MODULAR FORMS AND THE KAC-WAKIMOTO CONJECTURE

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ABSTRACT. In this paper, we review the many intriguing properties of modular forms, functions that have transformation conditions and are holomorphic on the upper half plane and at ∞. We study its application to the Kac-Wakimoto Conjecture on the number of representations of any positive integer as a sum of triangular numbers.

1. INTRODUCTION

Modular forms are fascinating functions that have specific transformation properties, which we shall see shortly, and are holomorphic on the upper half plane (consisting of all points with imaginary part greater than 0) and at ∞. They are very important tools with applications in many fields including number theory and topology, and they are utilized for problems like sphere packing and finding the number of representations of a number as a sum of squares. Here, after examining the fundamental properties of modular forms, we cover a different intriguing application of modular forms to the Kac-Wakimoto Conjecture. A very interesting result, it provides a formula for the number of ways of writing any positive integer n as the sum of m triangular numbers - those of the form $a(a-1)/2$ for any positive integer a - given a specific condition on m, where m is a positive integer. This was proved by Zagier in 2000 (see [1]). The proof of this conjecture relies heavily on modular forms, and we will review it in this paper.

Theorem 1.1 (Kac-Wakimoto Conjecture)**.** *Let $\Delta_m(n)$ be the number of ways to write a positive integer n as the sum of m triangular numbers, for some positive integer m. Then*

$$\Delta_{4s^2}(n) = \sum_{\substack{r_1, a_1, \ldots, r_s, a_s \in \mathbb{N}_{odd} \\ r_1 a_1 + \cdots + r_s a_s = 2n + s^2}} P_s(a_1, \ldots, a_s),$$

where s is a positive integer, $\mathbb{N}_{odd}$ is the set of positive odd integers, and P_s is a polynomial given by

$$P_s(a_1, \ldots a_s) = \frac{\prod_i a_i \cdot \prod_{i<j} (a_i^2 - a_j^2)^2}{4^{s(s-1)} s! \prod_{j=1}^{2s-1} j!}.$$

We begin with some definitions relating to modular forms, and we follow [2] and [3].

2. DEFINITIONS

Definition 2.1. The *modular group* is the group of all 2 x 2 matrices with integer entries and determinant 1, and defined as

$$SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

The *upper half plane* $\mathbb{H}$ is the set of all complex numbers that have imaginary part greater than 0.

Definition 2.2. A *möbius transformation* γ of $SL_2(\mathbb{Z})$ on the upper half plane $\mathbb{H}$ is

$$\gamma(z) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}.$$

For example, if $\gamma = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}$, then $\gamma(3) = \frac{4\cdot 3+3}{5\cdot 3+4} = \frac{15}{19}$. It follows that the imaginary part of $\gamma(z)$ satisfies the relation

$$\Im(\gamma(z)) = \frac{\Im(z)}{|cz+d|^2}.$$

We can check that this holds by letting $z = r + si$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to obtain

$$\begin{aligned}
\gamma(z) &= \frac{az+b}{cz+d} \\
&= \frac{a(r+si)+b}{c(r+si)+d} \\
&= \frac{(ar+b)+asi}{(cr+d)+csi} \\
&= \frac{((ar+b)+asi)((cr+d)-csi)}{|cz+d|^2} \\
&= \frac{(ar+b)(cr+d)+acs^2+(ad-bc)si}{|cz+d|^2}.
\end{aligned}$$

This implies that

$$\Im(\gamma(z)) = \Im\left(\frac{(ar+b)(cr+d)+acs^2+(ad-bc)si}{|cz+d|^2}\right) = \frac{\Im(z)}{|cz+d|^2},$$

since $ad-bc = 1$. We also define the following transformation property, one of the conditions for a function to be a modular form.

Definition 2.3. Let $f : \mathbb{H} \to \mathbb{C}$ be a meromorphic function and k be an integer. If $f(\gamma(z)) = (cz+d)^k f(z)$, for any matrix γ in the modular group $SL_2(\mathbb{Z})$ and point z in the upper half plane $\mathbb{H}$, then f is considered *weakly modular of weight k*.

Definition 2.4. A *modular form* is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ that is weakly modular and holomorphic on the upper half plane $\mathbb{H}$ and at ∞.

If the modular form is weakly modular of weight k, then it is called a *modular form of weight k*, and the set of all such modular forms is defined to be $M_k(SL_2(\mathbb{Z}))$.

3. Properties of Modular Forms

With these definitions, we can now look at some properties of modular forms. Note that any open subset $\mathcal{F}$ of the upper half plane is a fundamental domain of some group Γ if it satisfies the following: any two points that are distinct are not equivalent under Γ, and for any point in $\mathcal{F}$, there is a point in $\overline{\mathcal{F}}$, the closure of $\mathcal{F}$, that is equivalent to it under Γ. We can now find this for the modular group.

