Nevanlinna–Pick Interpolation

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A fascinating problem that arises from the the structure of certain types of functions is the problem of interpolation: given a data points $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$, how can we construct a function satisfying certain conditions that passes through all of these points? One famous example of this type of construction is Lagrange polynomial interpretation, which finds the unique polynomial of degree less than n that passes through all n points. The Nevanlinna–Pick Problem explores interpolation on the unit disk over holomorphic functions.

Problem 0.1 (Nevanlinna–Pick Interpolation Problem)**.** *Given n distinct input points $z_1, z_2, \ldots, z_n$ in the unit disk $\mathbb{D}$ and n output points $w_1, \ldots, w_n$ in $\overline{\mathbb{D}}$, does there exist a holomorphic function $f : \mathbb{D} \to \overline{\mathbb{D}}$ such that $f(z_k) = w_k$ for all k with $1 \leq k \leq n$?*

1 Pick's Approach

Pick's approach [1] inductively characterizes sets of points that could be interpolated. We begin by defining a few preliminary concepts. We first introduce the definition of positive semi-definiteness of a matrix with entries in $\mathbb{C}$, as this concept shows up in the statement of Pick's Interpolation Theorem.

Definition 1.1. A Hermitian matrix is a square matrix A satisfying $A^\intercal = \overline{A}$. A $n \times n$ Hermitian matrix B is called positive semi-definite if and only if for all $\mathbf{t} \in \mathbb{C}^n$, we have $\mathbf{t}^* B \mathbf{t} \geq 0$.

The Maximum Modulus Principle is a helpful tool for understanding the extrema of such functions, and we will use it in the proof.

Theorem 1.2 (Maximum Modulus Principle)**.** *Let D be a connected open subset of $\mathbb{C}$, and let $f : D \to \mathbb{C}$ be a holomorphic function. If $|f(z_0)| \geq |f(z)|$ for all z in a neighborhood of z_0, then f must be constant. Similarly, a holomorphic function $g : \overline{D} \to \mathbb{C}$ attains a maximum for $|g(z)|$ at some point $z \in \partial D$, the boundary of D.*

These preliminary concepts allow present an overview of Pick's inductive approach to Problem 0.1.

Theorem 1.3 (Pick, 1916)**.** *Problem 0.1 has a solution if and only if*

$$X_n = \begin{bmatrix} \frac{1-\overline{w_1}w_1}{1-\overline{z_1}z_1} & \frac{1-\overline{w_1}w_2}{1-\overline{z_1}z_2} & \cdots & \frac{1-\overline{w_1}w_n}{1-\overline{z_1}z_n} \\ \frac{1-\overline{w_2}w_1}{1-\overline{z_2}z_1} & \frac{1-\overline{w_2}w_2}{1-\overline{z_2}z_2} & \cdots & \frac{1-\overline{w_2}w_n}{1-\overline{z_2}z_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1-\overline{w_n}w_1}{1-\overline{z_n}z_1} & \frac{1-\overline{w_n}w_2}{1-\overline{z_n}z_2} & \cdots & \frac{1-\overline{w_n}w_n}{1-\overline{z_n}z_n} \end{bmatrix}$$

is positive semi-definite.

Proof. Proceed by induction on n. When $n = 1$ (we wish to find $f : z_1 \mapsto w_1$), consider the following two Blaschke factors:

$$B_1(z) = \frac{z - z_1}{1 - \overline{z_1}z} \qquad \text{and} \qquad B_2(z) = \frac{z - w_1}{1 - \overline{w_1}z}.$$

Since $B_1 : z_1 \mapsto 0$ and $B_2^{-1} : 0 \mapsto w_1$, $f = B_1 \circ B_2^{-1}$ takes $z_1 \mapsto w_1$ is a holomorphic function on $\mathbb{D} \to \overline{\mathbb{D}}$ regardless of choice of w_1 and z_1. On the other hand,

$$X_1 = \left[\frac{1-\overline{w_1}w_1}{1-\overline{z_1}z_1}\right] = \left[\frac{1-|w_1|^2}{1-|z_1|^2}\right].$$

Since $|w_1| \leq 1$, the only entry is a real number greater than or equal to 0. Hence, for any $\mathbf{t} \in \mathbb{C}^n$, we indeed have $\mathbf{t}^* X_1 \mathbf{t} \geq 0$, so X_1 is always positive semi-definite as desired.

