Fourier Theory

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Introduction to Fourier Series: Foundations and Applications

Representing functions as sums of simpler functions is extremely useful for both reducing complexity and providing accurate methods for approximation. Linear factors/power series are understandably an attractive choice for this given their simplicity. However, typically with the case of Taylor/Mclaurin series, each individual coefficient a_n does not tell us much about the whole function other than maybe some info about it's magnitude (if n is small). However, if we represent a function as a series of factors with different periods, each coefficient will tell us how much each period is expressed, informing us of a property that will exist across the whole function.

Thus, there ought to be a more efficient way to represent functions that possess some aspect of periodicity.

Preliminaries: Arriving at a Fourier series

Given two functions f, g we can define them as orthogonal if they satisfy

$$\int_{-\pi}^{\pi} f(x)g(x)dx = 0$$

Sine and cos are great candidates for this, so consider $\sin(nx), \cos(mx)$ with $n, m \in \mathbb{Z}^+$ and pair them 3 possible different ways:

1. $$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)dx = 0$$

   Since integrand is odd

2. $$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(x(n-m)) + \cos(x(n+m))dx$$
   $$= \frac{1}{2}\Big[\frac{\sin(x(n-m))}{n-m} + \frac{\sin(x(n+m))}{n+m}\Big]_{-\pi}^{\pi}$$

   The denominators introduce mildly interesting cases to consider. If no problems arise $\sin(\pi k) = 0, k \in \mathbb{Z}$ and the whole thing is 0. If both $m, n = 0$ using $\lim_{x\to 0}\sin(x) = x$ we get

   $$= \frac{1}{2}\Big[\frac{\sin(x(0))}{0} + \frac{\sin(x(0))}{0}\Big]_{-\pi}^{\pi} = \frac{1}{2}[2x]_{-\pi}^{\pi} = 2\pi$$

If non-zero and $m-n=0$ ($m+n \neq 0$ since $m,n \geq 0$), the fraction unaffected will be zero and with the other we similarly get

$$= \frac{1}{2}\Big[\frac{\sin(x(0))}{(0)} - 0\Big]_{-\pi}^{\pi} = \frac{1}{2}[x]_{-\pi}^{\pi} = \pi$$

Thus

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \begin{cases} 2\pi & m=n=0 \\ \pi & m=n\neq 0 \\ 0 & m\neq n \end{cases}$$

3.

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(x(n-m)) - \cos(x(n+m))dx$$
$$= \frac{1}{2}\Big[\frac{\sin(x(n-m))}{n-m} - \frac{\sin(x(n+m))}{n+m}\Big]_{-\pi}^{\pi}$$

Again we got cases. If no problems then $=0$ again. If $n=m=0$ this time we have

$$= \frac{1}{2}\Big[\frac{\sin(x(0))}{0} - \frac{\sin(x(0))}{0}\Big]_{-\pi}^{\pi} = 0$$

If $m=n\neq 0$, similar to above one fraction is zero and the other simplifies

$$= \frac{1}{2}\Big[\frac{\sin(x(0))}{0} - 0\Big]_{-\pi}^{\pi} = \pi$$

Thus

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = \begin{cases} \pi & m=n\neq 0 \\ 0 & else \end{cases}$$

We can now use these to find the Fourier coefficients to a 2π periodic function that can be represented by a sum of sines and cosines. Note the $n=0$ case has $\cos(0)=1$ and $\sin(0)=0$ hence the a_0 term out front

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

If we want b_k, times both sides by $\sin(kx)$ and integrate across a period $[-\pi,\pi]$.

$$\int_{-\pi}^{\pi} f(x)\sin(kx)dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\sin(kx) + \sum_{n=1}^{\infty} a_n\cos(nx)\sin(kx) + b_n\sin(nx)\sin(kx)\ dx$$

From above, every integral on the right goes to zero except the $n = k$ term

$$= \cdots + 0 + b_k \int_{-\pi}^{\pi} \sin(kx)\sin(kx)dx + 0 + \dots$$
$$= b_k \pi$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx$$

With the same working, for a_k times both sides by $\cos(kx)$ and integrate, using orthogonality again to yield

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx$$

Except the case where $k = 0$ where we ought to divide by 2π not π, which is why the original formula has the term $\frac{a_0}{2}$ instead of a_0 to accommodate this.

There is also a complex version of the Fourier series consisting of a sum of e^{ixn} terms. We could certainly derive this from what we already have by expanding with $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$ and $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$, but I'll show a more direct approach instead.

