Analytic Combinatorics

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Abstract

Many combinatorial objects, such as the number of involutions of length n, are easily described but do not have a simple explicit formula. However, using techniques from complex analysis, a simple expression for the asymptotic growth of such objects can be obtained. In this paper, we first outline methods of describing and finding generating functions for labeled and unlabeled combinatorial classes. We then provide techniques, with applications, for analyzing the asymptotic behaviour of coefficients of generating functions with poles and algebraic singularities. Finally, we do the same for entire generating functions.

1 Symbolic Methods

1.1 Combinatorial Classes

Definition 1.1. A combinatorial class, A, is defined as a set with a size function such that the size of any element is a non-negative integer, and there are a finite number of elements of any given size. We denote the size of an element $\alpha \in A$ by $|\alpha|_A$ or just $|\alpha|$ when the class is evident.

Definition 1.2. The counting sequence of the class is the sequence $(a_n)_{n\geq 0}$ where a_n is the number of objects in the class with size n. We call

$$A(z) = \sum_{n=0}^{\infty} a_n z^n$$

the corresponding ordinary generating function.

Definition 1.3. We let $[z^n]f(z)$ represent the coefficient of z^n in the power series $f(z)$.

Example. For example, we can consider the combinatorial class of binary words, that is sequences of characters a and b, where the size of an element is the length of the word. The counting sequence for binary words of size n is then 2^n, as each of the n letters could be either a or b and we make this choice n times. The ordinary generating function of this class is thus $\sum_{n=0}^{\infty} 2^n z^n = \frac{1}{1-2z}$, because it is a geometric series.

Definition 1.4. Two combinatorial classes A and B are isomorphic (we write $A \cong B$) if and only if they have the same counting sequence.

1.2 Constructing Combinatorial Classes

Definition 1.5. We let E be a neutral class with one element of size 0. We let Z be an atomic class with one element of size 1, which we call the *atom*.

Definition 1.6. The Cartesian product of two classes $A = B \times C$ is the set of ordered pairs $A = \{\alpha = (\beta, \gamma) | \beta \in B, \gamma \in C\}$ with the size of an element given by $|\alpha|_A = |\beta|_B + |\gamma|_C$. We denote A^n as the product of A with itself n times.

Definition 1.7. The disjoint sum of two classes $A = B + C$, is the union $B \times E_\square \cup C \times E_\diamond$ with the size of an element the same as it was in B and C.

Remark 1.8. The neutral elements serve as markers to distinguish elements of B and C in order to formalize the construction for sets which are not disjoint. We are, in effect, taking the disjoint union of the sets.

Definition 1.9. For a class A with no object of size 0, we define the sequence class as the infinite sum, $\mathrm{SEQ}(A) = E + A + A^2 + A^3 + \cdots$. We denote $\mathrm{SEQ}_k(A)$ as the sequences with k components (the generating function for this is A^k-since it is the ordered sequence of k objects).

Example. We can consider binary words as $\mathrm{SEQ}(\{a, b\})$. For instance, the sequences of size 4 ($\mathrm{SEQ}_4(\{a, b\})$)are $aaaa, aaab, aaba, aabb, abaa, abab, abba, abbb, baaa, baab, baba, babb, bbaa, bbab, bbba, bbbb$.

Definition 1.10. For a class A with no object of size 0, we define the cycle of A as $\mathrm{CYC}(A) = (\mathrm{SEQ}(A) \setminus E)/S$, where S is the equivalence relation between circular shifts of sequences.We define $\mathrm{CYC}_k(A)$ as $\mathrm{SEQ}_k(A)/S$

Example. If we consider $\mathrm{CYC}(\{a, b\})$, the cycles of size 4 are $aaaa, aaab, abab, aabb, abbb, bbbb$.

Definition 1.11. The multiset is defined as the quotient $\mathrm{MSET}(A) = \mathrm{SEQ}(A)/R$ where R is the equivalence relation between two sequences which are permutations of each other. We define $\mathrm{MSET}_k(A)$ similarly as $\mathrm{SEQ}_k(A)/R$.

Example. For $\mathrm{CYC}(\{a, b\})$, the multisets of size 4 are $aaaa, aaab, aabb, abbb, bbbb$.

Proposition 1.12. *We have the following:*

1. *If $A = B \times C$, then $A(z) = B(z)C(z)$.*
2. *If $A = B + C$, then $A(z) = B(z) + C(z)$.*
3. *If $A = SEQ(B)$, then $A(z) = \frac{1}{1-B(z)}$.*

Proof.

1. Suppose $A = B \times C$. Then we have

$$a_n = \sum_{n_1+n_2=n} b_{n_1} c_{n_2}$$

because an object of size n is made up of a pair of objects of size n_1 and n_2 from B and C with $n_1 + n_2 = n$. This is precisely the relationship between $[z^n]A(z)$ and $[z^n]B(z) \times C(z)$, so $A(z) = B(z)C(z)$.

2. Suppose $A = B + C$. We then have $a_n = b_n + c_n$ because an object in a is either one of b_n elements in B or c_n elements in C. Thus we have $A(z) = B(z) + C(z)$.

3. If $A = \mathrm{SEQ}(B)$, we have $A = E + B + B^2 + B^3 + \cdots$. By the previous parts of this proposition, we have $A(z) = 1 + B(z) + B(z)^2 + B(z)^3 + \cdots$. Summing this geometric series yields $A(z) = \frac{1}{1-B(z)}$.

□

Example. We can represent the positive integers as the combinatorial class $I = SEQ_{\geq 1}(Z)$. The generating function is $\frac{z}{1-z}$.

Definition 1.13. A *labeled object* of size n is a graph of n vertices, with an injective labeling function from the vertices to the integers. The object is *well-labeled* if the labels are the integers from 1 to n. A combinatorial class is a *labeled class* if it is made up of well-labeled objects. We define the counting sequence again as $a_n = |\{x \in A : |x| = n\}|$. The atomic class Z for labeled objects is the graph with one vertex labeled 1.

When counting labeled classes, we use exponential generating functions.

Definition 1.14. The exponential generating function of a counting sequence is

$$A(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}.$$

Example. Permutations are the class of labeled, directed paths. They have the counting sequence $P_n = n!$ and consequently the exponential generating function $\sum_{n=0}^{\infty} \frac{n! z^n}{n!} = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

Example. Urns are the class of totally disconnected graphs. There is thus one possible labeling for each object. They have the counting sequence $U_n = 1$ and consequently the exponential generating function $\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$.

Example. Cycles are the class of circular graphs with positive orientation. They have the counting sequence $C_n = (n-1)!$ and consequently the exponential generating function $\sum_{n=1}^{\infty} \frac{(n-1)! z^n}{n!} = \sum_{n=1}^{\infty} \frac{z^n}{n} = \log(\frac{1}{1-z})$.

Definition 1.15. The labeled product $A \star B$ is the set of ordered pairs (a, b) where a and b are consistent relabelings of structures from A and B respectively (i.e. a and b preserve the relative ordering of the structures). We defiine A^k as the labeled product of A with itself k times.

We can define SEQ, CYC, and SET for labeled classes.

Definition 1.16. We define $\mathrm{SEQ}(A)$ as before, as $E + A + A^2 + \cdots$ but with a labeled product instead of the Cartesian product. Again, we define $\mathrm{SEQ}_k(A) = A^k$.

Definition 1.17. We define $\mathrm{SET}(A)$ as the sequences of A under the equivalence relation of sequences being the same if they are permutations of each other. $\mathrm{SET}_k(A)$ are the sets formed from k-sequences of A.

Definition 1.18. We define CYC(A) as the sequences of A (excluding the neutral class) under the equivalence relation of sequences being the same if they are cyclic rotations of each other. $\text{CYC}_k(A)$ are the cycles formed from k-sequences of A.

Example. We can consider our previous examples in the light of these new constructions. Symbolically, we have that permutations are SEQ(Z), urns are SET(Z), and cycles are CYC(Z).

