L-FUNCTIONS

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Abstract. The goal of this paper is to cover the analytic continuation of L-functions and to prove the functional equation along with some interesting uses to other topics in math. The sources we have used are: [Apo98] [RS21].

1. Introduction

The Dirichlet L-function was originally made to help understand the distribution of prime numbers. However, it also has really interesting properties which are useful in other areas of math. More specifically, in this paper, we will talk about how L-functions relate to complex analysis and prove the analytic continuation and functional equation of $L(s,\chi)$. The Dirichlet L-function is defined as is defined as the sum of

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

It was originally introduced when Peter Gustav Lejeune Dirichlet used it to prove Dirichlet's theorem on arithmetic progressions, which states that there are an infinite number of primes in the form of $qk+r$ if $\gcd(k,r)=1$. It has since been used in many other proofs, such as the Prime number theorem for arithmetic progressions [Sel50], and, more recently, the Modularity theorem. This paper will prove the functional equation for the Dirichlet L-function which states that

$$L(1-s,x) = \frac{k^{s-1}\Gamma(s)}{(2\pi)^s}\{e^{-\pi is/2} + \chi(-1)e^{\pi is/2}\}G(1,\chi)L(s,\chi),$$

and give an analytic continuation for $L(s,x)$ using $\zeta(s,a)$.

2. Outline of Proof

- Prove the contour integral representation of $C(s,a)$ to give an analytic continuation of $\zeta(s,a)$.
- Use the formula that associates $\zeta(s,a)$ with $L(s,\chi)$ to prove the analytic continuation of $L(s,\chi)$.
- Define and Prove Hurwitz Formula.
- Use Hurwitz Formula to prove the functional equation for $\zeta(s,a)$.

Date: April 15, 2021.

- Get the functional equation for $L(s, x)$.

3. Definitions

Definition 1. The Dirichlet character $\chi(s)$ is any function from $\mathbb{Z} \implies \mathbb{C}$ that satisfies the following properties:

- There exists a k such that $\chi(n) = \chi(n+k)$.
- If $\gcd(n, k) > 1$ then $\chi(n) = 0$.
- If $\chi(mn) = \chi(m)\chi(n)$.

The Dirichlet Character $\chi(s)$ will be used in a series for the definition for $L(s, \chi)$. In this paper, we will call k the mod for χ.

Definition 2. The Dirichlet L-function $L(s, \chi)$ is defined as

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

The Hurwitz Zeta function is a series that is closely related to L-functions.

Definition 3. The Hurwitz Zeta function $\zeta(s, a)$ is defined as

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$

The Riemann Zeta function is a special case of the Hurwitz Zeta function when $a = 0$.

Definition 4. The Riemann Zeta function is defined as

$$\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{n^s}$$

Now take $L(s, \chi)$ and let k be the mod for χ.

$$\begin{aligned} L(s, x) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \\ &= \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{\chi(qk+r)}{(qk+r)^s} \\ &= \frac{1}{k^s} \sum_{r=1}^{k} \chi(r) \sum_{k=0}^{\infty} \frac{1}{(q+\frac{r}{k})^s} \\ &= k^{-s} \sum_{r=1}^{k} \chi(r) \zeta\left(s, \frac{r}{k}\right). \end{aligned}$$

Note that this is true because $\chi(qk+r) = \chi(r)$ because k is the mod for χ. This fact will be important later on in the paper.

4. Analytic Continuation of $\zeta(s,a)$

Theorem 1. *The series $\zeta(s,a)$ converges absolutely when $\Re(s) > 1$. The series converges uniformly in every half-plane where $\Re(s) \geq 1+\varepsilon$, $\varepsilon > 0$, so $\zeta(s,a)$ is analytic.*

Proof. This is true because of the fact that

$$\sum_{n=1}^{\infty} |(n+a)^{-s}| = \sum_{n=1}^{\infty} (n+a)^{-\Re(s)} \leq \sum_{n=1}^{\infty} (n+a)^{-(1+\varepsilon)}$$

Since $\zeta(1+\varepsilon, a)$ converges, we can use the Weierstrass M-test to prove that the series converges uniformly. The Weierstrass M-test states that if the sum of the absolute value converges, the sum converges uniformly. And since this fact is true, the sum converges uniformly and absolutely. □

The theorem we just proved will be used to prove the integral representation of the Hurwitz zeta function.

