Fourier Theory

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1 The Fourier Transform

Exponentials are central to the study of calculus. Their properties with respect to integration and differentiation make them uniquely easy-to-study objects in differential equations, complex analysis, algebra, and more. The Fourier transform and Fourier series allow us to break down a function into an integral or sum of these exponentials, which we can then analyze individually.

Definition 1.1. The *Fourier transform* $F : \mathbb{R} \to \mathbb{C}$ of a function $f : \mathbb{R} \to \mathbb{C}$ is defined by the relation

$$F(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i \xi x} f(x) \, \mathrm{d}x \,.$$

Remark 1.2. *The Fourier transform has many analytic properties, but there is no convenient characterization of functions with a well-behaved Fourier transform. Thus, we will work with functions f satisfying the following conditions, which are sufficient to show the most impressive facts about the Fourier transform while still applying to most practical functions:*

1. *f is continuous.*
2. *$\int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x < \infty$. Such functions are called* absolutely integrable.
3. *The Fourier transform F of f is absolutely integrable.*

With this set of hypotheses, we can invert the Fourier transform.

Lemma 1.3 (Fourier inversion theorem)**.** *For a function f and its Fourier transform F obeying the hypotheses in Remark 1.2, we have*

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i \xi x} F(\xi) \, \mathrm{d}\xi \,.$$

Proof. Since f is absolutely integrable, $F(\xi)$ exists for all ξ and since F is absolutely integrable, the integral given in the statement of Lemma 1.3 also converges. Expanding the definition of F and distributing the exponential prefactor gives:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{2\pi i \xi x} F(\xi) \, \mathrm{d}\xi &= \int_{-\infty}^{\infty} e^{2\pi i \xi x} \int_{-\infty}^{\infty} e^{-2\pi i \xi y} f(y) \, \mathrm{d}y \, \mathrm{d}\xi \\ &= \iint_{\mathbb{R}^2} e^{2\pi i \xi (x-y)} f(y) \, \mathrm{d}y \, \mathrm{d}\xi \,. \end{aligned}$$

From this point, we would really like to exchange the two summations, but doing so would not guarantee convergence. Instead, we will include a decay factor in the integrand: $e^{-\varepsilon \xi^2}$. Taking the limit as $\varepsilon \to 0$ will then allow us to recover the original integral:

$$\begin{aligned}
\iint_{\mathbb{R}^2} e^{2\pi i\xi(x-y)} f(y)\,\mathrm{d}y\,\mathrm{d}\xi &= \lim_{\varepsilon\to 0} \iint_{\mathbb{R}^2} e^{2\pi i\xi(x-y)} \cdot e^{-\varepsilon\xi^2} f(y)\,\mathrm{d}y\,\mathrm{d}\xi \\
&= \lim_{\varepsilon\to 0} \iint_{\mathbb{R}^2} e^{2\pi i\xi(x-y)} \cdot e^{-\varepsilon\xi^2} f(y)\,\mathrm{d}y\,\mathrm{d}\xi \\
&= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} e^{2\pi i\xi(x-y)-\varepsilon\xi^2}\,\mathrm{d}\xi\,\mathrm{d}y \\
&= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} e^{-\varepsilon\left(\xi^2-2\left(\frac{\pi i(x-y)}{\varepsilon}\right)\xi\right)}\,\mathrm{d}\xi\,\mathrm{d}y \\
&= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} e^{-\varepsilon\left(\left(\xi-\frac{\pi i(x-y)}{\varepsilon}\right)^2+\left(\frac{\pi(x-y)}{\varepsilon}\right)^2\right)}\,\mathrm{d}\xi\,\mathrm{d}y \\
&= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} e^{-\frac{\pi^2(x-y)^2}{\varepsilon}} f(y) \int_{-\infty}^{\infty} e^{-\varepsilon\left(\xi-\frac{\pi i(x-y)}{\varepsilon}\right)^2}\,\mathrm{d}\xi\,\mathrm{d}y\,.
\end{aligned}$$