Proposition 1.19. *We have that*

1. *If* $A = B \star C$, $A(z) = B(z)C(z)$
2. *If* $A = SEQ(B)$, $A(z) = \frac{1}{1-B(z)}$
3. *If* $A = SET(B)$, $A(z) = e^{B(z)}$
4. *If* $A = CYC(B)$, $A(z) = \log(\frac{1}{1-B(z)})$.

Proof.

1. Suppose $A = B \star C$. The number of relabelings of a pair of objects b and c are given by $\binom{|b|+|c|}{|b|}$. This is because the relabeling will use all the numbers from 1 to $|b| + |c|$, and a relabeling is entirely determined by which numbers are chosen to label b because the relative order of the numbers will fully define it. Thus we conclude

$$a_n = \sum_{n_1+n_2=n} \binom{n}{n_1} b_{n_1} c_{n_2}$$

because we sum over all possible sizes of objects n_1, n_2, all possible objects with such sizes, and all possible relabellings.

When multiplying EGFs, if $a(z) = b(z)c(z)$, by expanding out the product and considering the z^n coefficient, we have $\frac{a_n}{n!} = \sum_{k=0}^{n} \frac{1}{k!} b_k \frac{1}{(n-k)!} c_{n-k}$. Multiplying by $n!$, we get $a_n = \sum_{k=0}^{n} \binom{n}{k} b_k c_{n_k}$, which is exactly the relation between a_n and the sequences b_n and c_n in a labeled product. Thus $a(z) = b(z)c(z)$ if $A = B \star C$.

2. Let $A = \text{SEQ}(B)$. SEQ is defined in terms of labeled products and disjoint sums, so which we can convert to generating functions to get $A(z) = 1 + B(z) + B(z)^2 + \cdots = \frac{1}{1-B(z)}$ by summing geometric series.

3. Because our elements of the sequence are labeled, they are distinct, so we have a $k!$ to 1 correspondence between sequences and sets of B (one for every permutation), so if $A = \text{SET}_k(B)$, $A(z) = \frac{1}{k!} B(z)^k$. Thus, we have that, if $A = \text{SET}(B)$, $A(z) = 1 + B(z) + \frac{B(z)^2}{2!} + \frac{B(z)^3}{3!} + \cdots = e^{B(z)}$, as desired.

4. Similarly, we have a k to 1 correspondence between k-sequences and k cycles of B (one for every cycle), so if $A = \text{CYC}_k(B)$, $A(z) = \frac{1}{k} B(z)^k$. We also have, if $A = \text{CYC}(B)$, $A(z) = B(z) + \frac{B(z)^2}{2} + \frac{B(z)^3}{3} + \cdots = \log(\frac{1}{1-B(z)})$, as desired.

$\square$

This gives us a way to symbolically find the generating function of classes like urns, permutations, and cycles without directly calculating the counting sequence.

Example. Another way of thinking about permutations is as being made up of disjoint cycles. This can be represented as SET(CYC(Z)). Under this construction, the EGF is $e^{\log(\frac{1}{1-z})} = \frac{1}{1-z}$, which is what we found before, when considering permutations as SEQ(Z).

1.3 Examples

Example. Fibonacci numbers can be constructed the number of ways of representing n as the sum of 1 and 2. We can thus consider SEQ($I_{\{1,2\}}$) which has counting sequence F_{n+1}. Because $I_{\{1,2\}}$ has generating function $z + z^2$, we have that the generating function for the F_{n+1} is $\frac{1}{1-z-z^2}$ (or, shifting, the generating function for F_n is $\frac{z}{1-z-z^2}$).

We can use this generating function to find the explicit formula for the Fibonacci numbers. We can use partial fractions, to find

$$F(z) = \frac{z}{1-z-z^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi z} - \frac{1}{1-\psi z}\right)$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. Expanding $\frac{1}{1-\phi z}$ and $\frac{1}{1-\psi z}$ as geometric series, we have that $F(z) = \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}(\phi^n - \psi^n)x^n$. Thus $F_n = [z^n]F(z) = \frac{1}{\sqrt{5}}(\phi^n - \psi^n)$.

Example. The Catalan Numbers are generated by the class of rooted full binary trees with the size of a tree being the number of internal nodes (the nodes which have children).

We can consider elements of the combinatorial class: they are either just a single external node with size 0 (i.e. a neutral element) or made up of an internal root (a single structure of size 1) with two trees (the two subtrees). We thus have $C \cong E + Z \times C \times C$. By converting this to generating functions, we have $C(z) = 1 + zC(z)^2$. Which we can solve with the quadratic formula to give $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ (the other root would have a $\frac{1}{z}$ term).

We can use this formula to derive an explicit formula. Expanding with Newton's generalized binomial theorem, and simplifying, we have that

$$\begin{aligned}
C(z) &= \frac{1}{2z} - \frac{1}{2z}\sum_{k=0}^{\infty}\binom{1/2}{k}(-4z)^k \\
&= -\frac{1}{2z}\sum_{k=1}^{\infty}\frac{(1/2)(-1/2)(-3/2)\cdots(-k+3/2)}{k!}(-4)^k z^k \\
&= \sum_{k=1}^{\infty}\frac{(1)(3)\cdots(2k-3)2^{k-1}}{k!}z^{k-1} \\
&= \sum_{k=0}^{\infty}\frac{(1)(3)\cdots(2k-1)2^k k!}{(k+1)!k!}z^k \\
&= \sum_{k=0}^{\infty}\frac{(2k)!}{(k+1)k!k!}z^k,
\end{aligned}$$

so $C_n = [z^n]C(z) = \frac{1}{k+1}\frac{(2k)!}{k!k!} = \frac{1}{k+1}\binom{2k}{k}$.

Example. The combinatorial class T of rooted labeled trees is isomorphic to $Z \star \mathrm{SET}(T)$ (a root with a set of trees), so we have $T(z) = ze^{T(z)}$. We will be able to solve this later with the help of the Lagrange Inversion formula.

Example. The combinatorial class of 2-regular graphs is an unordered collection of disjoint undirected cycles. Furthermore, the cycles must be of size 3 or more, because cycles of size 1 and 2 are not 2-regular. The generating function for undirected cycles of length greater than 2 is half of that for directed cycles of length greater than 2, since you can choose two directions for the cycle. We let $UCYC_{>2}$ be the class of undirected cycles of length 3 or more. Note that the counting sequence (and consequently generating function) of $UCYC_{>2}$ is half of $\mathrm{CYC}_{>2}$ Thus this class can be represented as $\mathrm{SET}(UCYC_{>2}(Z))$. Thus the generating function is

$$e^{\frac{1}{2}\left(\log \frac{1}{1-z} - z - \frac{z^2}{2}\right)} = \frac{1}{\sqrt{1-z}} e^{-\frac{z}{2} - \frac{z^2}{4}}.$$

We will be able to approximate coefficients with Darboux's theorem.

Example. Derangements are permutations with no cycles of size 1. We can thus construct them as $\mathrm{SET}(\mathrm{CYC}_{>1}(Z))$. The EGF is thus $e^{\log(\frac{1}{1-z}) - z} = \frac{e^{-z}}{1-z}$. Expanding out the generating function, we find that

$$[z^n]\frac{e^{-z}}{1-z} = [z^n]\Big(1 - z + \frac{z^2}{2} - \frac{z^3}{6} + \cdots\Big)(1 + z + z^2 + \cdots) = \sum_{i=0}^{n} (-1)^i \frac{1}{n!}.$$

Thus the number of derangements of n is $n!\big(\sum_{i=0}^{n}(-1)^i \frac{1}{n!}\big) \approx \frac{n!}{e}$

Example. Involutions are permutations with cycles of size 1 and 2. They can thus be constructed as $\mathrm{SET}(\mathrm{CYC}_{1,2}(Z))$. Thus their EGF is $e^{z + \frac{z^2}{2}}$. We will see how to asymptotically approximate the coefficients with Hayman's method.