Theorem 2. *For $\Re(s) > 1$,*

$$\Gamma(s)\zeta(s,a) = \int_0^{\infty} \frac{x^{s-1}e^{-ax}}{1-e^{-x}} dx.$$

Proof. First we will prove it for real $s > 1$, then use analytic continuation to extend it. Our strategy will be too convert the integral for $\Gamma(s)$ into a sum involving $\zeta(s,a)$. First, we take the integral representation for $\Gamma(s)$ and make the substitution $x = (n+a)t$

$$\Gamma(s) = \int_0^{\infty} e^{-x}x^{s-1}dx = (n+a)^s \int_0^{\infty} e^{-(n+a)t}t^{s-1}dt.$$

Or, in other words

$$\Gamma(s)(n+a)^{-s} = \int_0^{\infty} e^{-nt}e^{-at}t^{s-1}dt.$$

If we sum over $n = 0$ to ∞ we get

$$\zeta(s,a)\Gamma(s) = \sum_{n=0}^{\infty} \int_0^{\infty} e^{-nt}e^{-at}t^{s-1}dt.$$

Now if we switch the sum and the integral we get

$$\begin{aligned}
\zeta(s,a)\Gamma(s) &= \int_0^{\infty} \sum_{n=0}^{\infty} e^{-nt}e^{-at}t^{s-1}dt \\
&= \int_0^{\infty} t^{s-1}e^{-at} \sum_{n=0}^{\infty} e^{-nt}dt \\
&= \int_0^{\infty} \frac{t^{s-1}e^{-at}}{1-e^{-t}}dt.
\end{aligned}$$

Figure 1

This proves the integral representation for the Hurwitz Zeta Function. □

The integral representation will help prove the contour integral representation. The next theorem shows a contour integral representation for the Hurwitz Zeta function. We will integrate over figure 1, which we will call γ. In γ, the circle is C_2, which goes counter clockwise and the middle is at $(0,0)$, the bottom line is C_1, which goes toward the circle, and the above line is C_3, which goes away from the circle. The radius of the circle (C_2) is defined as $c \leq 2\pi$. In the proof, we will take the limit as $c \to 0$ and prove the theorem below. The length of bottom and top line is ∞.

Theorem 3. *If $0 \leq a \leq 1$, then*

$$I(s,a) = \frac{1}{2\pi i}\int_\gamma \frac{z^{s-1}e^{az}}{1-e^z}dz$$

and $I(s,a)$ is an entire function of s. We also have

$$\zeta(s,a) = \Gamma(1-s)I(s,a).$$

Remark. We can use this theorem to prove the analytic continuation of the Hurwitz Zeta function. We can also define $\zeta(s,a)$ as $\Gamma(1-s)I(s,a)$ for $\Re(s) \leq 1$.

Proof. In the integral, z^s is $r^s e^{-\pi i s}$ on C_1 and $r^s e^{i\pi s}$ on C_3. We consider a compact disk $|s| \leq M$ and prove that C_1 and C_2 converges uniformly on every disk. Since the integrand is an entire function of s this will prove that $I(s,a)$ is entire (use Morera's Theorem). For C_1, we have, for $r \geq 1$,

$$|z^{s-1}| = r^{\Re(s)-1}|e^{-\pi i(\Re(s)-1+it)}| = r^{\Re(s)-1}e^{\pi t} \leq r^{M-1}e^{-\pi M}.$$