Now substituting,

$$n = \sqrt{\varepsilon}\left(\xi - \frac{\pi i(x-y)}{\varepsilon}\right),$$

we find

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} e^{-\frac{\pi^2(x-y)^2}{\varepsilon}} f(y) \int_{-\infty}^{\infty} e^{-\varepsilon\left(\xi-\frac{\pi i(x-y)}{\varepsilon}\right)^2}\,\mathrm{d}\xi\,\mathrm{d}y &= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} e^{-\frac{\pi^2(x-y)^2}{\varepsilon}} f(y) \int_{-\infty}^{\infty} e^{-n^2}\left(\frac{\mathrm{d}n}{\sqrt{\varepsilon}}\right)\mathrm{d}y \\
&= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{\varepsilon}} e^{-\frac{\pi^2(x-y)^2}{\varepsilon}} f(y)\,\mathrm{d}y\,.
\end{aligned}$$

The prefactor $\sqrt{\frac{\pi}{\varepsilon}} e^{-\frac{\pi^2(x-y)^2}{\varepsilon}}$ integrates to 1, and as $\epsilon \to 0$, it grows more and more sharply concentrated about $x = y$. Then, since f is continuous, the limiting value is simply

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{\varepsilon}} e^{-\frac{\pi^2(x-y)^2}{\varepsilon}} f(y)\,\mathrm{d}y = f(x)$$

as expected. □

The Fourier inversion theorem is really how we should intuitively think of the Fourier transform. F describes the amplitudes of the various complex exponentials which "integrate together" to form f. The Fourier inversion theorem is a more natural *definition* of F and the statement given in Definition 1.1 follows as a consequence, but we treat the Fourier inversion theorem as a result formally because Definition 1.1 allows us to define the Fourier transform even when the conditions of Remark 1.2 are not met.

2 Fourier Series

For functions defined over a finite domain $s : [a, b] \to \mathbb{C}$, we instead define a Fourier series consisting of countably many terms:

Definition 2.1. The *Fourier series* of a function $f : [a, b] \to \mathbb{C}$ with $f(a) = f(b)$ is an infinite sequence $\ldots, S_{-1}, S_0, S_1, \ldots \in \mathbb{C}$ is defined by the relation

$$S_n = \frac{1}{b-a} \int_a^b e^{-2\pi i n \frac{x-a}{b-a}} f(x)\,\mathrm{d}x\,.$$

Remark 2.2. *Analogous to the continuous case, it is easiest to analyze the Fourier series when the following conditions hold:*

1. *f is continuous.*

2. *The sum $\sum_{n=-\infty}^{\infty} S_n$ is absolutely convergent.*

We only need discrete Fourier coefficients to describe intervals because the Fourier series of an interval is analogous to the Fourier transform of that interval if it were repeated to fill the whole real line. The latter interpretation is not mathematically rigorous without generalizing the definition of functions to objects known as *distributions* (which are beyond the scope of this paper), but we can still analyze intervals with discrete Fourier series.

We can freely convert between absolutely convergent Fourier series and continuous functions. We will first require some helper lemmas.

Lemma 2.3. *If $f : [a, b] \to \mathbb{C}$ with $f(a) = f(b)$ is a continuous function whose Fourier transform is the infinite sequence S of zeroes, f is identically zero as well.*

Proof. For contradiction, let there be some $a < u < b$ such that $f(u) = v \neq 0$. Then, since f has a Fourier series which is identically 0, we must have

$$S_0 = \frac{1}{b-a} \int_a^b f(x) \, \mathrm{d}x \, .$$

In general, if we have some function

$$g(x) = \sum_{n=-\infty}^{\infty} e^{2\pi i n \frac{x-a}{b-a}} T_n,$$

with only finitely many nonzero T_n, we can then write the integral of $f(x)g(x)$ as a sum of Fourier coefficients which are all 0:

$$\begin{aligned}
\int_a^b f(x)g(x) \, \mathrm{d}x &= \int_a^b \sum_{n=-\infty}^{\infty} e^{2\pi i n \frac{x-a}{b-a}} T_n f(x) \, \mathrm{d}x \\
&= \sum_{n=-\infty}^{\infty} T_n \int_a^b e^{2\pi i n \frac{x-a}{b-a}} f(x) \, \mathrm{d}x \\
&= (b-a) \sum_{n=-\infty}^{\infty} T_n S_{-n} \\
&= 0.
\end{aligned}$$