Proposition 3.1. *The fundamental domain of the modular group $SL_2(\mathbb{Z})$ is*

$$\mathcal{F} = \{z \in \mathbb{H} : |z| > 1, |\Re(z)| < 1/2\}.$$

To prove this proposition, we first need to show that every point in the upper half plane can be mapped by some möbius transformation γ in $SL_2(\mathbb{Z})$ to a point in $\overline{\mathcal{F}}$. Secondly, we need to show that for any two distinct points, they are not equivalent under $SL_2(\mathbb{Z})$.

Proof. Let z be a point in the upper half plane. Consider the lattice $L = \{mz+n : m, n \in \mathbb{Z}\}$. Call the point of minimal modulus on L (different from the origin) as $cz + d$, where c and d are relatively prime. Then there exists some a and b such that $ad - bc = 1$, which implies we can define

$$\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

as an element of $SL_2(\mathbb{Z})$. As we described in the previous section,

$$\Im(\gamma(z)) = \frac{\Im(z)}{|cz + d|^2},$$

for any γ in $SL_2(\mathbb{Z})$. Since $cz + d$ is a point of minimal modulus, $\Im(\gamma(z))$ is maximal when $\gamma = \gamma_1$. Define

$$z' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n (\gamma_1(z)) = \gamma_1(z) + n,$$

where $|\Re(z')| \leq \frac{1}{2}$. Then if $|z'| < 1$, this implies that $\Im\left(\frac{-1}{z'}\right) = \frac{\Im(z')}{|z'|^2} > \Im(z)$, which contradicts our earlier statement that $\Im(\gamma_1(z))$ is a maximal which implies that $\Im(z)$ is maximal. Therefore $|z'| \geq 1$, so $z' \in \overline{\mathcal{F}}$ and z is equivalent to z' under $SL_2(\mathbb{Z})$. Next, assume for the sake of contradiction, that there are two distinct points w and $w' = \gamma_1(w)$ in $mathcalF$ where $\gamma_1 \neq \pm 1$. Then setting

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n$$

implies that $w' = \gamma_1(w) = w+n$. This is a contradiction since we stated that $|\Re(z_1)|, |\Re(z_2)| < \frac{1}{2}$. Therefore, $c \neq 0$ for $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Notice that the smallest value for $\Im(z)$ for all $z \in \mathcal{F}$ is $\sqrt{3}/2$, when $\Re(z) = 1\backslash 2$. Thus we can write

$$\frac{\sqrt{3}}{2} \leq \Im(z_2) = \frac{\Im(z_1)}{|cz + d|^2} \leq \frac{\Im(z_1)}{c^2 \Im(z_1)^2} < \frac{2}{c^2\sqrt{3}}.$$

This holds when $c = \pm 1$. If $\Im(z_1) \leq \Im(z_2)$, and $|\pm z_1 + d| \geq |z_1| > 1$, which contradicts the following property: $\Im(\gamma(z)) = \Im(z)\backslash(|cz + d|^2)$. Hence, $\mathcal{F}$ is a fundamental domain. □

The fundamental domain of $SL_2(\mathbb{Z})$ is the gray region in Figure 1, which shows the tiling of fundamental domains of this modular group on the upper half plane.

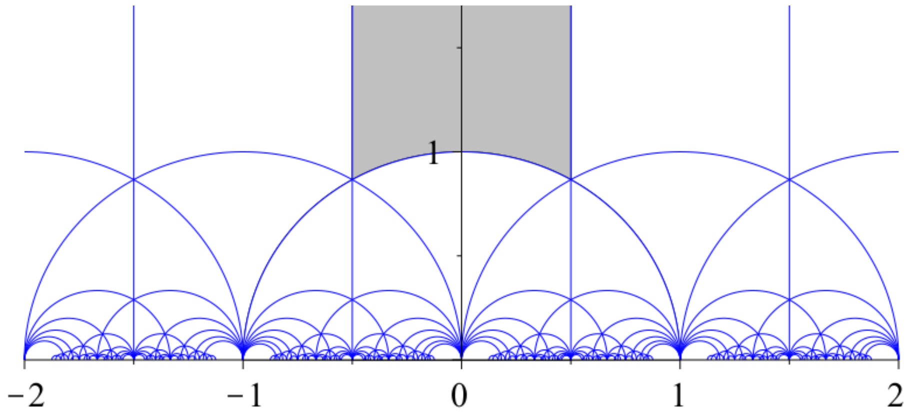

FIGURE 1. Fundamental domain tiling of the upper half plane. (Credit: A. Hulpke, Wikipedia; see [4].)