Example. Permutations with no cycles of size at most q can be constructed as $\mathrm{SET}(\mathrm{CYC}_{>q}(Z))$. Thus the EGF is

$$e^{\sum_{n>q} \frac{z^n}{n}} = e^{\log \frac{1}{1-z} - \sum_{1 \le n \le q} \frac{z^n}{n}} = \frac{1}{1-z} e^{-z + \cdots + z^q/q}.$$

We will see how to find the asymptotic value of the coefficients by analyzing poles.

Example. Stirling numbers of the first kind are permutations of n with k disjoint cycles, denoted ${n \brack k}$. We can consider it as $\mathrm{SET}(U \times \mathrm{CYC}(Z))$, where U serves to count the cycles. The generating function is thus $S(u,z) = e^{u \log(\frac{1}{1-z})} = (\frac{1}{1-z})^u$ with ${n \brack k} = n![u^k][z^n]S(u,z)$.

Bell numbers are the number of ways of partitioning the numbers from 1 to n into nonempty subsets. It thus can be represented as $\mathrm{SET}(\mathrm{SET}_{\geq 1}(Z))$. The generating function $B(z)$ is thus $e^{e^z - 1}$. Differentiating gives $B'(z) = B(z)e^z$. Multiplying out the right hand side of the equality and equating the z^n coefficients gives $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$ as a recurrence for the Bell numbers. We can modify the combinatorial construction for the Bell numbers by counting the counting the number of ways of partitioning n objects into k nonempty subsets; these are known as Stirling numbers of the second kind and denoted ${n \brace k}$. Letting U again count the size of a partition, we find that this combinatorial class equals $\mathrm{SET}(U \times \mathrm{SET}_{\geq 1}(Z))$, for which the generating function is $S(u,z) = e^{u(e^z - 1)}$, with ${n \brace k} = n![u^k][z^n]S(u,z)$.

2 Lagrange Inversion

Theorem 2.1. *Let $f(u)$ and $\phi(u)$ be formal power series in u, with $\phi(0) = 1$. Then there is a unique formal power series $u = u(t)$ that satisfies $u = t\phi(u)$. Further, the value of $f(u(t))$ of f at $u(t)$, when expanded in a power series in t about $t = 0$, satisfies*

$$[t^n]\{f(u(t))\} = \frac{1}{n}[u^{n-1}]\{f'(u)\phi(u)^n\}$$

Proof. We prove this for polynomials f and ϕ. This will imply it for the full formal power series because discarding the upper terms with degree greater than n of the power series does not affect our equation.

Consider $t = \frac{u}{\phi(u)}$. Since ϕ equals 1 at 0, it must be nonzero on some neighborhood of 0, so that $\frac{u}{\phi(u)} = u + O(u^2)$. The derivative (with respect to u) is nonzero at 0, so $t = \frac{u}{\phi(u)}$ must have some inverse mapping on a neighborhood of 0, in which u can be expressed as an analytic function in terms of t.

Because $u = t\phi(u)$, we have

$$\frac{1}{n}[u^{n-1}]\{f'(u)\phi(u)^n\} = \frac{1}{n}[u^{n-1}]\left\{f'(u)\left(\frac{u}{t}\right)^n\right\} = \frac{1}{n}[u^{-1}]\left\{\frac{f'(u)}{t^n}\right\}.$$

We can use the residue theorem to find that this last expression equals $\frac{1}{2\pi i}\int_C \frac{f'(u)}{t(u)^n}$, with C a small contour encircling the origin.

Because u is an analytic function of t, we are justified in making a change of variables from u to t in the integral in question, giving us

$$\frac{1}{n}\frac{1}{2\pi i}\int_C \frac{f'(u)}{t(u)^n}\,du = \frac{1}{n}\frac{1}{2\pi i}\int_C \frac{f'(u(t))u'(t)}{t^n}\,dt = \frac{1}{n}[t^{n-1}]\{f'(u(t))u'(t)\} = [t^n]f(u(t)),$$

as desired. □

Example. We previously showed the generating function for rooted labeled trees satisfies $T(z) = ze^{T(z)}$. We use Lagrange Inversion with $f(u) = u$, $t = z$ and $\phi(u) = e^u$, so that $T(z) = f(u(z))$. Then

$$[z^n]T(z) = \frac{1}{n}[u^{n-1}]e^{nu} = \frac{1}{n}\frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{n!}.$$

Consequently, there are n^{n-1} rooted labeled trees with n vertices. Furthermore, as there is an n to 1 correspondence between rooted labeled trees and labeled trees, there are n^{n-2} labeled trees.

3 Asymptotic Growth Rate: Poles

Theorem 3.1. *(Growth rate theorem) Let $f(z) = \sum a_n z^n$ be analytic in some region containing the origin, and let z_0 be a singularity of smallest modulus not zero, and let $\epsilon > 0$. Then for sufficient large n we have $|a_n| < \left(\frac{1}{|z_0|} + \epsilon\right)^n$, and for infinitely many n we have $|a_n| > \left(\frac{1}{|z_0|} - \epsilon\right)^n$.*

Proof. It is well-known that the radius of convergence of a Taylor series is equal to the radius of the largest open disc with the given center (in this case zero) on which the given function is holomorphic. Thus, the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is is $|z_0|$, and by the Cauchy-Hadamard theorem we have $\frac{1}{|z_0|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$, which is equivalent to the desired result. $\square$

The main method of this section is to find a simple function with similar singularities of the generating function, and use growth rate of said simple function as an estimate from the growth rate theorem.

Let R be the minimum modulus such that f has a singularity of this modulus. Consider all singularities on this circle. We first consider meromorphic functions.

Definition 3.2. If f is meromorphic and z_0 is a pole of order r, then the Laurent expansion in a punctured disk is $f = \sum_{j=1}^{r} a_{-j}(z-z_0)^{-j} + \sum_{j=0}^{\infty} a_j(z-z_0)^j$. The *Principle Part*, denoted by $PP(f : z_0)$ is the former sum.

For a meromorphic function $f(z)$, near a pole z_0, it turns out that f is well approximated by the principal part of its Laurent expansion. We will use this fact to come up with an approximation for the asymptotic behavior of the coefficients of f. Letting the singularities with minimum modulus R be $z_0, \ldots, z_s$, we have that since $f - PP(f : z_0)$ is analytic at z_0,

$$h = f - PP(f : z_0) - PP(f : z_1) - \cdots - PP(f : z_s)$$

is analytic on all of $|z| = R'$, for some $R' > R$.

So, by the growth rate theorem, the power series coefficients of h about the origin cannot grow faster than $\left(\frac{1}{R'} + \epsilon\right)^n$ for sufficiently large n. This means that if $f = \sum_{n=0}^{\infty} a_n x^n$ and if $g(z) = \sum_{k=0}^{s} PP(f; z_k) = \sum_{n=0}^{\infty} b_n x^n$, then $a_n = b_n + O\left(\left(\frac{1}{R'} + \epsilon\right)^n\right)$ as $n \to \infty$.

This brings us to the following theorem:

Theorem 3.3. *Let f be meromorphic in a neighborhood containing the origin. Let $R > 0$ be the modulus of the pole(s) $z_0, \ldots, z_l$ of smallest modulus. Let $R' > R$ be the modulus of the pole(s) of f of next smallest modulus, and let $\epsilon > 0$. Then*

$$[z^n]f(z) = [z^n]\sum_{j=0}^{l} PP(f; z_j) + O\left(\left(\frac{1}{R'} + \epsilon\right)^n\right).$$

Proof. If we subtract $PP(f; z_0)$ from $f(z)$, then the resulting function $g(z)$ is analytic at z_0. We claim that subtracting $PP(f; z_1)$ from g is the same as subtracting $PP(g; z_1)$ from g, that is, that $PP(f - PP(f; z_0); z_1) = PP(f; z_1)$.