Our strategy now will be to show that there is some trigonometric function $g(x)$ which is so sharply peaked around u that the integral of $f(x)g(x)$ must be nonzero. WLOG, let $v > 0$. Since f is continuous, we can choose some $\delta, \varepsilon > 0$ such that for x satisfying $|x - u| \leq \delta$ we have $f(x) > \varepsilon$. Let B be the maximum value of $|f(x)|$, which exists since f is continuous and defined over a compact interval. Let

$$g_1(x) = \cos\left(\frac{x-u}{b-a}\right) + \epsilon$$

where ϵ is chosen such that $f(x) < 1 - \epsilon/2$ for $|x - u| > \delta$. Choose some δ' such that $f(x) > 1 + \epsilon/2$ whenever $|x - u| < \delta'$. Let $g_k(x) = g_1^k(x)$. Each g can be expressed as a sum of trigonometric functions. However,

$$\begin{aligned}
\int_a^b f(x)g_k(x) \, \mathrm{d}x &\geq \int_{|x-y|<\delta} f(x)g_k(x) \, \mathrm{d}x - \int_{|x-y|>\delta} |f(x)g_k(x)| \, \mathrm{d}x \\
&\geq \int_{|x-y|<\delta'} (1+\epsilon/2)^k \varepsilon \, \mathrm{d}x - \int_{|x-y|>\delta} (1-\epsilon/2)^k B \, \mathrm{d}x \\
&\geq 2\delta'(1+\epsilon/2)^k \varepsilon - (b-a)(1-\epsilon/2)^k B.
\end{aligned}$$

For large enough k, the integral of $f(x)g_k(x)$ is necessarily positive, giving us a contradiction. □

We now show a simple version of the Fourier inversion formula for series.

Lemma 2.4. *For an infinite series S and an associated function*

$$f(x) = \sum_{n=-\infty}^{\infty} e^{2\pi i n \frac{x-a}{b-a}} S_n$$

satisfying the hypotheses of Remark 2.2, we can recover S_n via the formula

$$S_n = \frac{1}{b-a} \int_a^b e^{-2\pi i n \frac{x-a}{b-a}} f(x)\, \mathrm{d}x\,.$$

Proof. Expanding the definition:

$$\begin{aligned}
\frac{1}{b-a} \int_a^b e^{-2\pi i n \frac{x-a}{b-a}} f(x)\, \mathrm{d}x &= \frac{1}{b-a} \int_a^b e^{-2\pi i n \frac{x-a}{b-a}} \sum_{m=-\infty}^{\infty} e^{2\pi i m \frac{x-a}{b-a}} S_m \,\mathrm{d}x \\
&= \sum_{m=-\infty}^{\infty} S_m \frac{1}{b-a} \int_a^b e^{2\pi i (m-n) \frac{x-a}{b-a}} \,\mathrm{d}x \\
&= \sum_{m=-\infty}^{\infty} S_m \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \\
&= S_n.
\end{aligned}$$

□

In essence, Lemma 2.4 shows how to derive the Fourier transformation/Fourier series: integrating against an exponential with a suitable frequency extracts the desired frequency and kills all other frequencies. A similar result holds in the other direction.

Lemma 2.5. *For a function f and its Fourier transform S satisfying the hypotheses of Remark 2.2, we have*

$$f(x) = \sum_{n=-\infty}^{\infty} e^{2\pi i n \frac{x-a}{b-a}} S_n.$$

Proof. Let

$$g(x) = \sum_{n=-\infty}^{\infty} e^{2\pi i n \frac{x-a}{b-a}} S_n$$

so that we wish to show that $f(x) = g(x)$. By hypothesis, S is the Fourier series for f. By Lemma 2.4, S is also the Fourier series for g. Examining Definition 2.1, we see that the Fourier series is linear in the input, in that the Fourier series of $f - g$ is the Fourier series of f minus that of g. Therefore, the Fourier series of $f - g$ is identically 0. By Lemma 2.3, $f - g = 0$ and $f = g$. □

3 Differential Equations

The Fourier transformation and Fourier series are central to the study of differential equations. Since exponentials have clean differential properties, we can solve many forms of linear differential equations by writing solutions as the sum of exponentials.

Example 3.1. *Waves on a string of length L can be represented by the height $y(x,t)$ of the string at a position x and time t. Under certain idealized conditions, y obeys the wave equation:*

$$\ddot{y} = v^2 y'',$$

where $\ddot{y}$ is the second derivative of y with respect to time, y'' is its second derivative with respect to position, and v is a constant. Initially, we hold the string in place such that

$$y(x,0) = \begin{cases} x & 0 \le x \le L/4, \\ L/2 - x & L/4 \le x \le 3L/4 \\ x - L & 3L/4 \le x \le L. \end{cases}$$

At time $t=0$, we let go of the string but hold on to the ends (so that $y(0,t)=y(L,t)=0$ for all t). Find the height of the string at all times and positions.