For C_3, for $r \geq 1$, we have

$$|z^{s-1}| = r^{\Re(s)-1}|e^{\pi i(\Re(s)-1+it)}| = r^{\Re(s)-1}e^{-\pi t} \leq r^{M-1}e^{\pi M}.$$

Using this, we can get, on either C_1 or C_3, that

$$|\frac{z^{s-1}e^{az}}{1-e^z}| \leq \frac{r^{M-1}e^{\pi M}e^{-ar}}{1-e^r} = \frac{r^{M-1}e^{\pi M}e^{(1-a)r}}{e^r-1}.$$

Since $e^r - 1 < e^r/2$ when $r > \log(2)$ so the integrand is bounded by $Ar^{M-1}e^{-ar}$ where A is a constant depending on M. Since $\int_c^\infty r^{M-1}e^{-ar}$ converges when $c > 0$, C_1 and C_3 converge uniformly on every compact disk $|s| \leq M$. Now that we have the convergence right, our plan for the proof will be to split the integral into 2 parts and prove that 1st part goes to $\Gamma(s)\zeta(s,a)$, and the second part goes to 0. Now, let's take

$$2\pi i I(s,a) = \left(\int_{C_1} + \int_{C_2} + \int_{C_3}\right) z^{s-1} g(z) dz,$$

where $g(z) = \frac{e^{az}}{1-e^z}$. On C_1 and C_3, $g(z) = g(-r)$ and on C_2, we let $z = ce^{i\theta}$. We have:

$$\begin{aligned} 2\pi i I(s,a) &= \int_c^\infty r^{s-1} e^{-\pi i s} g(-r) dr + i \int_\pi^{-\pi} c^{s-1} e^{(s-1)i\theta} c e^{i\theta} g(ce^{i\theta}) d\theta + \int_c^\infty r^{s-1} e^{\pi i s} g(-r) dr \\ &= 2i \sin(\pi s) \int_c^\infty r^{s-1} g(-r) dr + \int_\pi^{-\pi} e^{is\theta} g(ce^{i\theta}) d\theta. \end{aligned}$$

Dividing by $2i$, we get

$$\pi I(s,a) = \sin(\pi s) I_1(s,c) + I_2(s,c)$$

where I_1 and I_2 are functions of c and s that replace the integrals. Now letting $c \to 0$ gives us

$$\lim_{c \to 0} I_1(s,c) = \int_0^\infty \frac{r^{s-1} e^{-ar}}{1 - e^r} dr = \Gamma(s)\zeta(s,a)$$

for real part > 1. Now we must show that as $c \to 0$, $I_2(s,c) \to 0$. To do that we use the fact that $g(z)$ is analytic in $|z| < 2\pi$ except for a simple pole at $z = 0$. Therefore $zg(z)$ is analytic everywhere inside $|s| < 2\pi$ which means it is bounded. We have $|g(z)| \leq \frac{A}{|z|}$, where $|z| = c < 2\pi$ and A is constant. This means that

$$I_2(s,c) \leq \frac{c^{\Re(s)}}{2} \int_{-\pi}^{\pi} e^{i\theta} \frac{A}{c} d\theta \leq A e^{\pi |t|} c^{\Re(s)-1}.$$

If $\Re(s) > 1$ and $c \to 0$ we get that $I_2(s,c) \to 0$ which means that $\sin(\pi s)\Gamma(s)\zeta(s,a)$, and since $\Gamma(s)\Gamma(1-s) = \pi sin(\pi s)$, this proves that $\zeta(s,a) = \Gamma(1-s)I(s,a)$ which completes the proof. □

We can now use this theorem to prove the functional equation later on. But first note that in this theorem, $\Gamma(1-s)I(s,a)$ is well defined for $\Re(s) \leq 1$, so therefore we can use this to define $\zeta(s,a)$ for $\Re(s) \leq 1$.

