\eta)$ (where t_1 and t_2 are the above-defined maps for $\mathcal{U}_1$ and $\mathcal{U}_2$ respectively), then $\zeta \sim \eta$.

Definition 3.7 (1st Cohomology Group of X)**.** We can now define the more general cohomology group of a topological space as

$$H^1(X, \mathcal{F}) = \left(\bigsqcup_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{F}) \right) \Big/ \sim .$$

The group operation is naturally defined via taking two elements of $H^1(X, \mathcal{F})$ and sending their representatives to a common refined group, and then reverting back to the original cover. As previously, this is independent of the choice of cover and the choice of map (τ). Moreover, this group is abelian as the original cohomology groups are abelian.

4. Exact Sequences

We begin this section by defining a sheaf homomorphism in the natural way:

Definition 4.1 (Homomorphism of Sheaves)**.** Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be sheaves over X. A *sheaf homomorphism* $\alpha : \mathcal{F} \to \mathcal{G}$ is a family of group homomorphisms, such that for each open set $U \subset X$ there exists $\alpha_U \in \alpha$ with

$$\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

and if $V \subset U \subset X$, $\alpha_U(f)|V = \alpha_V(f|V)$ for any $f \in \mathcal{F}(U)$. Namely, the group homomorphisms are compatible with the restrictions. We can define an *isomorphism of sheaves* as a homomorphism of sheaves where each group homomorphism is itself an isomorphism.

Example (Inclusions). Let $\mathcal{P}$ be the sheaf of functions on a topological space X, and $\mathcal{O}$ be the sheaf of holomorphic functions. Then, the inclusion $\mathcal{P} \hookrightarrow \mathcal{O}$ is a sheaf homomorphism.

We can also naturally define the kernel of a sheaf homomorphism.

Definition 4.2 (Kernel of a Sheaf Homomorphism)**.** Let X be a topological space, $\mathcal{F}, \mathcal{G}$ sheaves, and $\alpha : \mathcal{F} \to \mathcal{G}$ a homomorphism. The kernel $\mathcal{H}$ of α is defined with

$$\mathcal{H}(U) = \ker \left(\mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \right),$$

which can be verified to be a sheaf.

Example (Holomorphic Functions). Once again, let $\mathcal{O}$ be the sheaf of holomorphic functions on $\mathbb{C}$ and let $\mathcal{O}^*$ be the sheaf of holomorphic functions with the operation of multiplication such that the holomorphic functions are nonzero everywhere. Let α be the homomorphism having $\alpha(f) = \exp(2\pi i f)$ for any $f \in U \subset X$. Then,

$$\ker \left(\mathcal{O} \xrightarrow{\alpha} \mathcal{O}^* \right) = \mathbb{Z}.$$

We now proceed to define an exact sequence of sheaf homomorphisms.

Definition 4.3 (Exact Sequence)**.** Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a sheaf homomorphism over X. Let $x \in X$. Then α induces a homomorphism $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ on the stalks of the sheaves. For sheaves $\mathcal{F}, \mathcal{G}, \mathcal{H}$, and homomorphisms α, β, the sequence

$$\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H}$$

is *exact* if the following sequence satisfies $\ker \beta_x = \operatorname{im} \alpha_x$ for any $x \in X$:

$$\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x.$$

More generally, if

$$\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \cdots \longrightarrow \mathcal{F}_n$$

has

$$\mathcal{F}_k \longrightarrow \mathcal{F}_{k+1} \longrightarrow \mathcal{F}_{k+2}$$

exact for all k between 1 and $n-2$, then we call this whole sequence exact.

Definition 4.4 (Mono- and Epimorphisms)**.** The following two definitions are analogs of one-to-one functions and onto functions. A sheaf homomorphism $\alpha : \mathcal{F} \to \mathcal{G}$ is a monomorphism if $0 \to \mathcal{F} \to \mathcal{G}$ is exact, and an epimorphism if $\mathcal{F} \to \mathcal{G} \to 0$ is exact.

We can see from the sheaf axioms that the group homomorphisms induced by a sheaf monomorphism are injective, but this is not necessarily true for epimorphisms and being surjective (although they are analogous concepts).

Definition 4.5 (Short Exact Sequences)**.** If the sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is exact, then the sequence is called a *short exact sequence*.

Lemma 4.6. *If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves on X, then for any open $U \subset X$, the sequence of groups $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ is exact.*

Proof. We can see the first three terms form an exact sequence, so we need only show $\mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ is exact. Let α be the homomorphism from $\mathcal{F} \to \mathcal{G}$ and β be from $\mathcal{G} \to \mathcal{H}$. Let $g \in \alpha(f), f \in \mathcal{F}(U)$. By exactness of stalks, for any $x \in U$ there is V_x open in U with $\beta(g)|V_x = 0$, so by gluing we get that $\beta(g) = 0$, or $g \in \ker \beta$, so $\operatorname{im} \alpha \in \ker \beta$.

Now suppose $\beta(g) = 0$ for $g \in \mathcal{G}(U)$. We know for any $x \in U$, $\ker \beta_x = \operatorname{im} \alpha_x$, so if we let $(V_i)_{i \in I}$ be an open cover of U, we have $f_i \in \mathcal{F}(V_i)$ with $\alpha(f_i) = g|V_i$. We can intersect and glue once again to get an $f \in \mathcal{F}(U)$ with $f|V_i = f_i$. Since α is a homomorphism, we have $\alpha(f)|V_i = \alpha(f|V_i) = g|V_i$, so by gluing yet again, $\alpha(f) = g$. ■

5. The Riemann-Roch Theorem

We need to define a divisor as a method of keeping track of zeros on a Riemann surface. Namely, we have:

Definition 5.1 (Divisor)**.** A divisor on a Riemann surface X is a function

$$D : X \to \mathbb{Z}$$

such that if K is compact in X then $D(x) = 0$ for all but finitely many points. The divisors form an abelian group $\operatorname{Div}(X)$.