Now, suppose that the Nevanlinna–Pick Interpolation Problem is true if we replace n with $n-1$. That is, there exists a holomomorphic function $f_{n-1} : \mathbb{D} \to \overline{\mathbb{D}}$ that interpolates any $n-1$ distinct input points to any $n-1$ output points if and only if K_{n-1} is positive semi-definite. We consider two separate cases:

Case 1. Let $|w_n| < 1$. Define the two Blaschke factors

$$B_1(z) = \frac{z - z_n}{1 - \overline{z_n}z} \qquad \text{and} \qquad B_2(z) = \frac{z - w_n}{1 - \overline{w_n}z}$$

taking $B_1 : z_n \mapsto 0$ and $B_2 : w_n \mapsto 0$. Moreover, let

$$z'_i = B_1(z_i) = \frac{z_i - z_n}{1 - \overline{z_n}z_i} \qquad \text{and} \qquad w'_i = B_2(w_i) = \frac{w_i - w_n}{1 - \overline{w_n}w_i}$$

for all $i = 1, 2, \ldots, n$. Notice that $\{(z_1, w_1), (z_2, w_2), \ldots, (z_n, w_n)\}$ has an suitable interpolation f if and only if $\{(z'_1, w'_1), (z'_2, w'_2), \ldots, (z'_n, w'_n) = (0, 0)\}$ also has an suitable interpolation g, where f and g are related by $f = B_2^{-1} \circ g \circ B_1$ or equivalently $g = B_2 \circ f \circ B_1^{-1}$. We claim that X_n is positive semi-definite if and only if

$$X'_n = \begin{bmatrix} \frac{1-\overline{w'_1}w'_1}{1-\overline{z'_1}z'_1} & \frac{1-\overline{w'_1}w'_2}{1-\overline{z'_1}z'_2} & \cdots & \frac{1-\overline{w'_1}w'_n}{1-\overline{z'_1}z'_n} \\ \frac{1-\overline{w'_2}w'_1}{1-\overline{z'_2}z'_1} & \frac{1-\overline{w'_2}w'_2}{1-\overline{z'_2}z'_2} & \cdots & \frac{1-\overline{w'_2}w'_n}{1-\overline{z'_2}z'_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1-\overline{w'_n}w'_1}{1-\overline{z'_n}z'_1} & \frac{1-\overline{w'_n}w'_2}{1-\overline{z'_n}z'_2} & \cdots & \frac{1-\overline{w'_n}w'_n}{1-\overline{z'_n}z'_n} \end{bmatrix}$$

is positive semi-definite. Consider the expression

$$\begin{aligned} \mathbf{t}^* X'_n \mathbf{t} &= \begin{bmatrix} \overline{t_1} & \overline{t_2} & \cdots & \overline{t_n} \end{bmatrix} \begin{bmatrix} x'_{11} & x'_{12} & \cdots & x'_{1n} \\ x'_{21} & x'_{22} & \cdots & x'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x'_{n1} & x'_{n2} & \cdots & x'_{nn} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{t_i} t_j x'_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{t_i} t_j \left(\frac{1 - \overline{w'_i}w'_j}{1 - \overline{z'_i}z'_j} \right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{t_i} t_j \left(\frac{1 - \left(\frac{\overline{w_i}-\overline{w_n}}{1-w_n\overline{w_i}}\right)\left(\frac{w_j-w_n}{1-\overline{w_n}w_j}\right)}{1 - \left(\frac{\overline{z_i}-\overline{z_n}}{1-z_n\overline{z_i}}\right)\left(\frac{z_j-z_n}{1-\overline{z_n}z_j}\right)} \right). \end{aligned}$$