Definition 5. If $\Re(s) \leq 1$, we can define $\zeta(s,a)$ as

$$\zeta(s,a) = \Gamma(1-s)I(s,a).$$

This equation proves the analytic continuation of $\zeta(s,a)$ in the whole plane. Now, we must show the analytic continuation of $L(s,x)$.

5. Analytic Continuation of L(s, x)

To prove the analytic continuation of $L(s,x)$, we will use what we proved at the start of the paper:

$$L(s,x) = k^{-s} \sum_{r=1}^{k} \chi(r)\zeta\left(s, \frac{r}{k}\right).$$

Theorem 4. *The following is true:*

1. *$\zeta(s,a)$ is analytic on all s except for a simple pole at $s=1$ with residue 1.*
2. *For the principle character χ_1 mod k, $L(s,\chi_1)$ is analytic except for s=1 with residue $\varphi(k)/k$.*
3. *If $\chi \neq \chi_1$, $L(s,x)$ is entire function of s.*

Remark. In this theorem, χ_1 is defined as 1 is $\gcd(n,k)=1$, and 0, if $\gcd(n,k)>1$ where k is the mod.

Proof. We will first begin by proving (1). Since $I(s,a)$ is entire, the only possible poles are at $s=1,2,3,\ldots$ when $\Gamma(1-s)=\infty$. But theorem 1 shows that $\zeta(s,a)$ is analytic at $s=2,3,4,\ldots$, so $s=1$ is the only possible pole. Now, we must show that the pole is residue 1. When s is an integer, the integrals evaluated at C_1 and C_3 cancel each other leaving only C_2. This means that if s is an integer we have (using the Cauchy Residue Theorem):

$$I(s,a) = \frac{1}{2\pi i}\int_{C_2} \frac{z^{s-1}e^{az}}{1-e^z}dz = \text{Res}\ \frac{z^{s-1}e^{az}}{1-e^z}.$$

Letting $s=1$ gives us the following:

$$I(1,a) = \text{Res}_{\ z=0}\ \frac{e^{az}}{1-e^z} = \lim_{z\longrightarrow 0}\frac{ze^{az}}{1-e^z} = \lim_{z\longrightarrow 0}\frac{1}{1-e^z} = \lim_{z\longrightarrow 0}\frac{-1}{e^z} = -1.$$

Now, to prove the residue at $s=1$ for $\zeta(s,a)$, we compute the limit

$$\lim_{s\longrightarrow 1}(s-1)\zeta(s,a) = -\lim_{s\longrightarrow 1}(1-s)I(s,a)\Gamma(1-s) = -I(1,a)\lim_{s\longrightarrow 1}\Gamma(2-s) = \Gamma(1) = 1$$

which proves (1). To prove (2) and (3), we use the fact that

$$\sum_{r (\text{mod } k)} \chi(r) = \{0,\ \text{if } \chi \neq \chi_1;\ \phi(k),\ \text{if } \chi = \chi_1\}.$$

Since $\zeta(s, \frac{r}{k})$ has a simple pole at $s = 1$, $\chi(r)\zeta(s, \frac{r}{k})$ also has a simple pole at $s = 1$ with residue $\chi(r)$. Therefore, calculating the residue of $L(s, x)$, we have:

$$\begin{aligned}\mathrm{Res}_{s=1} L(s,x) &= \lim_{s \longrightarrow 1} (s-1) L(s,x) \\ &= \lim_{s \longrightarrow 1} (s-1) k^{-s} \sum_{r=1}^{k} \chi(r) \zeta\left(s, \frac{r}{k}\right) \\ &= \frac{1}{k} \sum_{r=1}^{k} \chi(r) \\ &= \{0, \text{ if } \chi \neq \chi_1; \ \frac{\phi(k)}{k}, \text{ if } \chi = \chi_1\}.\end{aligned}$$