For any function f, the *order of vanishing* of f at any point z in $SL_2(\mathbb{Z})$, given by $\mathrm{ord}_z(f)$, is the order of the poles of f at z. On the other hand, the order of vanishing of f at ∞, given by $\mathrm{ord}_\infty(f)$, is defined in terms of the Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n q^n,$$

for some coefficients a_i and where $q = e^{2\pi iz}$ for any z in the upper half plane. Specifically, $\mathrm{ord}_\infty(f)$ is the smallest integer n such that the Fourier coefficients are non-zero. 5 Modular forms are categorized into two types: *Eisenstein series* and *cusp forms*. Eisenstein series with weight greater than 2 are holomorphic and are given by

$$G_k(z) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k},$$

for any z in the upper half plane. Cusp forms are modular forms that vanish at the point ∞ and have a non-zero leading coefficient in their Fourier expansion. They do, however, converge to a non-zero holomorphic function in $\mathbb{H}$. An interesting example of a cusp form is the *discriminant function* $\Delta(z)$, given by

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24},$$

which has weight 12 on $SL_2(\mathbb{Z})$.

Proposition 3.2. *If f is a non-zero modular form of weight k on $SL_2(\mathbb{Z})$ for some integer k, then*

$$\sum_{P\in SL_2(\mathbb{Z})\backslash\mathbb{H}} \frac{1}{n_P} ord_P(f) + ord_\infty(f) = \frac{k}{12}.$$

Corollary 3.3. *If either $k < 0$ or k is odd, then the dimension of $M_k(SL_2(\mathbb{Z}))$ is 0, and if $k \geq 0$ and k is even, the dimension of $M_k(SL_2(\mathbb{Z}))$ is bounded above by $(k/12)+1$ for $k \not\equiv 2 \pmod{12}$, and bounded above by $(k/12)$ for $k \equiv 2 \pmod{12}$.*

Define $SL_2(\mathbb{R})$ be the group of 2x2 matrices with real entries and determinant 1. We now state a very important result that describes the relation between modular forms in $SL_2(\mathbb{R})$.

Proposition 3.4. *For any discrete subgroup Γ of $SL_2(\mathbb{R})$ such that $\Gamma \backslash \mathbb{H}$ have finite volume V, the dimension of $M_k(\Gamma)$ is bounded above by $(kV/4\pi) + 1$ for all $k \in \mathbb{Z}$.*

4. Kac-Wakimoto Conjecture

We can now use the properties of modular forms and the Fourier expansion for the proof of the Kac-Wakimoto Conjecture, which we stated in Theorem 1.1. The idea of the proof is as follows. Let $F(z)$ be defined as a linear combination of $g_{h_1}(z) \cdots g_{h_s}(z)$ for each choice of $h_1, \dots, h_s \geq 1$ such that $h_1 + \cdots + h_s = s^2$, where $g_{h_i}(z)$ is a modular form of weight $2h_i$ under $\Gamma_0(4)$ (the group of matrices γ where $4|c$) given by $g_{h_i}(z) = \sum_{r,a \in \mathbb{N}_{\text{odd}}} a^{2h_i - 1} q^{ra}$. Then it turns out that

$$\sum_{\substack{r_1, a_1, \dots, r_s, a_s \in \mathbb{N}_{odd} \\ r_1 a_1 + \cdots + r_s a_s = 2n + s^2}} P_s(a_1, \dots, a_s)$$

which is a polynomial of degree $2s^2 - s$ is the coefficient of the q^{2n+s^2} term in $F(z)$. This implies that $F(z)$ must also be a modular form, but of weight $2s^2$ under $\Gamma_0(4)$. The Fourier expansion of $F(z)$ is of the form $q^{s^2}\mathbb{Q}[[q^2]]$, and by applying Proposition 3.2, we will find that the only functions in $M_{2s^2}(\Gamma_0(4))$ are of the form $F(z)$ multiplied by a constant. The function $\theta_F(z)^{4s^2}$, where

$$\theta_F(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} q^{n^2} = 2q^{\frac{1}{4}} + 2q^{\frac{9}{4}} + 2q^{\frac{25}{4}} + \cdots,$$

also has a Fourier expansion of the form $q^{s^2}\mathbb{Q}[[q^2]]$, and it can be shown that $\theta_F(z)^{4s^2} = \Delta_{4s^2}(n)$. Since functions of the form $F(z)$ times a constant are the only ones in $M_{2s^2}(\Gamma_0(4))$, $\theta_F(z)^{4s^2}$ must be equal to $F(z)$ times some constant. If we plug in $n = 0$ we will find that this constant is one. Therefore,

$$\Delta_{4s^2}(n) = \sum_{\substack{r_1, a_1, \dots, r_s, a_s \in \mathbb{N}_{odd} \\ r_1 a_1 + \cdots + r_s a_s = 2n + s^2}} P_s(a_1, \dots, a_s),$$

as desired.

We refer the reader to [2] for an overview of the proof, and to [1] for the detailed version.

References

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