The numerator expression can be evaluated as

$$1 - \left(\frac{\overline{w_i}w_j - \overline{w_n}w_j - \overline{w_i}w_n + |w_n|^2}{1 - w_n\overline{w_1} - \overline{w_n}w_j + |w_n|^2\overline{w_i}w_j} \right)$$

$$= \frac{1 - \overline{w_i} w_j - |w_n|^2 + |w_n|^2 \overline{w_i} w_j}{1 - w_n \overline{w_1} - \overline{w_n} w_j + |w_n|^2 \overline{w_i} w_j} = \frac{(1 - \overline{w_i} w_j)(1 - |w_n|^2)}{(1 - w_n \overline{w_i})(1 - \overline{w_n} w_j)},$$

and we can thus write

$$\begin{aligned}
\mathbf{t}^* X_n' \mathbf{t} &= \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{t_i} t_j \left(\frac{1 - \left(\frac{\overline{w_i} - \overline{w_n}}{1 - w_n \overline{w_i}} \right) \left(\frac{w_j - w_n}{1 - \overline{w_n} w_j} \right)}{1 - \left(\frac{\overline{z_i} - \overline{z_n}}{1 - z_n \overline{z_i}} \right) \left(\frac{z_j - z_n}{1 - \overline{z_n} z_j} \right)} \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{t_i} \left(\frac{1 - z_n \overline{z_i}}{1 - w_n \overline{w_i}} \right) t_j \left(\frac{1 - \overline{z_n} z_j}{1 - \overline{w_n} w_j} \right) \left(\frac{1 - \overline{w_i} w_j}{1 - \overline{z_i} z_j} \right) \left(\frac{1 - |w_n|^2}{1 - |z_n|^2} \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{\left(\sqrt{\frac{1 - |w_n|^2}{1 - |z_n|^2}} \left(\frac{1 - \overline{z_n} z_i}{1 - \overline{w_n} w_i} \right) t_i \right)} \left(\sqrt{\frac{1 - |w_n|^2}{1 - |z_n|^2}} \left(\frac{1 - \overline{z_n} z_j}{1 - \overline{w_n} w_j} \right) t_j \right) x_{ij}.
\end{aligned}$$

Define $\mathbf{t}' \in \mathbb{C}^n$ such that for all $i = 1, 2, \ldots, n$,

$$t_i' = \sqrt{\frac{1 - |w_n|^2}{1 - |z_n|^2}} \left(\frac{1 - \overline{z_n} z_i}{1 - \overline{w_n} w_i} \right) t_i.$$

Then, we can write

$$\begin{aligned}
\mathbf{t}^* X_n' \mathbf{t} &= \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{\left(\sqrt{\frac{1 - |w_n|^2}{1 - |z_n|^2}} \left(\frac{1 - \overline{z_n} z_i}{1 - \overline{w_n} w_i} \right) t_i \right)} \left(\sqrt{\frac{1 - |w_n|^2}{1 - |z_n|^2}} \left(\frac{1 - \overline{z_n} z_j}{1 - \overline{w_n} w_j} \right) t_j \right) x_{ij} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{t_i'} t_j' x_{ij} = \mathbf{t}'^* X_n \mathbf{t}'.
\end{aligned}$$

It follows easily that X_n is positive semi-definite if and only if X_n' is positive semi-definite. Therefore, it suffices to consider the corresponding problem in g, where $g(0) = 0$. Notice that since $g(0) = 0$, we can define the function $h(z) = g(z)/z$, which is also holomorphic and since z itself is a Blaschke factor, h is well-defined on $\mathbb{D} \to \overline{\mathbb{D}}$. Moreover, h interpolates the set of $n - 1$ points

$$\{(z_1', w_1'/z_1'), (z_2', w_2'/z_2'), \ldots, (z_n', w_n'/z_n')\}$$

By the inductive hypothesis, such an h exists if and only if the matrix

$$Y_{n-1} = \begin{bmatrix} \frac{1 - \overline{(w_1'/z_1')}(w_1'/z_1')}{1 - \overline{z_1'} z_1'} & \frac{1 - \overline{(w_1'/z_1')}(w_2'/z_2')}{1 - \overline{z_1'} z_2'} & \cdots & \frac{1 - \overline{(w_1'/z_1')}(w_{n-1}'/z_{n-1}')}{1 - \overline{z_1'} z_{n-1}'} \\ \frac{1 - \overline{(w_2'/z_2')}(w_1'/z_1')}{1 - \overline{z_2'} z_1'} & \frac{1 - \overline{(w_2'/z_2')}(w_2'/z_2')}{1 - \overline{z_2'} z_2'} & \cdots & \frac{1 - \overline{(w_2'/z_2')}(w_{n-1}'/z_{n-1}')}{1 - \overline{z_2'} z_{n-1}'} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1 - \overline{(w_{n-1}'/z_{n-1}')}(w_1'/z_1')}{1 - \overline{z_{n-1}'} z_1'} & \frac{1 - \overline{(w_{n-1}'/z_{n-1}')}(w_2'/z_2')}{1 - \overline{z_{n-1}'} z_2'} & \cdots & \frac{1 - \overline{(w_{n-1}'/z_{n-1}')}(w_{n-1}'/z_{n-1}')}{1 - \overline{z_{n-1}'} z_{n-1}'} \end{bmatrix}$$