This proves (2), because if we plug in $\chi = \chi_1$, we get $\frac{\phi(k)}{k}$, and it proves (3), because if we let $\chi \neq \chi_1$, we get 0, and because $L(s, x)$ has no other poles, this proves that $L(s, x)$ is an entire function of s. □

We can now use the analytic continuation of $\zeta(s, a)$ on $L(s, \chi)$ using the fact that

$$L(s, \chi) = k^{-s} \sum_{r=1}^{k} \chi(r) \zeta(s, \frac{r}{k})$$

and plugin $\zeta(s, \frac{r}{k})$. Now, Since $\zeta(s, a)$ converges, $L(s, \chi)$ converges.

6. Functional Equation for the Hurwitz Zeta Function

To prove the functional equation for L-functions, we will first need to prove Hurwitz's formula, which is another interpretation of $\zeta(s, a)$ that makes sense for $\Re(s) < 0$. After that, we will prove the functional equation for $\zeta(s, a)$ which will then be used to prove the functional equation for the Dirichlet L-function in the next chapter.

Lemma 5. *Let $S(r)$ donate the region that remains when we remove all open disks with radius r, for $0 \leq r \leq \pi$, with centers at $z = 2\pi n i$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. Then when $0 < a \leq 1$ the function*

$$g(z) = \frac{e^{az}}{1 - e^z}$$

is bounded in $S(r)$ (the bound depends on r).

Proof. Let $z = x + iy$ and consider the punctured rectangle $Q(r) = \{z : |x| \leq 1, |y| \leq \pi, |z| \geq r\}$, so in other words there is a $1 \times \pi$ rectangle with a circular hole in the middle at (0,0) with radius r. This is a compact set so g is bounded on $Q(r)$. Since $|g(z + 2\pi i)| = g(z)$, g is bounded in the punctured infinite strip

$$\{z : |x| \leq 1, |z - 2n\pi i| \geq r, n = 0, \pm 1, \pm 2, \pm 3, \ldots\}.$$

Now we show that g is bounded outside the strip. Let $|x| \geq 1$ and take

$$|g(z)| = \left|\frac{e^a z}{1-e^z}\right| = \frac{e^{ax}}{|1-e^z|} \leq \frac{e^{ax}}{|1-e^x|}.$$

For $x \geq 1$ we have $|1-e^x| = e^x - 1$ and $e^a x \leq e^x$, so

$$|g(z)| \leq \frac{e^x}{e^x-1} \leq \frac{1}{1-e^{-x}} \leq \frac{1}{1-e^{-}1} = \frac{e}{e-1}.$$

Now we need to prove it for $x \leq -1$. We have $|1-e^x| = 1-e^x$ so

$$|g(z)| \leq \frac{e^a x}{1-e^x} \leq \frac{1}{1-e^{-x}} \leq \frac{1}{1-\frac{1}{e}} = \frac{e}{e-1}.$$

We have now proven that $g(z)$ is bounded on $|x| \leq 1$, $x \geq 1$, and $x \leq -1$. This completes the proof. □

We can now prove a theorem called Hurwitz's formula. For this theorem we will use a function called $F(x,s)$ given by

$$F(x,s) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s},$$

where x is real and $\Re(s) > 1$. Note that x is periodic by 1, so $F(x,s) = F(x+1,s)$. Also note that $F(1,s) = \zeta(s)$ and that the series converges absolutely for $\Re(s) > 1$ due to theorem 1. If x is not an integer the series also converges for $\Re(s) > 0$. From now on, we will refer to $F(x,s)$ as the periodic zeta function.