We also define divisors for meromorphic functions on X. The divisor of a meromorphic function is simply its order at a point x, and we let this be denoted as (f) for a function f.

We say two divisors D and D' are *equivalent* if there exists a meromorphic f with $(f) = D - D'$.

Definition 5.2 (Degree of a Divisor)**.** Suppose X is a compact Riemann surface, so that if D is a divisor of X, then there are only finitely many $x \in X$ with $D(x)$ nonzero, so we can call

$$\deg D = \sum_{x \in X} D(x),$$

which is a homomorphism from $\mathrm{Div}(x)$ to $\mathbb{Z}$. Note that for meromorphic f, $\deg(f) = 0$, as a meromorphic function has equally many zeroes and poles.

Let D be a divisor on X, U open in X. We define a sheaf $\mathcal{O}_D(U)$ as the set of meromorphic functions on U whose orders at any $x \in U$ are at least $-D(x)$.

Lemma 5.3. *Let D be a divisor on X with $\deg D < 0$. Then $H^0(X, \mathcal{O}_D) = 0$.*

Proof. Suppose we have $0 \neq f \in H^0(X, \mathcal{O}_D)$. Then $\deg(f) \geq -\deg D > 0$, which is a contradiction. ∎

Definition 5.4 (Skyscraper Sheaf)**.** Let $P \in X$ where X is a Riemann Surface. Define the *skyscraper sheaf* $\mathbb{C}_P$ as

$$\mathbb{C}_P(U) = \begin{cases} \mathbb{C} & P \in U \\ 0 & P \notin U. \end{cases}$$

We can see that $H^0(X, \mathbb{C}_P) \cong \mathbb{C}$ and $H^1(X, \mathbb{C}_P) = 0$.

Let D be a divisor on X and let $P \in X$ with P also a divisor that is the indicator, such that $P(P) = 1$, and P evaluated elsewhere is zero, so that we have an inclusion $\mathcal{O}_D \to \mathcal{O}_{D+P}$. Now let V be an open set about P and ψ be a chart with $\psi(P) = 0$. We define a sheaf homomorphism $\beta : \mathcal{O}_{D+P} \to \mathbb{C}_P$ with $\beta_U \equiv 0$ when $P \notin U$ and if $P \in U, f \in \mathcal{O}_{D+P}(U)$, then f is meromorphic and thus has a Laurent series expansion, and we let $\beta_U(f)$ be the term with the least (i.e. most negative) power, which is a complex number and thus an element of $\mathbb{C}_P(U)$. We can then see that sequence

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_{D+P} \xrightarrow{\beta} \mathbb{C}_P \longrightarrow 0$$

is short exact. Then, in a result following from the work in the previous section (proof omitted, see Forster 15.12), we get that

$$0 \to H^0(X, \mathcal{O}_D) \to H^0(X, \mathcal{O}_{D+P}) \to \mathbb{C} \to H^1(X, \mathcal{O}_D \to H^1(X, \mathcal{O}_{D+P}) \to 0$$

is exact.

Lemma 5.5. *Let X be a compact Riemann surface, and $D \leq D'$ be divisors. Then we have an epimorphism*

$$H^1(X, \mathcal{O}_D) \hookrightarrow H^1(X, \mathcal{O}_{D'} \to 0.$$

Proof. Since X is compact, we can write $D' = D + P_1 + \cdots + P_k$ where the P_i's are indicator functions as defined above. Inductively, we are done by the previous proof. ∎

We state without proof that $H^1(X, \mathcal{O})$ is a finite-dimensional vector space, and we call this dimension g, for *genus*. However, we are now finally ready to prove Riemann-Roch!

Theorem 5.6 (Riemann-Roch)**.** *Let X be a compact Riemann surface of genus g, and let D be a divisor. Then $H^0(X,\mathcal{O}_D), H^1(X,\mathcal{O}_D)$ are finite-dimensional vector spaces satisfying*

$$\dim H^0(X,\mathcal{O}_D) - \dim H^1(X,\mathcal{O}_D) = 1 - g + \deg D.$$

Proof. Suppose $D \equiv 0$. Then, $H^0(X,\mathcal{O}_D) = \mathcal{O}(X)$ which has dimension 1 by compactness and $\dim H^1 = g$ by definition.

Now suppose D is a divisor and $D' = D + P$. From here, we write $V = \mathrm{im}(H^0(X,\mathcal{O}_{D'}) \to \mathbb{C})$, and $W = \mathbb{C}/V$, so that $\dim V + \dim W = \deg D' - \deg D = 1$ and we have the short exact sequences (by the above)

$$0 \to H^0(X,\mathcal{O}_D) \to H^0(X,\mathcal{O}_{D'}) \to V \to 0, \quad 0 \to W \to H^1(X,\mathcal{O}_D) \to H^1(X,\mathcal{O}_{D'} \to 0.$$

Thus $\dim H^0(X,\mathcal{O}_{D'}) = \dim H^0(X,\mathcal{O}_D) + \dim V$, $\dim H^1(X,\mathcal{O}_{D'}) = \dim H^1(X,\mathcal{O}_D) + \dim W$. This gives us

$$\dim H^0(X,\mathcal{O}_{D'}) - \dim H^1(X,\mathcal{O}_{D'}) - \deg D' = \dim H^0(X,\mathcal{O}_D) - \dim H^1(X,\mathcal{O}_D) - \deg D,$$

which means that if Riemann-Roch holds for D, it holds for D'. But since every divisor on a compact Riemann surface has finite degree, we can simply induct from the zero divisor, and we are done! ■

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