We have

$$f(z) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Consider the integral of e^{inx} across one period 2π. First if $n = 0$

$$\int_0^{2\pi} e^{inz}dz = \int_0^{2\pi} 1dx = 2\pi$$

Otherwise

$$\int_0^{2\pi} e^{inz}dz = \left[\frac{e^{inz}}{in}\right]_0^{2\pi} = \frac{e^0 - e^{2\pi in}}{in} = \frac{1-1}{in} = 0$$

So to find a c_k in our complex Fourier series, times both sides by e^{-ikx} then integrate across one period 2π

$$\int_0^{2\pi} f(x)e^{-ikx}dx = \int_0^{2\pi} \sum_{n=-\infty}^{\infty} c_n e^{ix(n-k)}dx$$

Split the integral and take the constants out

$$= \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{ix(n-k)}dx$$

From above, the integrals on the right where $n - k \neq 0$ become zero, leaving only the $n = k$ term

$$= c_k \int_0^{2\pi} e^{k-k} dx$$

$$= 2\pi c_k$$

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx$$

Convergence of Fourier Series

There are several conditions necessary for a Fourier series to exist which, for the most part, are intuitively reasonable things to ask for from a function.

If a function satisfies the Dirichlet conditions then it's Fourier series converges. These conditions are that the function must have:

1. A finite number of maxima/minima in a single period
2. A finite number of points of discontinuity in a single period
3. It must be absolutely integrable across one period, i.e

$$\int_P |f(x)| dx < \infty$$

Conveniently, since we are describing periodic functions, we need only be sure any conditions hold across the span of one period. Further, this lends another advantage over power series as convergence need only be assessed for a period to inform us of the whole function, as opposed to a radius (Taylor series) or annulus (Mclaurin series) of convergence.

The other obvious benefit is the ability to represent discontinuous functions since we only need continuity on each piece as opposed to the whole function (though there still must be a finite number of pieces per period). The Fourier series of a function f converges at a point a to

$$\frac{\lim_{x \to a^-} f(x) + \lim_{x \to a^+} f(x)}{2}$$

Evidently if a is not a point of discontinuity $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ and it is simply $f(a)$. Otherwise, at points of discontinuity, it converges to the average of f at either end of the discontinuity.

If we want to strengthen our conclusion from convergence to absolute convergence, we additionally need to satisfy

$$\sum_{n=1}^{\infty} |a_n \cos(nx) + b_n \sin(nx)| < \infty$$

By triangle inequality on the left we see

$$\leq \sum_{n=1}^{\infty} |a_n \cos(nx)| + |b_n \sin(nx)| \leq \sum_{n=1}^{\infty} |a_n| + |b_n|$$

So it is sufficient to show that

$$\sum_{n=1}^{\infty} |a_n| + |b_n| < \infty$$

for absolute convergence.

This also satisfies the Weierstrass M-test, which states that if a sequence of functions $g_n(x)$ are all each bounded by some value $M_n \geq |g_n(x)|$ and that the sum of these bounds converges $\sum_{n=1}^{\infty} M_n < \infty$, then the sum of the sequence $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly.

Proof. Here let $g_n(x) = a_n \cos(nx) + b_n \sin(nx)$, from triangle inequality above these are each bounded by $M_n = |a_n| + |b_n|$, and we also know

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} |a_n| + |b_n| < \infty$$

So it satisfies both conditions of the M-test, $f(x) = \sum_{n=1}^{\infty} g_n(x)$ converges uniformly.

Thus, absolute convergence of a Fourier series implies it is uniformly convergent (the converse is not necessarily true though). □

Fourier transform and inverse

Until now we have defined the series for 2π periodic functions, though for this following explanation it's beneficial to clarify that these are functions whose smallest (fundamental) period is at most 2π (e.g. $4\pi, 6\pi, \ldots$ are also periods). The understandable follow up is then to generalise to other fundamental periods.

This can be achieved simply by subbing $x = \frac{\pi x}{L}$ scaling x by L/π, shifting the interval $[-\pi, \pi]$ to $[-L, L]$,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})$$

And the coefficients via a sub $x \to \frac{x\pi}{L}$ become

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{\frac{-in\pi x}{L}} dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{n\pi x}{L}) dx, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{n\pi x}{L}) dx$$

Where $f(x)$ is now a function with a base period of $2L$. All previous theorems still hold with these new expressions.