is positive semi-definite. Thus, g exists if and only if Y_{n-1} is positive semi-definite. We are interested in finding a relationship between Y_{n-1} and X_n'. Notice now that since

$w'_n = z'_n = 0$, we have

$$x'_{in} = \frac{1 - \overline{w'_i} w'_n}{1 - \overline{z'_i} z'_n} = 1 \qquad \text{and} \qquad x'_{ni} = \frac{1 - \overline{w'_n} w_i}{1 - \overline{z'_n} z_i} = 1.$$

Again, we consider the following expression:

$$\begin{aligned}
\mathbf{t}^* X'_n \mathbf{t} &= \begin{bmatrix} \overline{t_1} & \overline{t_2} & \cdots & \overline{t_n} \end{bmatrix} \begin{bmatrix} x'_{11} & x'_{12} & \cdots & x'_{1(n-1)} & 1 \\ x'_{21} & x'_{22} & \cdots & x'_{2(n-1)} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x'_{(n-1)1} & x'_{(n-1)2} & \cdots & x'_{(n-1)(n-1)} & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix} \\
&= \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \overline{t_i} t_j x'_{ij} \right) + \left(\sum_{i=1}^{n-1} \overline{t_n} t_i \right) + \left(\sum_{i=1}^{n-1} \overline{t_i} t_n \right) + \overline{t_n} t_n \\
&= \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \overline{t_i} t_j (x'_{ij} - 1) \right) + \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{t_i} t_j \right).
\end{aligned}$$

Consider the expression within the first summation for $1 \le i, j \le n-1$:

$$x'_{ij} - 1 = \frac{1 - \overline{w'_i} w'_j}{1 - \overline{z'_i} z'_j} - 1 = \frac{\overline{z'_i} z'_j - \overline{w'_i} w'_j}{1 - \overline{z'_i} z'_j} = \overline{z'_i} z'_j \left(\frac{1 - \overline{(w'_i / z'_i)} \left(w'_j / z'_j \right)}{1 - \overline{z'_i} z'_j} \right) = \overline{z'_i} z'_j y_{ij}.$$

This is where Y_{n-1} starts to show up in the expression. We can rewrite the second summation as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{t_i} t_j = \left(\sum_{i=1}^{n} \overline{t_i} \right) \left(\sum_{i=1}^{n} t_i \right) = \overline{\left(\sum_{i=1}^{n} t_i \right)} \left(\sum_{i=1}^{n} t_i \right) = \left| \sum_{i=1}^{n} t_i \right|^2.$$

Using these simplifications, we can write

$$\begin{aligned}
\mathbf{t}^* X'_n \mathbf{t} &= \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \overline{t_i} t_j (x'_{ij} - 1) \right) + \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{t_i} t_j \right) \\
&= \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \overline{t_i z'_i} t_j z'_j y_{ij} \right) + \left| \sum_{i=1}^{n} t_i \right|^2.
\end{aligned}$$

Define $\mathbf{t}'' \in \mathbb{C}^n$ such that $t''_i = t_i z'_i$ for all $i = 1, 2, \ldots, n$. We can write

$$\mathbf{t}^* X'_n \mathbf{t} = \mathbf{t}''^* Y_{n-1} \mathbf{t}'' + \left| \sum_{i=1}^{n} t_i \right|^2.$$

Now we claim that Y_{n-1} is positive semi-definite if and only if X_n is positive semi-definite. If Y_{n-1} is positive semi-definite, then for all $\mathbf{t} \in \mathbb{C}$, we have

$$\mathbf{t}^* X'_n \mathbf{t} = \mathbf{t}''^* Y_{n-1} \mathbf{t}'' + \left| \sum_{i=1}^{n} t_i \right|^2 \ge \left| \sum_{i=1}^{n} t_i \right|^2 \ge 0.$$

On the other hand, if X_n is positive semi-definite, then for all $\mathbf{t}'' \in \mathbb{C}$, there exists $\mathbf{t} \in \mathbb{C}$

so that $\mathbf{t}^* X_n' \mathbf{t} \geq 0$. Hence, Y_{n-1} is positive semi-definite if and only if X_n is positive semi-definite, and we are done.