To see this, note that $PP(f - PP(f; z_0); z_1) = PP(f; z_1) - PP(PP(f; z_0); z_1)$, but the second term on the right vanishes as $PP(f; z_0)$ is analytic at z_1. By induction, we can extend this to all of f's poles of smallest modulus. The result then follows from the growth rate theorem, as we have shown that subtracting from $f(z)$ the sum of all of its principal parts of its singularities of smallest modulus R yields a function that is analytic on a larger disk $|z| < R'$. $\square$

The following lemma gives us a means to practically calculate the asymptotics of these principal parts.

Lemma 3.4. *We have* $PP(f : z_0) = \sum_{n\geq 0} z^n \sum_{j=1}^{r} \frac{(-1)^j a_{-j}}{z_0^{n+j}} \binom{n+j-1}{j-1}$*, where* r *is the order of the pole at* z_0*.*

Proof. We calculate:

$$\begin{aligned} PP(f; z_0) &= \sum_{j=1}^{r} \frac{a_{-j}}{(z-z_0)^j} \\ &= \sum_{j=1}^{r} \frac{(-1)^j a_{-j}}{z_0^j (1-(z/z_0))^j} \\ &= \sum_{j=1}^{r} \frac{(-1)^j a_{-j}}{z_0^j} \sum_{n\geq 0} \binom{n+j-1}{j-1} (z/z_0)^n \\ &= \sum_{n\geq 0} z^n \sum_{j=1}^{r} \frac{(-1)^j a_{-j}}{z_0^{n+j}} \binom{n+j-1}{j-1} \end{aligned}$$

□

In other words, a pole of order r at z_0 contributes $\sum_{j=1}^{r} \frac{(-1)^j a_{-j}}{z_0^{n+j}} \binom{n+j-1}{j-1}$ to the coefficients z^n of f.

Example. Ordered Bell numbers.

We investigate the asymptotic behavior of the ordered bell numbers, which are defined as: a set of n elements has $\left\{ {n \atop k} \right\}$ (ie. Stirling numbers of the second kind) into k classes. If we pick one of these classes in particular, but do not care about the order of the elements within the classes, then we see that $[n]$ has $k! \left\{ {n \atop k} \right\}$ ordered partitions into k classes. The ordered bell number $b(n)$ is the total number of ordered partitions of $[n]$, ie. $\sum_k k! \left\{ {n \atop k} \right\}$.

We first derive an identity on the Stirling numbers of the second kind, which as stated are the number of partitions of a set of n elements into k classes. For the set Ω of objects we take the collection of all k^n ways of arranging n distinguishable balls in k labeled boxes. Further, such an arrangement will have property P_i if box i is empty. Then $k! \left\{ {n \atop k} \right\}$ is the number of objects with no properties.

Let S be some set of properties. If $S \subseteq N$ and there are N arrangement of balls in boxes have at least the set S of properties. Then n counts the arrangements of n labeled balls into just $k - |S|$ labeled. boxes, because all of the boxes that are labeled by S must be empty. There are clearly $(k - |S|)^n$ such arrangements. Hence, $N = (k - |S|)^n$ if $|S| \leq k$, and 0 otherwise.

If we sum over all sets S of r properties, we obtain for $r \leq k$, $N_r = \binom{k}{r}(k-r)^n$, whose opsgf is $N(x) = \sum_{0\leq r\leq k} \binom{k}{r}(k-r)^n x^r$. We can now invoke the sieve to find that the number of arrangements that have exactly t empty cells is the coefficient of x^t in $N(x-1)$. On the other hand, the number of arrangements that have exactly t empty cells is $\binom{k}{t}(k-t)! \left\{ {n \atop k-t} \right\} = \frac{k!}{t!} \left\{ {n \atop k-t} \right\}$, which results in the identity $\sum_{0\leq r\leq k} \binom{k}{r}(k-r)^n (x-1)^r = k! \sum_{0\leq t\leq k} \left\{ {n \atop k-t} \right\} \frac{x^t}{t!}$.

Now we can return to approximating the ordered bell numbers. We want to determine the growth rate of $\{b(n)\}$ as n grows large. We can multiply both sides of the identity for Stirling numbers by e^{-y} from 0 to ∞. This yields $\tilde{b}(n) = \sum_{r\geq 0} \frac{r^n}{2^{r+1}}$. Thus, the EGF of the ordered Bell numbers is $f(z) = \sum_{n>0} \frac{\tilde{b}(n)}{n!} z^n = \frac{1}{2-e^z}$.

Luckily, this only has simple poles, at the points $\log 2 \pm 2k\pi i$. The principle part at $z_0 = \log 2$ is $(-1/2)/(z - \log 2)$, which contributes $\frac{1}{2(\log 2)^{n+1}}$ to the coefficients of z^n. There are no other singularities of $f(z)$ with modulus $\log 2$, so $h(z) = f(z) - \frac{(-1/2)}{z - \log 2}$ is analytic in the larger circle of radius $|\log 2 + 2\pi i|$, which equals $\rho = \sqrt{(\log 2)^2 + 4\pi^2}$, which is roughly 6.32. Thus, the coefficients of $h(z)$ are $O((0.16)^n)$. Thus, the ordered Bell numbers $\tilde{b}(n)$ are of the form $\tilde{b}(n) = \frac{1}{2(\log 2)^{n+1}} n! + O((0.16)^n n!)$.

The result can be improved by taking more terms of the asymptotic exapnsion from the principal parts of $f(z)$ at its remaining poles, taken in increasing order of their absolute values.

Example. Permutations with no small cycles.

We found earlier that the EGF of the permutations on n elements with all cycles of length greater than q is $\frac{1}{1-z} e^{-z+\cdots+z^q/q}$. Let q be fixed, and $f_q(n)$ denote the number of such permutations on n elements.

Note that the only singularity of this function is a pole at $z = 1$ with principle part $\frac{e^{-H_q}}{(1-z)}$, where H_q is the qth harmonic number.

In this case, the difference between the function and the its principal part at $z = 1$ is

$$h(z) = f_q(z) - \frac{e^{-H_q}}{1-z} = \frac{e^{-z+\cdots+z^q/q} - e^{-H_q}}{1-z}$$

which is an entire function, as $z = 1$ is a zero of the function in the numerator. This means that the estimate will be very accurate.

From the growth rate theorem, the n-th coefficient of $h(z)$ is $O(\epsilon^n)$ as $n \to \infty$ for every $\epsilon > 0$. This yields the estimate

$$\frac{f(n,q)}{n!} = e^{-H_q} + O(\epsilon^n),$$

as $n \to \infty$.

The error term here is noticeably small, suggesting that the probability that a randomly chosen permutation of order n having only cycles of length strictly greater than q, ie. $\frac{f(n,q)}{n!}$ is actually very close to being independent of n, an interesting result.

4 Asymptotic Growth Rate: Algebraic singularities

Now we move on to consider algebraic singulaities of f, ie. the singularity z_0 of f with smallest modulus is a branch point: $f(z) = (z_0 - z)^\alpha g(z)$ for some non-integer real α, and g is analytic at z_0.

In this section, we derive Darboux's lemma, which allows us to deduce asymptotics for generating functions with algebraic singularities. It turns out that the process is very similar to the case of meromorphic functions, but the proof is nontrivial.

By considering $f(zz_0)$ instead of f, we can assume WLOG that $z_0 = 1$, so f is analytic in the unit disk with a branch point at $z = 1$. Suppose that $z_0 = 1$ is the only singularity of f on some disk $|z| < 1 + \rho$. We can expand g in a power series $g(z) = \sum_{k \geq 0} g_k (1-z)^k$ that converges in a neighborhood of $z = 1$. Plugging this into our expression for f, we obtain the expansion $f(z) = \sum_{k \geq 0} g_k (1-z)^{k+\alpha}$.