Let's consider an analogous situation where the height starts as a complex exponential. Consider a function $h(x,t)$ obeying the same differential equation given in Example 3.1, where

$$h(x,0)=h_0 e^{2\pi i n \frac{x}{L}}.$$

Then,

$$\ddot{h}=v^2 h''=\left(\frac{2\pi i n v}{L}\right)^2 h=-\left(\frac{2\pi n v}{L}\right)^2 h$$

at $t=0$. Now, we notice that this expression for h'' is valid as long as h_0 does not depend on x. Therefore, we can include a multiplicative time-dependent term while still obeying

$$v^2 h''=-\left(\frac{2\pi n v}{L}\right)^2 h.$$

We then need to satisfy the equation

$$\ddot{h}=-\left(\frac{2\pi n v}{L}\right)^2 h.$$

We can do this by adding a time-dependent cos term:

$$h(x,t)=h_0 e^{2\pi i n \frac{x}{L}}\cos\left(\frac{2\pi n v}{L}t\right).$$

We used a cos term here instead of a complex exponential to account for the fact that the string starts at rest. With the cosine, we have $\dot{h}=0$ at $t=0$. Since this formula satisfies our initial conditions and the differential equation, it is the unique solution.

Now, we know how a single complex exponential evolves over time. If we can write the initial string position as a sum of various complex exponentials, we can determine how each constituent evolves and thus, what happens to the function as a whole. Of course, this decomposition is exactly what Fourier series do for us! The Fourier series of $y(x,0)$ is given by

$$\begin{aligned}
S_n &= \frac{1}{L}\int_0^L e^{-2\pi i n \frac{x}{L}} y(x,0)\,\mathrm{d}x \\
&= \frac{1}{L}\int_0^L e^{-2\pi i n \frac{x}{L}}\,\mathrm{d}x \begin{cases} x & 0\le x\le L/4, \\ L/2-x & L/4\le x\le 3L/4 \\ x-L & 3L/4\le x\le L \end{cases} \\
&= \frac{1}{L}\int_0^{L/4} x\left(e^{-2\pi i n \frac{x}{L}}+e^{-2\pi i n \frac{L/2-x}{L}}-e^{-2\pi i n \frac{x+L/2}{L}}-e^{-2\pi i n \frac{L-x}{L}}\right)\mathrm{d}x \\
&= \frac{1}{L}\int_0^{L/4} x\left(e^{-2\pi i n \frac{x}{L}}+(-1)^n e^{2\pi i n \frac{x}{L}}-(-1)^n e^{-2\pi i n \frac{x}{L}}-e^{2\pi i n \frac{x}{L}}\right)\mathrm{d}x \\
&= -i\frac{4}{L}\int_0^{L/4} x\sin\left(\frac{2\pi n x}{L}\right)\mathrm{d}x \begin{cases} 0 & 2|n \\ 1 & 2\nmid n \end{cases} \\
&= -i\frac{L}{4}\int_0^1 x\sin\left(\frac{\pi n x}{2}\right)\mathrm{d}x \begin{cases} 0 & 2|n \\ 1 & 2\nmid n \end{cases} \\
&= -i\frac{L}{4}\left(-\frac{x\cos(\pi n x/2)}{\pi n/2}+\frac{\sin(\pi n x/2)}{(\pi n/2)^2}\right)\Bigg|_0^1 \begin{cases} 0 & 2|n \\ 1 & 2\nmid n \end{cases} \\
&= \frac{L}{(\pi n)^2}\begin{cases} 0 & n\equiv 0,2 \pmod 4, \\ -i & n\equiv 1 \pmod 4, \\ i & n\equiv 3 \mod 4. \end{cases}
\end{aligned}$$

Define

$$\chi_4 = \begin{cases} 0 & n \equiv 0, 2 \pmod 4, \\ 1 & n \equiv 1 \pmod 4, \\ -1 & n \equiv 3 \pmod 4. \end{cases}$$

Then, we can rewrite $y(x, 0)$ as

$$\begin{aligned} y(x, 0) &= \sum_{n=-\infty}^{\infty} S_n e^{2\pi i n \frac{x}{L}} \\ &= \frac{L}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{\chi_4(n)}{n^2} \cdot \frac{e^{2\pi i n \frac{x}{L}}}{i} \\ &= \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{\chi_4(n)}{n^2} \sin\left(\frac{2\pi n x}{L}\right). \end{aligned}$$

Then, we see that

$$y(x, t) = \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{\chi_4(n)}{n^2} \sin\left(\frac{2\pi n x}{L}\right) \cos\left(\frac{2\pi n v}{L} t\right).$$

If we were really ambitious, we could now go and actually compute this trigonometric sum. Such a calculation is beyond the scope of this paper, but we can visualize the result using a computer.