Case 2. If $|w_n| = 1$, then $|f(z_n)| = 1 \geq |f(z)|$ for all $z \in \mathbb{D}$, so by the Maximum Modulus Principle, $f(z) = f(z_n)$ for all $z \in \mathbb{D}$, meaning all of the w_i's are equal to w_n. Thus f exists if and only if $w_1 = w_2 = \cdots = w_n$. If $w_1 = w_2 = \cdots = w_n$, then every numerator in X_n evaluates to $1 - \overline{w_n} w_n = 0$, making X_n the zero matrix, which is positive semi-definite. On the other hand, suppose X_n is positive semi-definite. Define a map $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\langle a, b \rangle = a^* X_n b.$$

This map is a positive semi-definite Hermitian form, as it satisfies

- Conjugate symmetry:

$$\begin{aligned}
\overline{\langle b, a \rangle} &= \overline{b^* X_n a} \\
&= \overline{\left(\begin{bmatrix} \overline{b_1} & \overline{b_2} & \cdots & \overline{b_n} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \right)} \\
&= \overline{\left(\begin{bmatrix} \sum_{i=1}^n \overline{b_1} x_{i1} & \sum_{i=1}^n \overline{b_2} x_{i2} & \cdots & \sum_{i=1}^n \overline{b_n} x_{in} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \right)} \\
&= \overline{\sum_{i=1}^n a_i \sum_{j=1}^n \overline{b_j} x_{ij}} = \sum_{i=1}^n \sum_{j=1}^n \overline{a_i \overline{b_j} x_{ij}}.
\end{aligned}$$

  Since X_n is Hermitian, $x_{ij} = \overline{x_{ji}}$ and thus

$$\sum_{i=1}^n \sum_{j=1}^n \overline{a_i \overline{b_j} x_{ij}} = \sum_{i=1}^n \sum_{j=1}^n a_i \overline{b_j} x_{ji} = a^* X_n b = \langle a, b \rangle.$$

- Linearity in the first argument:

$$\langle c_1 x + c_2 y, z \rangle = (c_1 x + c_2 y)^* X_n z = c_1 (x^* X_n z) + c_2 (y^* X_n z) = c_1 \langle x, z \rangle + c_2 \langle y, z \rangle.$$

- Positive semi-definiteness:

$$\langle a, a \rangle = a^* X_n a \geq 0,$$

  due to the positive semi-definiteness of X_n.

Hence, we can use the Cauchy-Schwarz Inequality to obtain

$$\langle e_n, e_n \rangle \langle e_i, e_i \rangle \geq \langle e_i, e_n \rangle^2,$$

where e_i denotes the elementary basis vector with 1 in the ith position and 0 elsewhere. Note that $e_n^* X_n e_n = (X_n)_{nn} = \frac{1 - |w_n|^2}{1 - |z_n|^2} = 0$, so $\langle e_i, e_n \rangle^2 \leq 0$, so $\langle e_i, e_n \rangle = 0$. Thus,

$$\frac{1 - \overline{w_i} w_n}{1 - \overline{z_i} z_n} = (X_n)_{in} = e_i^* X_n e_n = 0,$$

from which we obtain $w_i = (\overline{w_n})^{-1} = w_n$ for each $i = 1, 2, \ldots, n-1$. Hence, $w_1 = w_2 = \cdots = w_n$, as desired. □

2 Nevanlinna's Approach

Nevanlinna [2] took a separate approach in his 1919 paper motivated by Schur's algorithm, used to solve similar problems such as Carathéodory's Interpolation Problem, which identifies conditions in which a polynomial can be interpolated by a holomorphic function f bounded on $\mathbb{D}$. But Nevanlinna was not just interested in determining when a set of poitns can be interpolated. In addition to giving a recursive condition to characterize what sets of n points can be interpolated by a holomorphic function $f : \mathbb{D} \to \overline{\mathbb{D}}$, Nevanlinna produced a representation of all functions that interpolated such sets of n points. We start by exploring the very basic $n = 1$ base case of Problem 0.1 as illustrated in [3].