It turns out that, like the case of meromorphic functions, each successive term in the above series generates the next term in the asymptotic exapansion of the coefficients of f. Furthermore, the function $g_0(1-z)^\alpha$ has for its coefficient of z^n, the main contribution to that coefficient of f. To prove these results, and the following Darboux theorem, we proceed with a few lemmas.

Lemma 4.1. *Let $\{a_n\}, \{b_n\}$ be two sequences that satisfy (a)$a_n = O(n^{-\gamma})$ and (b)$b_n - O(\theta^n)$ where $0 < \theta < 1$. Then, $\sum_k a_k b_{n-k} = O(n^{-\gamma}$.*

Proof. We first have

$$\begin{aligned}\left|\sum_{0\le k\le n/2} a_k b_{n-k}\right| &\le \{\max_{0\le k\le n/2} |a_k|\}\{\sum_{0\le k\le n/2} C\theta^{n-k}\} \\ &\le \max C, Cn^{-\gamma}\{C'\theta^{n/2}\} \\ &\le C''\tilde{\theta}^n\end{aligned}$$

where $(0 < \tilde{\theta} < 1)$.

Additionally, we have

$$\begin{aligned}\left|\sum_{n/2<k\le n}\right| &\le \{\max_{n/2<k\le n} |a_k|\}\{\sum_{n/2<k\le n} \theta^{n-k}\} \\ &\le C'''n^{-\gamma}\end{aligned}$$

proving the lemma. ☐

Lemma 4.2. *If β is a nonreal integer, then $[z^n](1-z)^\beta \sim \frac{n^{-\beta-1}}{\Gamma(-\beta)}$.*

Proof. We have

$$[z^n](1-z)^\beta = \binom{\beta}{n}(-1)^n = \binom{n-\beta-1}{n} = \frac{(n-\beta-1)!}{n!(-\beta-1)!} = \frac{\Gamma(n-\beta)}{\Gamma(-\beta)\Gamma(n+1)}.$$

By stirling's formula, ie. $\Gamma(n+1) \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$, the result easily follows. ☐

Lemma 4.3. *Let $u(z) = (1-z)^\gamma v(z)$, where $v(z)$ is analytic in some disk $|z| < 1+\rho$. Then, $[z^n]u(z) = O(n^{-\gamma-1})$.*

Proof. Apply lemma with $a_n = [z^n](1-z)^\gamma$ and $b_n = [z^n]v(z)$. Since v is analytic in a disk $|z| < 1+\rho$, we have $b_n = O(\theta^n)$, and the result follows from the second lemma. ☐

We are now ready to prove Darboux's Theorem:

Theorem 4.4. *(Darboux): Let $v(z)$ be analytic in some disk $|z| < 1+\rho$, and suppose that in a neighborhood of $z=1$ it has the expansion $v(z) = \sum v_j(1-z)^j$. For some noninteger real β, we then have*

$$[z^n]\{(1-z)^\beta v(z)\} = [z^n]\{\sum_{j=0}^m v_j(1-z)^{\beta+j}\} + O(n^{-m-\beta-2}) = \sum_{j=0}^m \binom{n-\beta-j-1}{n} + O(n^{-m-\beta-2}).$$

Proof. We have

$$(1-z)^\beta v(z) - \sum_{j=0}^{m} v_j(1-z)^{\beta+j} = \sum_{j>m} v_j(1-z)^{\beta+j} = (1-z)^{\beta+m+1}\tilde{v}(z)$$

and the regions of analyticity of $\tilde{v}$ and v are the same. The result follows from the third lemma. ☐

Example. 2-regular graphs.

Recall that the EGF of the number of 2-regular graphs on n vertices is $f(z) = \frac{e^{-z/2-z^2/4}}{\sqrt{1-z}}$, which has a branch point at $z = 1$. Applying Darboux's Lemma, we have $\beta = -1/2$ and $v(z) = \exp(-z/2 - z^2/4)$. Let $\gamma(n)$ denote the number of

Expanding this about $z = 1$ yields

$$e^{-z/2-z^2/4} = e^{-3/4} + e^{-3/4}(1-z) + \frac{1}{4}e^{-3/4}(1-z)^2 + \cdots,$$

and from Darboux, this expansion leads to an asymptotic formula for the coefficients of $f(z)$, which are $\gamma(n)/n!$, where $\gamma(n)$ is the number of 2-regular graphs of n vertices. Using $m = 2$ in Darboux yields

$$\frac{\gamma(n)}{n!} = e^{-3/4}\binom{n-1/2}{n} + e^{-3/4}\binom{n-3/2}{n} + \frac{1}{4}e^{-3/4}\binom{n-5/2}{n} + O(n^{-7/2}).$$

This can be further simplified with the asymptotic exansion of the binomial coefficient, that $\binom{n-\alpha-1}{n}$ is roughly equal to $\frac{n^{-\alpha-1}}{\Gamma(-\alpha)} \cdot \left(1 + \frac{\alpha(\alpha+1)}{2n} + \frac{\alpha(\alpha+1)(\alpha+2)(3\alpha+1)}{24n^2}\right)$,

and substituting this into our expresion yields

$$\gamma(n) \approx \frac{n!e^{-3/4}}{\sqrt{n\pi}}\{1 - \frac{5}{8n} + \frac{1}{128n^2} + \cdots\}.$$

Darboux's method can be extended to when there are more branch points on the circle of convergence; the result is due to Szego.

5 Hayman's Method

We now introduce *Hayman's Method*, which applies to the case where the generating function in question is entire, that is, it has no singularities to which we can apply the aforementioned analyses. The motivating example of this section will be the exponential e^z. In fact, applying Hayman's Method to obtain the asymptotic growth rate of its coefficients will precisely yield Stirling's Formula.

We first consider a simple yet suboptimal approach. By Cauchy's Integral Formula, we have

$$\frac{1}{n!} = \frac{1}{2\pi i}\int_\Gamma \frac{e^z}{z^{n+1}}dz,$$

where Γ is some path enclosing the origin, which we let be a circle with radius r. Applying the triangle and ML inequalities then yields

$$\begin{aligned}\frac{1}{n!} &\leq \frac{1}{2\pi}\int_\Gamma \left|\frac{e^z dz}{z^{n+1}}\right| \\ &\leq \frac{1}{2\pi}\cdot 2\pi r \cdot \max_{|z|=r}\frac{|e^z|}{|z|^{n+1}} \\ &= \frac{e^r}{r^n}.\end{aligned}$$

e^z has no singularities, so r can be chosen arbitrarily. In particular, we can choose it to minimize this upper bound. To do this, we set the derivative of $\frac{e^r}{r^n}$ to 0 and solve for r:

$$\begin{aligned}\frac{d}{dr}\frac{e^r}{r^n} =& \frac{e^r(r^n - nr^{n-1})}{r^{2n}} = 0 \\ &\implies r = n \\ &\implies \min_{r\geq 0}\frac{e^r}{r^n} = \frac{e^n}{n^n}.\end{aligned}$$

Hence, the bound obtained from this argument is $\frac{1}{n!} \leq (\frac{e}{n})^n$. Comparing to Stirling's Formula, $\frac{1}{n!} \sim \frac{1}{\sqrt{2\pi n}}(\frac{e}{n})^n$, we see that this simple argument gives the correct exponential growth, though we are off by a factor of $\frac{1}{\sqrt{2\pi n}}$.