Theorem 6. *(Hurwitz's Formula) If* $0 < a \leq 1$ *and* $\Re(s) > 1$*, then*

$$\zeta(1-s,a) = \frac{\Gamma(s)}{(2\pi)^s}\{e^{-\pi i s/2}F(a,s) + e^{\pi i s/2}F(-a,s)\}$$

Proof. For this proof, we will consider the function $I_N(s,a)$ and prove that it is related to $I(s,a)$, then we will use the fact that $\Gamma(s)\zeta(s,a) = I(s,a)$ to prove the functional equation. Consider the function

$$I_N(s,a) = \frac{1}{2\pi i}\int_{C(N)} \frac{z^{s-1}e^{az}}{1-e^z}dz$$

where $C(N)$ is the contour shown in Figure 2 (page 10), where the radius of the inside circle is $c < \pi$, and the radius of the outside circle is $(2n+1)\pi$. In the smaller circle, the we are rotating counter clock-clockwise, in the bigger circle, we are rotation clockwise, in the line in the upper half plane, we are moving away from $(0,0)$, and in the line from the bottom half plane, we are moving toward $(0,0)$. Note that N must be an integer. First, we must prove that $I(s,a)$ (all the way back on theorem 2) is the same as $I_N(s,a)$ as $N \longrightarrow \infty$ and with $\Re(s) < 0$. To do this, it suffices to show that the outer circle tends to 0 as $N \longrightarrow \infty$. On the outer circle, we have $z = Re^{i\theta}$, $-\pi \leq \theta \leq \pi$, so

$$|z^{s-1}| = |R^{s-1}e^{i\theta(s-1)}| = R^{\Re(s)-1}e^{-t\theta} \leq R^{\Re(s)-1}e\pi|t|.$$

Since the outer circle lies on the set of $S(r)$ of Theorem 5, the integrand is bounded by $Ae^{\pi|t|}R^{\Re(s)-1}$, where A is the bound for $g(z)$ because of Theorem 5. This means the integral is bounded by

$$2\pi Ae^{\pi|t|}R^{\Re(s)},$$

and this goes to 0 as $R \longrightarrow \infty$ if $\Re(s) < 0$. Therefore, replace s by $1-s$ we get that

$$\lim_{n\longrightarrow\infty} I_N(1-s,a) = I(1-s,a)$$

if $\Re(s) > 1$. Now we need to compute $I_N(1-s,a)$. We will use Cauchy's Residue Theorem. We have:

$$I_N(s,a) = -\sum_{n=-N, n\neq 0}^{N} R(n) = -\sum_{n=1}^{N}\{R(n) + R(-n)\}$$

where

$$R(n) = \text{Res}_{z=2\pi ni}\left(\frac{z^{s-1}e^{az}}{1-e^z}\right).$$

This means that

$$R(n) = \lim_{z\longrightarrow 2\pi ni}(z-2\pi ni)\frac{z^{-s}e^{az}}{1-e^z} = \frac{e^{2\pi nia}}{(2\pi ni)^s}\lim_{z\longrightarrow 2\pi ni}\frac{z-2\pi ni}{1-e^z} = -\frac{e^{2n\pi ia}}{(2n\pi i)^s},$$

hence this gives us

$$I_N(1-s,a) = \sum_{n=1}^{N}\frac{e^{2\pi nia}}{(2\pi ni)^s} + \sum_{n=1}^{N}\frac{e^{-2\pi nia}}{(-2\pi ni)^s}.$$

Now, we can take out the $(2\pi ni)^s$ and factor it out and leave n^s:

$$I_N(1-s,a) = \frac{e^{-\pi is/2}}{(2\pi)^s}\sum_{n=1}^{N}\frac{e^{2\pi nia}}{n^s} + \frac{e^{\pi is/2}}{(2\pi)^s}\sum_{n=1}^{N}\frac{e^{-2\pi nia}}{n^s}.$$

Now, we let $n \longrightarrow \infty$, so $I_N(s,a)$ would become $I(s,a)$ from what we proved previously, and we have

$$I(s,a) = \frac{e^{-\pi is/2}}{(2\pi)^s}F(a,s) + \frac{e^{\pi is/2}}{(2\pi)^s}F(-a,s).$$