But what if periodic functions aren't good enough, then you can just consider an aperiodic function as one whose fundamental period is infinity, i.e. it never repeats. Thus, taking limit as $L \to \infty$ generalises to non-periodic functions.

In its current form we are ill-prepared to deal with $L \to \infty$. So consider re-writing it with respect to $K_n = \frac{n\pi}{L}$, which produces values at regular intervals call them $\Delta K = \frac{\pi}{L}$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2L} \int_{-L}^{L} f(x) e^{-ixK} dx \ e^{ixK}$$

Since $\Delta K / \pi = 1/L$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-L}^{L} f(x) e^{-ixK} dx \ e^{iKx} \Delta K$$

Now interpret what is actually being done with respect to K here. You have some ΔK interval multiplied with some function at inputs K spaced at regular intervals of ΔK, summed from negative infinity to positive infinity. This is equivalent to taking a grainy approximation of the functions integral across $-\infty \to \infty$. It is practically a Riemann sum, we simply need to take $L \to \infty$ to make $\Delta K \to 0$ and our approximation is perfected. Use ξ for the integral.

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \lim_{L \to \infty} (\int_{-L}^{L} f(x) e^{-ix\xi} dx) \ e^{i\xi x} d\xi$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx \ e^{i\xi x} d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\xi) e^{i\xi x} d\xi$$

Where $F(\xi) = \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx$ is the Fourier transform. Since it is derived from the periodic term e^{inx} coefficients, which previously lent insight for periodic functions, it can be thought of as their continuation to a continuous spectrum of periods. In other words, it is a function which reports the amplitude (aka expression) from a continuous spectrum of period/frequencies that make up the original function. Hence it is considered to shift functions from their normal 'space domain' into a 'frequency domain' where it is easily identifiable the periods/frequencies that make up a certain function as well as how much each frequency contributes to the whole.

What we've also shown here is that if we compose a function's Fourier transform with a second transform we return to $f(x)$. This outer transform $F^{-1}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-\xi x} dx$ is called the Inverse Fourier transform (for obvious reason) and can be used to retrieve the original function or to compile an array of frequencies into their resulting function. $F(z)$ and $F^{-1}(z)$ have several similar-looking representations depending on the use case. An interesting family of functions with respect to Fourier transforms are the Gaussian functions $g(x) = e^{-ax^2 - bx - c}, a > 0$ as $F(g(x))$ always returns another Gaussian function. In the special case of $e^{-\pi z^2}$, it is its own Fourier transform.

Poisson summation formula

Poisson's Summation formula states that the sum of a function at regular intervals is equal to the sum of its Fourier transform across the same intervals.

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} F(n)$$

Proof. To prove this take the convention where $F(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx$ and consider the RHS,

$$RHS = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)e^{-2\pi i x n}dx$$

The sum is with respect to n only, so bring it inside

$$= \int_{-\infty}^{\infty} f(x) \sum_{n=-\infty}^{\infty} e^{-2\pi i x n}dx$$

Here we can make use of a convenient result from the Dirac Delta function, that

$$\sum_{n=\infty}^{\infty} e^{inx} = 2\pi \sum_{k=-\infty}^{\infty} \delta(x - 2\pi k)$$

with it we have

$$RHS = \int_{-\infty}^{\infty} f(x)2\pi \sum_{n=-\infty}^{\infty} \delta(-2\pi x - 2\pi k)dx$$

But first a detour, what is the Delta function? Let's first define its precursor, δ_ϵ for $\epsilon > 0$

$$\delta_\epsilon(x) = \begin{cases} 0, & x > \epsilon/2 \\ \frac{1}{\epsilon}, & -\epsilon/2 \leq x \leq \epsilon/2 \\ 0, & x < -\epsilon/2 \end{cases}$$

Evidently this is just a flat line segment of height $1/\epsilon$ and width ϵ centred at the origin, so its area is $\epsilon/\epsilon = 1$. As we take $\epsilon \to \infty$ we arrive at our proper δ function, defined by

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

with

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

This seemingly plain function holds some useful properties, the first of which being that if we have a function f, we know

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

Proof. For this consider its precursor δ_ϵ, since we know what inputs yield zero we can narrow our integral significantly

$$\int_{-\infty}^{\infty} f(x)\delta_\epsilon(x-a)dx = \int_{a-\epsilon/2}^{a+\epsilon/2} f(x)\delta_\epsilon(x-a)dx$$