Definition 5.1. Let $f(z) = \sum_{n\geq 0} a_n z^n$ be a function converging within some radius R, with $0 < R < \infty$. Letting $h(r) = \log(f(r))$, define $a(r) := rh'(r)$ and $b(r) := r^2h''(r) + rh'(r)$. Then we say $f(z)$ is *H-admissible* if:

1. $f(r) > 0$ on some interval (R_0, R).
2. There exists some function $\delta(r)$ defined on (R_0, R) such that $0 \leq \delta \leq \pi$ and such that as $r \to R$ uniformly for $|\theta| \leq \delta(r)$,

$$f(re^{i\theta}) \sim f(r)e^{i\theta a(r) - \frac{1}{2}\theta^2 b(r)}.$$

3. As $r \to R$, uniformly for $\delta(r) \leq |\theta| < \pi$,

$$f(re^{i\theta}) = o\left(\frac{f(r)}{\sqrt{b(r)}}\right)$$

4. $\lim_{r\to R} b(r) = \infty$.

Example. $f(z) = e^z$ is admissible with $\delta(r) = r^{-2/5}$:

1. This condition is clearly satisfied because $e^x > 0$ for all $x \in \mathbb{R}$.
2. We let $R = \infty$, and in order that $\delta(r) = r^{-2/5} < \pi$ for all $r \in (R_0, R)$ be satisfied, we let $R_0 = 1$. Since $a(r)$ and $b(r)$ are both r, we wish to show that for all ε, there

exists an X such that for all $r > X$, for all $|\theta| \leq r^{-2/5}$, $\left|\frac{e^{re^{i\theta}}}{e^{r(1+i\theta-\theta^2/2)}} - 1\right| < \varepsilon$. We let $X = (3\log(1+\varepsilon))^{-5}$ and verify that this choice works:

$$\begin{aligned}
\left|\frac{e^{re^{i\theta}}}{e^{r(1+i\theta-\theta^2/2)}} - 1\right| &= \left|e^{r((1+i\theta-\theta^2/2-i\theta^3/3!+\theta^4/4!+\dots)-(1+i\theta-\theta^2/2))} - 1\right| \\
&= \left|e^{r(-i\theta^3/3!+\theta^4/4!+\dots)} - 1\right| \\
&= \left|\sum_{k=1}^{\infty} \frac{(r(-i\theta^3/3! + \theta^4/4! + \dots))^k}{k!}\right| \\
&\leq \sum_{k=1}^{\infty} \frac{|r(-i\theta^3/3! + \theta^4/4! + \dots)|^k}{k!} \\
&= e^{r|-i\theta^3/3!+\theta^4/4!+\dots|} - 1 \leq e^{r|-i\theta^3/6+\theta^4/24|} - 1 \\
&\leq e^{r(|\theta|^3/6+|\theta|^4/24)} - 1 \leq e^{r|\theta|^3/3} - 1 \qquad \text{(because } |\theta| \leq 1\text{)} \\
&\leq e^{r^{-1/5}/3} - 1 < \varepsilon,
\end{aligned}$$

as desired.

3. We calculate:

$$\begin{aligned}
\left|\frac{e^{re^{i\theta}}\sqrt{r}}{e^r}\right| &= e^{r\mathrm{Re}(e^{i\theta}-1)}\sqrt{r} \\
&= e^{r(\cos\theta-1)}\sqrt{r},
\end{aligned}$$

which tends to 0 for any nonzero θ, because it is well-known that the growth rate of exponentials is larger than that of polynomials with real exponents.

4. Clearly $\lim_{r\to\infty} r = \infty$.

For each of the below results, $f = \sum_{n\geq 0} a_n z^n$ is an admissible function with R, $a(r)$, $b(r)$ and $\delta(r)$ as in its definition above.

Lemma 5.2. *$\lim_{r\to R} \delta(r)^2 b(r) = \infty$.*

Proof. We have:

$$\begin{aligned}
\frac{|f(re^{i\delta})|}{f(r)} &\sim \left|e^{i\delta(r)a(r)-\frac{1}{2}\delta(r)^2 b(r)}\right| = e^{-\delta(r)^2 b(r)/2}, && \text{(by Condition 2)} \\
\frac{|f(re^{i\delta})|}{f(r)} &= o\left(\frac{1}{\sqrt{b(r)}}\right) && \text{(by Condition 3)} \\
&= o(1) && \text{(by Condition 4)} \\
\implies \lim_{r\to\infty} e^{-\delta(r)^2 b(r)/2} &= 0 \implies \lim_{r\to\infty} \delta(r)^2 b(r) = \infty,
\end{aligned}$$

as desired. ∎

Lemma 5.3. *We have*

$$a_n r^n = \frac{f(r)}{\sqrt{2\pi b(r)}}\left(\exp\left(-\frac{(a(r)-n)^2}{2b(r)}\right) + o(1)\right)$$

Proof. We prove this via an estimation of Cauchy's Integral

$$\begin{aligned}
a_n r^n &= \frac{1}{n!}\frac{d^n}{dz^n} f(rz)|^{z=0} \\
&= \frac{1}{2\pi i}\int_{|z|=1} \frac{f(rz)}{z^{n+1}} dz \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(re^{i\theta})}{e^{in\theta}} d\theta \\
&= \frac{1}{2\pi}\bigg(\int_{-\delta(r)}^{\delta(r)} + \int_{\delta(r)}^{2\pi-\delta(r)}\bigg) \frac{f(re^{i\theta})}{e^{in\theta}} d\theta
\end{aligned}$$

Applying the ML inequality and then Condition 4 of admissibility yields

$$\bigg|\int_{\delta(r)}^{2\pi-\delta(r)} \frac{f(re^{i\theta})}{e^{in\theta}} d\theta\bigg| \leq 2(\pi - \delta(r)) \max_{\delta(r)\leq\theta\leq 2\pi-\delta(r)} |f(re^{i\theta})| = \frac{o(f(r))}{\sqrt{b(r)}}$$

uniformly in n as $r \to R$. As for the second integral, from Condition 2 of admissibility, we have

$$\begin{aligned}
\int_{-\delta(r)}^{\delta(r)} \frac{f(re^{i\theta})}{e^{in\theta}} d\theta &= f(r)\int_{-\delta(r)}^{\delta(r)} (1+o(1)) \exp\bigg(i\theta(a(r)-n) - \frac{1}{2}\theta^2 b(r)\bigg) d\theta \\
&= f(r)\bigg(\int_{-\delta(r)}^{\delta(r)} \exp\bigg(i\theta(a(r)-n) - \frac{1}{2}\theta^2 b(r)\bigg) d\theta + o\bigg(\int_{\infty}^{\infty} \exp\bigg(-\frac{1}{2}b(r)\theta^2\bigg) d\theta\bigg)\bigg) \\
&= f(r)\bigg(\int_{-\delta(r)}^{\delta(r)} \exp\bigg(i\theta(a(r)-n) - \frac{1}{2}\theta^2 b(r)\bigg) d\theta + o\bigg(\frac{1}{\sqrt{b(r)}}\bigg)\bigg)
\end{aligned}$$

We approximate this last integral as

$$\begin{aligned}
&\int_{-\delta(r)}^{\delta(r)} \exp\bigg(i\theta(a(r)-n) - \frac{1}{2}\theta^2 b(r)\bigg) d\theta \\
=&\int_{-\delta(r)}^{\delta(r)} \exp\bigg(-\frac{1}{2}\bigg(\theta\sqrt{b(r)} - i\frac{a(r)-n}{\sqrt{b(r)}}\bigg)^2 - \frac{(a(r)-n)^2}{2b(r)}\bigg) d\theta \\
=&\frac{1}{\sqrt{b(r)}} \exp\bigg(-\frac{(a(r)-n)^2}{2b(r)}\bigg) \int\limits_{-\delta(r)\sqrt{b(r)}-i\frac{a(r)-n}{\sqrt{b(r)}}}^{\delta(r)\sqrt{b(r)}-i\frac{a(r)-n}{\sqrt{b(r)}}} e^{-t^2/2} dt \\
=&\frac{1}{\sqrt{b(r)}} \exp\bigg(-\frac{(a(r)-n)^2}{2b(r)}\bigg) \int\limits_{-\delta(r)\sqrt{b(r)}}^{\delta(r)\sqrt{b(r)}} e^{-t^2/2} dt \quad \text{(by Cauchy's Integral Theorem)} \\
=&\sqrt{\frac{2\pi}{b(r)}} \exp\bigg(-\frac{(a(r)-n)^2}{2b(r)}\bigg)(1+o(1)) \quad \text{(By Lemma 5.2).}
\end{aligned}$$