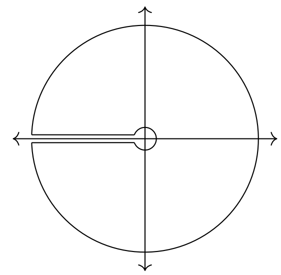
This gives

$$\zeta(1-s,a) = \Gamma(s)I(1-s,a) = \frac{\Gamma(s)}{(2\pi)^s}\{e^{-\pi is/2}F(a,s) + e^{\pi is2}F(-a,s)\},$$

which completes the proof of the theorem. □

Now, we use this to prove the functional equation for the Hurwitz Zeta Function. We can also prove the functional equation for the Riemann Zeta function by letting $s = 1$, but we will not do that here to keep the paper on topic.

Figure 2



Theorem 7. *If h and k are integers, $1 \leq h \leq k$, then for all s*

$$\zeta(1-s, \frac{h}{k}) = \frac{2\Gamma(s)}{(2\pi k)^s} \sum_{r=1}^{k} \cos\left(\frac{\pi s}{2} - \frac{2\pi r h}{k}\right) \zeta(s, \frac{r}{k}).$$

Proof. This comes from the fact that $F(x,s)$ is a linear combination of $\zeta(s,a)$ when x is rational. Assuming $x = \frac{h}{k}$ we can rearrange the terms according to residue classes mod k by writing

$$n = qk + r,$$

where $1 \leq r \leq k$ and $q = 1, 2, 3, \ldots$. This gives us, for $\Re(s) > 1$,

$$\begin{aligned} F(\frac{h}{k}, s) &= \sum_{n=1}^{\infty} \frac{e^{2\pi i n h/k}}{n^s} \\ &= \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{e^{2\pi i n h/k}}{(qk+r)^s} \\ &= \sum_{r=1}^{k} e^{2\pi i n h/k} \sum_{q=0}^{\infty} \frac{1}{(qk+r)^s} \\ &= k^{-s} \sum_{r=1}^{k} e^{2\pi i n h/k} \zeta(s, \frac{r}{k}). \end{aligned}$$

Now if we take $\frac{h}{k}$ in Hurwitz's Formula (Theorem 6) and use this association with $F(x,s)$ and $\zeta(s,a)$, we have:

$$\begin{aligned} \zeta(1-s, \frac{h}{k}) &= \frac{\Gamma(s)}{(2\pi k)^s} \sum_{r=1}^{k} (e^{-\pi i s/2} e^{2\pi i r h/k} + e^{\pi i s/2} e^{-2\pi i r h/k}) \zeta(s, \frac{r}{k}) \\ &= \frac{2\Gamma(s)}{(2\pi k)^s} \sum_{r=1}^{k} \cos\left(\frac{\pi s}{2} - \frac{2\pi r h}{k}\right) \zeta(s, \frac{r}{k}), \end{aligned}$$

which is true for real part > 1, but we can extend it analytically to all s which completes the proof. □

7. Functional Equation for $L(s,\chi)$

Now that we have finished the proof for the Hurwitz Zeta Functions, we can put everything together and get the functional equation for $L(s,\chi)$.

Theorem 8. *Let χ be a Dirichlet Character* $\bmod k$*, and let d be the induced modulus, and we have:*

$$\chi(n) = \psi(n)\chi_1(n)$$

where χ is a character $\bmod d$ *and χ_1 is the principle character. Then for all s, we have*

$$L(s,\chi) = L(s,\psi)\prod_{p|k}\left(1 - \frac{\psi(p)}{p^s}\right)$$

Proof. First let $\Re(s) > 1$ and use the Euler Product to get

$$L(s,x) = \prod_{p}\left(1 - \frac{\chi(p)}{p^s}\right).$$

Since $\chi(p) = \psi(p)\chi_1(p)$ and $\chi_1(p) = 0$ is $p|k$ and $\chi_1(p)$= 1 if $p \nmid k$, so we have

$$\begin{aligned} L(s,x) &= \prod_{p\nmid k}\frac{1}{1-\frac{\psi(p)}{p^s}} \\ &= \prod_{p}\frac{1}{1-\frac{\psi(p)}{p^s}} \times \prod_{p|k}\left(1 - \frac{\psi(p)}{p^s}\right) \\ &= L(s,\psi)\prod_{p|k}\left(1 - \frac{\psi(p)}{p^s}\right). \end{aligned}$$