We also know what value it takes on here

$$= \frac{1}{\epsilon}\int_{a-\epsilon/2}^{a+\epsilon/2} f(x)dx$$

By the mean value theorem for integrals, we know that for any $\int_a^b g(x)dx$ we can find a $g(c)$, where $a \leq c \leq b$, such that the rectangle $g(c)(b-a)$ is equal to the integral.
Using this we have for some $a-\epsilon/2 \leq c \leq a+\epsilon/2$ that

$$(a+\epsilon/2-(a-\epsilon/2))f(c) = \int_{a-\epsilon/2}^{a+\epsilon/2} f(x)dx$$

$$f(c) = \frac{1}{\epsilon}\int_{a-\epsilon/2}^{a+\epsilon/2} f(x)dx$$

$$f(c) = \int_{-\infty}^{\infty} f(x)\delta_\epsilon(x-a)dx$$

As we take $\epsilon \to \infty$ though, by squeeze theorem $a-\epsilon/2 \leq c \leq a+\epsilon/2$ resolves to $c = a$,

$$f(a) = \int_{-\infty}^{\infty} f(x)\delta(x-a)dx$$

□

Another useful result of the δ function is that

Proof.

$$|k|\delta(kx) = \delta(x)$$

For this recall our two requirements to define δ, the first is satisfied easily:

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Since dilating horizontally and vertically change nothing in this respect, we simply get

$$|k|\delta(kx) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

and for the second requirement that $\int_{-\infty}^{\infty} \delta(x)dx = 1$, we have

$$\int_{-\infty}^{\infty} |k|\delta(kx)dx$$

Sub $kx = u$

$$= \frac{|k|}{k} \int_{-\infty \times k}^{\infty \times k} \delta(u)du$$

If $k > 0$,

$$= \frac{k}{k} \int_{-\infty}^{\infty} \delta(x)dx = 1$$

and if $k < 0$

$$= \frac{-k}{k} \int_{\infty}^{-\infty} \delta(x)dx = \frac{k}{k} \int_{-\infty}^{\infty} \delta(x)dx = 1$$

As required, both conditions are satisfied. □

Ok, now back to what we were doing, we have

$$RHS = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)2\pi \ \delta(-2\pi x - 2\pi k)dx$$

since we now know $|k|\delta(kx) = \delta(x)$, with $k = -2\pi$

$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\delta(x + k)dx$$

and since we also know $\int_{-\infty}^{\infty} f(x)\delta(x - \xi)dx = f(\xi)$, we get

$$= \sum_{k=-\infty}^{\infty} f(-k)$$

Flip $k \to -k$

$$\sum_{k=-\infty}^{\infty} f(k) = LHS$$

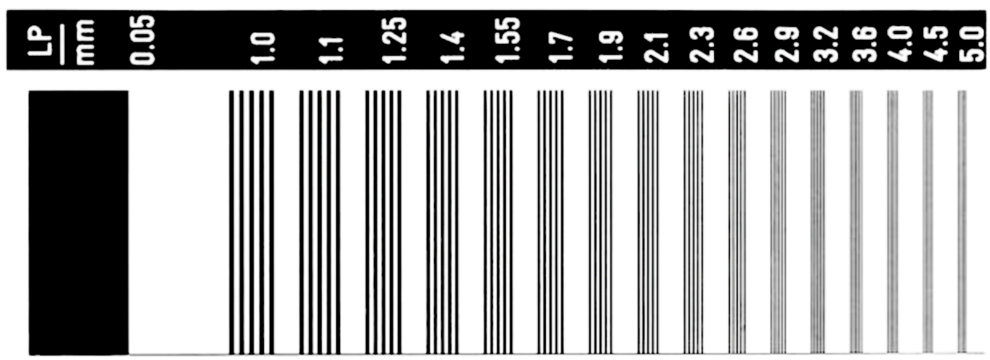
As required. □

This can lead to several other interesting proofs beyond the scope of this paper, such as the Nyquist-Shannon sampling theorem which states that a band-limited signal (one whose Fourier transform is zero beyond a certain frequency) can be perfectly reconstructed from its discrete samples if the sampling rate is sufficiently high (greater than twice the highest frequency component of the function).