Problem 2.1. *Let $z_1 \in \mathbb{D}$ and $w_1 \in \overline{\mathbb{D}}$. Characterize all holomorphic functions $f : \mathbb{D} \to \overline{\mathbb{D}}$ such that $f(z_1) = w_1$.*

To answer this question, we consider two separate cases. If $|w_1| = 1$, then by the Maximum Modulus Principle, f must be constant and therefore $f \equiv w_1$ is the only function that satisfies the condition. For the $|w_1| < 1$ case, notice that the function

$$B_1(f(z)) = \frac{f(z) - w_1}{1 - \overline{w_1} f(z)}$$

defined using the Blaschke factor taking $w_1 \mapsto 0$ is a holomorphic function on $\mathbb{D} \to \overline{\mathbb{D}}$ and thus can be written as $B_2(z)g(z)$ for some holomorphic function g on $\mathbb{D} \to \overline{\mathbb{D}}$, where the Blaschke factor B_2 maps $z_1 \mapsto 0$. This idea is similar to the one present in the inductive step of Pick's approach, but now we are able to specifically find all solutions rather than just whether a solution exists. Thus, all solutions can be written as

$$f(z) = B_1^{-1}(B_2(z)g(z)) = \frac{B_2(z)g(z) + w_1}{1 + \overline{w_1} B_2(z)g(z)}.$$

It is helpful to introduce a commonly used matrix representation of such transformations, which we define below.

Definition 2.2 (Möbius transformations)**.** Let $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. The function

$$f(z) = \frac{az + b}{cz + d}$$

is called a Möbius transformation, and can be represented by its matrix form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

When Möbius transformations are composed, it is easy to show that the matrix forms multiply.

Moreover, we would like the matrix representation to have a determinant equal to $B_2(z)$, so we write

$$f(z) = \frac{1}{\sqrt{1-|w_1|^2}} \begin{pmatrix} B_2(z) & w_1 \\ \overline{w_1} B_2(z) & 1 \end{pmatrix} g(z).$$

Return to Problem 0.1. This process could be repeated to find the family of all possible interpolations of a set of n points once we can already characterize the family of all possible interpolations of a subset of $n-1$ points, though we will use the shifted $w_n^{(n-1)}$ to account for the updated output value at z_n on the nth step. Continuing this process gives us a general solution of

$$f(z) = \left(\frac{1}{\sqrt{1-|w_1|^2}} \begin{pmatrix} B(z_1,a) & w_1 \\ \overline{w_1} B(z_1,a) & 1 \end{pmatrix} \right) \left(\frac{1}{\sqrt{1-\left|w_2^{(1)}\right|^2}} \begin{pmatrix} B(z_2,a) & w_2^{(1)} \\ \overline{w_2^{(1)}} B(z_2,a) & 1 \end{pmatrix} \right)$$

$$\cdots \left(\frac{1}{\sqrt{1-\left|w_n^{(n-1)}\right|^2}} \begin{pmatrix} B(z_n,a) & w_n^{(n-1)} \\ \overline{w_n^{(n-1)}} B(z_n,a) & 1 \end{pmatrix} \right) g(z).$$

Denote M_i to be the ith matrix in the expression above. We can thus write

$$f = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} g,$$

where the entries A_n, B_n, C_n, D_n of the matrix are rational functions known as Nevanlinna coefficients. Moreover, we have

$$\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} = \prod_{i=1}^{n} M_i,$$

which leads to several interesting properties of Nevanlinna coefficients. For example, notice that all poles of the Nevanlinna coefficients must be reciprocals of the conjugates of the input data. The ideas used in the Nevanlinna approach can also be used to tackle the Carathéodory Problem, providing a construction for a sequence of finite Blaschke products converging pointwise to any holomorphic function $f : \mathbb{D} \to \overline{\mathbb{D}}$.

References

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