This completes the proof for $\Re(s) > 1$ and we can extend it analytically to all s using analytic continuation. □

To associate $F(x,s)$ and $L(s,\chi)$, we need a to prove another theorem.

Theorem 9. *Let χ be a primitive character mod k. Then for $\Re(s) > 1$ we have*

$$G(1,\bar{\chi})L(s,\chi) = \sum_{h=1}^{k}\bar{\chi}(h)F\left(\frac{h}{k}, s\right),$$

where $G(m,\chi)$ is the Gauss Sum:

$$G(m,\chi) = \sum_{r=1}^{k}\chi(r)e^{2\pi i r m/k}.$$

Note that $\bar{\chi}$ is the complex conjugate of χ, and a primitive character means that you cannot turn χ mod n into χ mod m.

Proof. Take the sum

$$\sum_{h=1}^{k} \bar{\chi}(h)F\left(\frac{h}{k}, s\right) = \sum_{h=1}^{k}\sum_{n=1}^{\infty} \bar{\chi}(h)e^{2\pi nih/k}n^{-s}$$
$$= \sum_{n=1}^{\infty} n^{-s} \sum_{h=1}^{k} \chi(\bar{h})e^{2\pi nih/k}$$
$$= \sum_{n=1}^{\infty} n^{-s}G(n, \bar{\chi}).$$

But $G(n, \bar{\chi})$ is seperable because χ is primitive, so $G(n, \bar{\chi}) = G(1, \bar{\chi})\chi(n)$, so we have

$$\sum_{h=1}^{k} \bar{\chi}(h)F(\frac{h}{k}, s) = G(1, \bar{\chi}) \sum_{n=1}^{\infty} \chi(n)n^{-s} = G(1, \bar{\chi})L(s, \chi),$$

which completes the proof. □

Now using everything, we can finish proving the functional equation for the Dirichlet L-function.

Theorem 10. *The functional equation for Dirichlet L-function is defined as*

$$L(1-s, x) = \frac{k^{s-1}\Gamma(s)}{(2\pi)^s}\{e^{-\pi is/2} + \chi(-1)e^{\pi is/2}\}G(1, \chi)L(s, \chi)$$

Proof. We let $x = \frac{h}{k}$ in Hurwitz's formula then multiply each one by $\chi(h)$ and sum over h. This gives us

$$\sum_{h=1}^{k} \chi(h)F(1-s, \frac{-h}{k}) = \frac{\Gamma(s)}{(2\pi)^s}\left\{e^{-\pi is/2}\sum_{h=1}^{k} \chi(h)F\left(\frac{h}{k}, s\right) + e^{\pi is/2}\sum_{h=1}^{k} \chi(h)F\left(\frac{-h}{k}, s\right)\right\}.$$

Since $F(x, s)$ is periodic within x with period 1 and $\chi(h) = \chi(-1)\chi(-h)$ we can get

$$\sum_{h \bmod k} \chi(h)\zeta(1-s, \frac{h}{k}) = \frac{\Gamma(s)}{(2\pi)^s}\{e^{-\pi is/2} + \chi(-1)e^{\pi is/2}\} \sum_{h=1}^{k} \chi(h)F\left(\frac{h}{k}, s\right).$$

Now, when we multiply both sides by k^{s-1}, we get the functional equation for $L(s, \chi)$ which completes the proof. □

This finishes the paper.

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