Finally, combining each of these estimates gives us

$$
\begin{aligned}
a_n r^n &= \frac{f(r)}{2\pi}\sqrt{\frac{2\pi}{b(r)}}\exp\left(-\frac{(a(r)-n)^2}{2b(r)}\right)(1+o(1)) + \frac{o(f(r))}{\sqrt{b(r)}} \\
&= \frac{f(r)}{\sqrt{2\pi b(r)}}\exp\left(-\frac{(a(r)-n)^2}{2b(r)}\right) + \frac{f(r)}{\sqrt{2\pi b(r)}}\exp\left(-\frac{(a(r)-n)^2}{2b(r)}\right)o(1) + \frac{f(r)}{\sqrt{2\pi b(r)}}o(1) \\
&= \frac{f(r)}{\sqrt{2\pi b(r)}}\exp\left(-\frac{(a(r)-n)^2}{2b(r)}\right) + \frac{f(r)}{\sqrt{2\pi b(r)}}o(1)\left(\exp\left(-\frac{(a(r)-n)^2}{2b(r)}\right)+1\right) \\
&= \frac{f(r)}{\sqrt{2\pi b(r)}}\left(\exp\left(-\frac{(a(r)-n)^2}{2b(r)}\right)+o(1)\right),
\end{aligned}
$$

where the last equality follows from noting that $\exp\left(-\frac{(a(r)-n)^2}{2b(r)}\right)$ cannot approach infinity as $r \to R$, because $a(r)$ and $b(r)$ are nonzero for $r \in (R_0, R)$. □

Lemma 5.4. *There exists an $R_1 < R$ such that $a(r)$ is strictly monotonically increasing in (R_1, R). Furthermore, we have $\lim_{r\to R} a(r) = \infty$.*

Proof. The former statement follows immediately from the fact that $b(r) = ra'(r) \to \infty$ as $r \to R$. As for the latter statement, substituting $n = -1$ into Lemma 5.3 yields

$$
\begin{aligned}
0 =& f(r)\left(\exp\left(-\frac{(a(r)+1)^2}{2b(r)}\right)+o(1)\right) \\
\implies& \exp\left(-\frac{(a(r)+1)^2}{2b(r)}\right) = o(1) \\
\implies& \lim_{r\to R}\frac{(a(r)+1)^2}{2b(r)} = \infty,
\end{aligned}
$$

which, since $\lim_{r\to R} b(r) = \infty$, implies the desired result. □

Lemma 5.4 shows that, for sufficiently large integers n, the equation $a(r) = n$ has a unique solution r_n which satisfies $r_n \to R$ as $n \to \infty$. This observation, in light of Lemma 5.3, leads us to the main result of this section.

Theorem 5.5. *Let r_n be the positive real root of the equation $a(r_n) = n$, for $n \in Z^+$. Then*

$$a_n \sim \frac{f(r_n)}{r_n^n\sqrt{2\pi b(r_n)}} \text{ as } n \to +\infty.$$

Example. Since we have shown that $f(z) = e^z$ is admissible and that $a(r) = b(r) = r$, Stirling's Formula $\frac{1}{n!} \sim \frac{1}{\sqrt{2\pi n}}\left(\frac{e}{n}\right)^n$ follows immediately from substituting $r_n = n$.

Remark 5.6. Notice that once we know that a certain function is admissible, applying Theorem 5.5 is very straightforward. The "hard work" is in showing that a function is in fact admissible.

6 Saddle point method

The saddle point method helps to find good approximations to integrals of the form

$$\int_{\gamma(t)} F(z,t)dz,$$

where t is a real parameter, and F is analytic with respect to z in some domain $G(t) \in \mathbb{C}$ containing the path $\gamma(t)$. As we will see below, such integrals arise from applying Cauchy's Integral Formula to the coefficients of an EGF.

Before diving into an example, we provide a brief outline of the Saddle Point Method:

1. Substitute the path of integration with another one, call it $\sigma(t)$, without changing the value of the integral (i.e. without crossing singular points) such that along $\sigma(t)$, $|F(z,t)|$ has some sharp peaks and is small everywhere else.

2. Apply the Method of Laplace:

   (a) Choose neighborhoods of these peaks large enough that the main contribution to the value of the integral is being captured,

   (b) In these neighborhoods, substitute the integrand with simpler functions.

   (c) Asymptotically estimate the resulting integrals.

Remark 6.1. The name "saddle point method" comes from the way of finding sharp peaks when choosing an appropriate path: By the Maximum Modulus Principle, $|F(z,t)|$ does not have any maxima or minima other than zeroes in the interior of $G(t)$ (the latter of which comes from applying the Principle to $\frac{1}{F(z,t)}$). Thus, the only points where $\frac{d}{dz}|F(z,t)| = 0$ and $F(z,t) \neq 0$ are saddle points.

Suppose that $\zeta \in G(t)$ is a saddle point of $F(z) = F(z,t)$, that is, $F(\zeta) \neq 0, F'(\zeta) = \cdots = F^{(k-1)}(\zeta) = 0$ and $F^{(k)}(\zeta) \neq 0$ for some $k \geq 2$. Furthermore, suppose that $k = 2$, so that, for $|z-\zeta|$ small enough, the function $\log F(z)$ can be expanded as

$$\log F(z) = \log F(\zeta) + \frac{F''(\zeta)}{F(\zeta)}\frac{(z-\zeta)^2}{2} + O((z-\zeta)^3).$$

Since $|F(z)| = e^{\Re \log F(z)}$, $|F(z)|$ is of fastest decrease for the same z that $\Re \log F(z)$ is of fastest decrease. It follows that, since

$$\Re \log F(z) - \Re \log F(\zeta) = \Re\left(\frac{F''(\zeta)}{F(\zeta)}\frac{(z-\zeta)^2}{2}\right) + O((z-\zeta)^3)$$

is minimized when $\frac{(z-\zeta)^2}{2} = -\overline{\frac{F''(\zeta)}{F(\zeta)}}$, or $\arg(z-\zeta) = \pm\frac{1}{2}\left(\pi - \arg\frac{F''(\zeta)}{F(\zeta)}\right)$, $|F(z)|$ is of fastest decrease for

$$z = \zeta + r\exp\left(\frac{i}{2}\left(\pi - \arg\frac{F''(\zeta)}{F(\zeta)}\right)\right),$$

with $r \in \mathbb{R}$. This line is called the saddle point's *axis* or *direction of steepest descent*. Thus, for suitable functions, the path σ in Step 1 should be chosen such that the highest points of $|F(z)|$ along σ are also saddle points of $|F(z)|$, and in small neighborhoods of such a saddle point, σ approximates its axis.