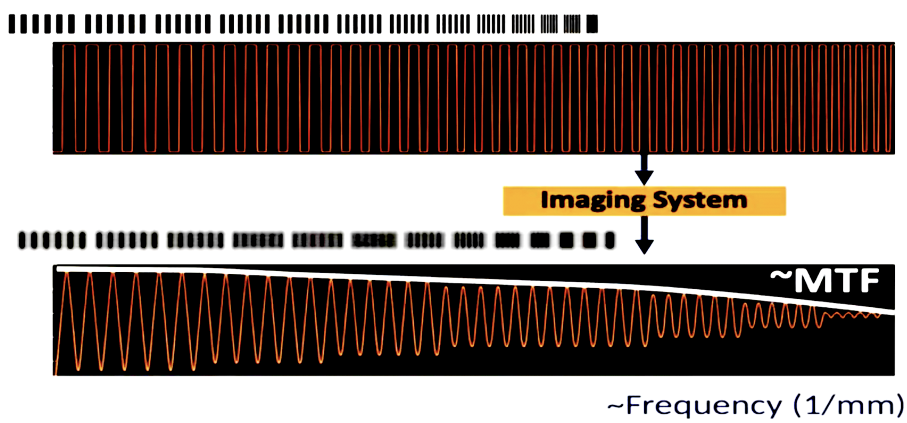
Applications of Fourier Theory: Assessing Resolution

How do we assess the resolution of an imaging system, i.e. how 'clearly' it can capture small features of the real world? While this may seem like a mildly interesting endeavor reserved for cinephiles, in higher-stakes scenarios where precision is key, such as medical x-ray imaging or military drone systems, the rest of us are also quickly motivated to find a precise solution.

A reasonably intuitive approach to measure resolution would be to quantify the degree of blur that occurs between nearby objects, as well as how close we can bring two objects before their edges are indistinguishable. A test for this is below, where line pairs (LP) are placed with higher spatial frequency towards the right.

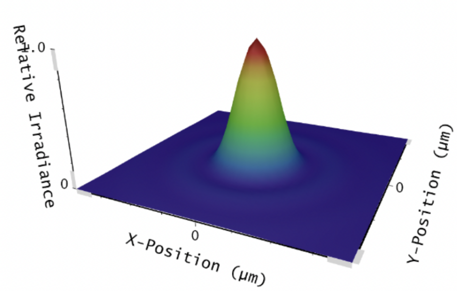


The qualitative method here is simple enough, ask people what the greatest spatial frequency is where they can still distinguish adjacent lines. This is obviously subjective, so instead we could set absolute white to 0, absolute black to 1, and plot the intensity of white as we move across. Performing this with the perfect signal as well as our image let's us compare the blur imposed by the system.

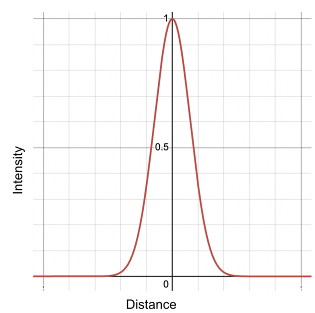


Here each peak captures a 'signal' and each quarter-period between a peak and a trough captures an 'edge response'. The amplitude of the curve across spatial frequencies captures this intuition of quantifying blur as signals get closer together, progressing towards indistinguishable grey with high frequency.

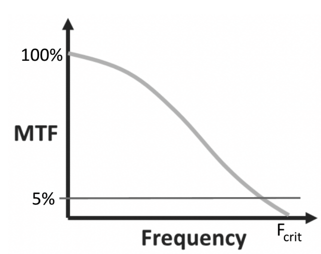
Let's formalise for continuity so as not to rely on sampling discrete spatial frequencies. Take a theoretically perfect signal that exists only at a point and run it through the imaging system in question, this will result in some degree of blur around the 'true' signal in the resulting image. Plot the observed intensity at each point in space:



This is the Point Spread Function (PSF), which describes the system's ability to capture a point source. From here, take a vertical slice that runs through the center, and plot the intensity as you move across the slice. Now take the radial average for all such possible slices, or in other words, for each radius from the center take an average of the intensity values as you rotate about the center. The resulting graph is the Line Spread Function (LSF)



Taking the Fourier Transform of this converts from the space domain, with distance against intensity, to the frequency domain, with signals per distance as the input. This is our Modulation Transfer Function (MTF), which describes how clearly each spatial frequency is captured by the imaging system and the decrease in clarity as image complexity increases.



The cutoff frequency for the imaging system is the frequency beyond which signals are no longer captured, though for human eyes we can't distinguish much past about 4-5%. In practical application, the only real difference in method to obtain the MTF is just to use a thin enough wire to approximate a point signal.

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