4 Poisson Summation Formula

The Poisson Summation Formula allows us to use the Fourier transform to compute periodic sums of a function.

Theorem 4.1. *Poisson Summation Formula Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function and $F : \mathbb{R} \to \mathbb{C}$ be its Fourier transform such that both are absolutely integrable and such that $f^{(n)}$ decays at infinity for all n. Then,*

$$\sum_{x=-\infty}^{\infty} f(x) = \sum_{\xi=-\infty}^{\infty} F(\xi).$$

Proof. Define $g : [0, 1] \to \mathbb{C}$ as

$$g(x) = \sum_{n=-\infty}^{\infty} f(x + n).$$

A precise treatment of convergence is beyond the scope of this paper, but $g(x)$ converges to a continuous function and is absolutely integrable due to the conditions on the decay of $f^{(n)}$. Since g is continuous and periodic we can find its Fourier series S:

$$g(x) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x} S_n.$$

Notably,

$$g(0) = \sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} S_n.$$

Expanding the definition of S_n:

$$\begin{aligned} S_n &= \int_0^1 e^{-2\pi inx} g(x)\,\mathrm{d}x \\ &= \int_0^1 e^{-2\pi inx} \sum_{m=-\infty}^{\infty} f(x+m)\,\mathrm{d}x \\ &= \sum_{m=-\infty}^{\infty} \int_m^{m+1} e^{-2\pi inx} f(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} e^{-2\pi inx} f(x)\,\mathrm{d}x \\ &= F(n). \end{aligned}$$

Therefore

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} S_n = \sum_{n=-\infty}^{\infty} F(n)$$

as expected. □

Let's use this formula to solve the Basel problem!

Example 4.2. *Find the value of*

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We would really like to apply the Poisson summation formula directly to $f(x) = 1/x^2$, but we cannot because it is not continuous. Instead, we will apply it to

$$f_a(x) = \frac{1}{x^2 + a^2}.$$

Then, we can recover $\zeta(2)$ via the relation

$$2\zeta(2) = \lim_{a\to 0} \left(-\frac{1}{a^2} + \sum_{n=-\infty}^{\infty} f_a(n) \right).$$

As we might expect, this approach only allows us to calculate even integer values of the zeta function. Now, let $F : \mathbb{R} \to \mathbb{C}$ be the Fourier transform of f. For an integer value n:

$$F(n) = \int_{-\infty}^{\infty} \frac{e^{-2\pi inx}}{x^2 + a^2}\,\mathrm{d}x\,.$$

First, assume $n > 0$. Then, we take the red contour shown in Figure 1, where we take the path

$$-\infty \to -N \to -N - iM \to N - iM \to N \to \infty.$$

The segments from $-\infty \to -N$ and $N \to \infty$ are bounded by $O\left(\frac{1}{N}\right)$. The two vertical segments are bounded by $O\left(\frac{M}{N^2}\right)$ and the segment from $-N - iM$ to $N - iM$ is bounded by $O\left(\frac{N\exp(-2\pi nM)}{M^2}\right)$. Thus, choosing $M = \log N$ ensures that the total contribution is $O\left(\frac{1}{N}\right)$. Thus, the integral along this contour goes to 0 as $N \to \infty$.

If $n = 0$, take the blue contour shown in Figure 1, which goes $-\infty$ to $-N$ along the real axis, $-N$ to N along a semicircle, and N to ∞ along the real axis. All three segments of this contour are bounded by $O\left(\frac{1}{N}\right)$, so the integral as $N \to \infty$ goes to 0.

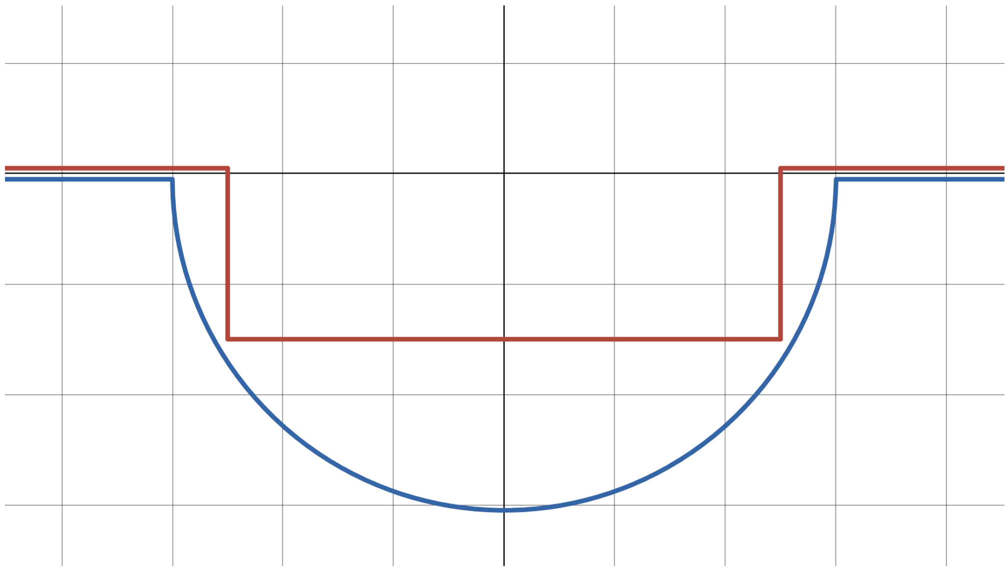

Figure 1: The contours used to Fourier transform $\frac{1}{x^2+a^2}$.

Both of these contours include the residue at $x = -ia$ whereas the contour along the real line excludes it. Therefore, we must account for the residue there, giving

$$F(n) = -2\pi i \frac{e^{-2\pi i n x}}{x - ia}\bigg|_{x=-ia} = \frac{\pi e^{-2\pi n a}}{a}$$

for nonnegative integer n. Note that negating n turns the integrand into its complex conjugate. Therefore, $F(-n) = \overline{F(n)}$. Since $F(n)$ is real, this relation shows that $F(-n) = F(n)$. Now,

$$\begin{aligned}
\zeta(2) &= \frac{1}{2}\lim_{a\to 0}\left(-\frac{1}{a^2} + \sum_{n=-\infty}^{\infty} F(n)\right) \\
&= \frac{1}{2}\lim_{a\to 0}\left(-\frac{1}{a^2} - \frac{\pi}{a} + 2\frac{\pi}{a}\sum_{n=0}^{\infty} e^{-2\pi n a}\right) \\
&= \frac{1}{2}\lim_{a\to 0}\left(-\frac{1}{a^2} + \frac{\pi}{a}\left(-1 + 2\frac{1}{1-e^{-2\pi a}}\right)\right) \\
&= \frac{1}{2}\lim_{a\to 0}\left(-\frac{1}{a^2} + \frac{\pi}{a}\cdot\frac{1+e^{-2\pi a}}{1-e^{-2\pi a}}\right) \\
&= \frac{1}{2}\lim_{a\to 0}\left(\frac{a\pi\left(1+e^{-2\pi a}\right) - \left(1-e^{-2\pi a}\right)}{a^2\left(1-e^{-2\pi a}\right)}\right) \\
&= \frac{1}{2}\lim_{a\to 0}\left(\frac{\pi\left(1+e^{-2\pi a}\right) - 2\pi^2 a e^{-2\pi a} - 2\pi e^{-2\pi a}}{2a\left(1-e^{-2\pi a}\right) + 2\pi a^2 e^{-2\pi a}}\right) \\
&= \frac{1}{2}\lim_{a\to 0}\left(\frac{2\pi^2 e^{-2\pi a} - 2\pi^2 e^{-2\pi a} + 4\pi^3 a e^{-2\pi a}}{2\left(1-e^{-2\pi a}\right) + 4\pi a e^{-2\pi a} + 4\pi a e^{-2\pi a} - 4\pi^2 a^2 e^{-2\pi a}}\right) \\
&= \frac{1}{2}\lim_{a\to 0}\left(\frac{4\pi^3 e^{-2\pi a}}{8\pi + 4\pi e^{-2\pi a}}\right) \\
&= \frac{1}{2}\cdot\frac{\pi^2}{3} \\
&= \frac{\pi^2}{6}.
\end{aligned}$$