Example. We seek to estimate the number of involutions of length n as $n \to \infty$. As shown in the sixth example of section 1.3, the EGF counting these involutions is

$$F(z) = \sum_{n=0}^{\infty} F_n \frac{z^n}{n!} = e^{z+z^2/2}.$$

The first step in our asymptotic analysis is to apply the residue theorem, which yields

$$\frac{F_n}{n!} = \frac{1}{2\pi i} \oint \frac{e^{z+z^2/2}}{z^{n+1}}\, dz,$$

where the path of integration encircles the origin exactly once, counterclockwise. We set the derivative of the integrand to 0 to determine the saddle points:

$$\begin{aligned}
\frac{d}{dz}\frac{e^{z+z^2/2}}{z^{n+1}} &= \frac{d}{dz}\exp\left(z + \frac{z^2}{2} - (n+1)\log z\right) \\
&= \exp\left(z + \frac{z^2}{2} - (n+1)\log z\right)\left(1 + z - \frac{n+1}{z}\right) = 0 \\
&\implies z^2 + z = n+1.
\end{aligned}$$

Thus, there are two saddle points, namely,

$$-\frac{1}{2} \pm \sqrt{\frac{5}{4} + n} = -\frac{1}{2} \pm \sqrt{n}\Big(1 + \frac{5}{8}n^{-1} + O(n^{-2})\Big).$$

It will turn out that we only need to work with one of these saddle points; let us therefore (arbitrarily) consider the saddle point at $\zeta_n = -\frac{1}{2} + \sqrt{\frac{5}{4} + n}$. Letting $h(z) = z + \frac{z^2}{2} - (n+1)\log z$ be the logarithm of the integrand, the power series expansion of h centered in ζ_n is given by

$$h(z) = h(\zeta_n) + \left(1 + \frac{n+1}{\zeta_n^2}\right)\frac{(z-\zeta_n)^2}{2} + \sum_{k=3}^{\infty}(-1)^k \frac{n+1}{k}\left(\frac{z-\zeta_n}{\zeta_n}\right)^k, \tag{6.1}$$

which converges for $|z - \zeta_n| < \zeta_n$. Because the coefficient of $(z-\zeta)^2$ is real, the axis of the saddle point ζ_n is perpendicular to the real line. We therefore show that the path $\gamma = \gamma_1 + \gamma_2$ given by

$$\begin{aligned}
\gamma_1 &= \{z : z = \zeta_n + it, -\delta_n \le t \le \delta_n\} \\
\gamma_2 &= \{z : |z|^2 = \zeta_n^2 + \delta_n^2, \arg(\zeta_n + i\delta_n) \le \arg(z) \le 2\pi - \arg(\zeta_n + i\delta_n)\},
\end{aligned}$$

with $\delta_n \in \mathbb{R}$ to be determined, can be used to estimate the integral in question. Specifically, we seek to do this by replacing $\int_{\gamma_1} e^{h(z)}$ with a sufficiently well-approximating complete Gaussian integral and showing that $\int_{\gamma_2} e^{h(z)}$ tends to 0 as $n \to \infty$. In order to successfully do the former, δ_n must be chosen such that for $z \in \gamma_1$, we have

1. $h(z) \sim h(\zeta_n) + h''(\zeta_n)(z-\zeta_n)^2$, and
2. $h''(\zeta_n)\delta_n^2 \to \infty$

as $n \to \infty$. The last sum of (6.1) can be rewritten as $-(n+1)\left(\frac{z-\zeta_n}{\zeta_n}\right)^3 \sum_{k=0}^{\infty} \frac{(-1)^k}{k+3}\left(\frac{z-\zeta_n}{\zeta_n}\right)^k$, and since $\zeta_n \sim \sqrt{n}$, we have $(n+1)(z-\zeta_n)^3\zeta_n^{-3} \sim (n^{-1/2}+n^{-3/2})(z-\zeta_n)^3$. So, the first condition is satisfied if δ is chosen small enough so that

$$\delta^3 = o(\sqrt{n}) \quad \text{as } n \to \infty. \tag{6.2}$$

The quantity $h''(\zeta_n) = 1 + \frac{n+1}{\zeta_n^2}$ tends to 2 as $n \to \infty$, because $\zeta_n^2 \sim n$. Thus, in order to satisfy the second condition, δ has to be chosen such that

$$\delta^2 \to \infty \quad \text{as } n \to \infty. \tag{6.3}$$

We will use $\delta = n^{1/8}$, since this choice satisfies both (6.2) and (6.3). First, let us show that the integral over γ_2 vanishes as $n \to \infty$. For $z \in \gamma_2$, we have

$$\begin{aligned}
|e^{h(z)}| = e^{\Re h(z)} &\le \exp\left(\zeta_n + \frac{\zeta_n^2}{2} - (n+1)\log\zeta_n\right) \\
&= \exp\left((n+1)(1-\log\zeta_n) - \frac{\zeta_n^2}{2}\right) \qquad (\text{Recalling } n+1 = \zeta_n^2 + \zeta_n) \\
&= \left(\frac{e}{\zeta_n}\right)^{n+1} e^{-\zeta_n^2/2}.
\end{aligned}$$

Applying the ML inequality to the integral over γ_2 then yields

$$\begin{aligned}
\left|\int_{\gamma_2} e^{h(z)}\,dz\right| &\le 2\pi\sqrt{\zeta_n^2+\delta^2}\left(\frac{e}{\zeta_n}\right)^{n+1} e^{-\zeta_n^2/2} \\
&= 2\pi\sqrt{1+\left(\frac{\delta}{\zeta_n}\right)^2}\left(\frac{e}{\zeta_n}\right)^{n} e^{-\zeta_n^2/2+1},
\end{aligned}$$

which indeed tends to zero as $n \to \infty$.

For $z \in \gamma_1$, we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma_1} e^{h(z)}dz &= \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{h(\zeta_n+it)}\,dt \\
&= \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{h(\zeta_n)-h''(\zeta_n)t^2/2}(1+O(n\delta^3\zeta_n^{-3}))\,dt \qquad (\text{since } e^{O(z)} = 1+O(z)) \\
&= \frac{e^{h(\zeta_n)}(1+O(n^{-1/8}))}{2\pi\sqrt{h''(\zeta_n)}} \int\limits_{-\delta\sqrt{h''(\zeta_n)}}^{\delta\sqrt{h''(\zeta_n)}} e^{-u^2/2}\,du \qquad \big(u = \sqrt{h''(\zeta_n)}t\big) \\
&= \frac{e^{h(\zeta_n)}(1+O(n^{-1/8}))}{2\pi\sqrt{h''(\zeta_n)}}\bigg(\int_{-\infty}^{\infty} e^{-u^2/2} + O\bigg(\int_{\delta}^{\infty} e^{-u^2/2}\,du\bigg)\bigg) \quad (\text{Since } \sqrt{h''(\zeta_n)} \to \sqrt{2} > 1) \\
&= \frac{e^{h(\zeta_n)}(1+O(n^{-1/8}))}{2\pi\sqrt{h''(\zeta_n)}}\bigg(\int_{-\infty}^{\infty} e^{-u^2/2} + O\bigg(\int_{\delta}^{\infty} -ue^{-u^2/2}\,du\bigg)\bigg) \\
&= \frac{e^{h(\zeta_n)}(1+O(n^{-1/8}))}{2\pi\sqrt{h''(\zeta_n)}}\bigg(\int_{-\infty}^{\infty} e^{-u^2/2} + O\Big(e^{-\delta^2/2}\Big)\bigg) \\
&= \frac{e^{h(\zeta_n)}}{\sqrt{2\pi h''(\zeta_n)}}\big(1+O(n^{-1/8})\big).
\end{aligned}$$

Since $\zeta_n^2 = n - \sqrt{n} + \frac{3}{2} + O(n^{-1/2})$ and since

$$\begin{aligned}
\log\zeta_n^{n+1} &= (n+1)\log\left(\sqrt{n}\left(1 - \frac{1}{2\sqrt{n}} + \frac{5}{8n} + O(n^{-2})\right)\right) \\
&= (n+1)\log\sqrt{n} - (n+1)\left(\frac{1}{2\sqrt{n}} - \frac{5}{8n} + O(n^{-2}) + \frac{1}{2}\left(\frac{1}{2\sqrt{n}} - \frac{5}{8n} + O(n^{-2})\right)^2 + O(n^{-3/2})\right) \\
&= (n+1)\log\sqrt{n} - (n+1)\left(\frac{1}{2\sqrt{n}} - \frac{5}{8n} + O(n^{-2}) + \frac{1}{2}\left(\frac{1}{4n} + O(n^{-3/2})\right)\right) \\
&= (n+1)\log\sqrt{n} - \left(\frac{\sqrt{n}}{2} - \frac{1}{2} + O(n^{-1/2})\right),
\end{aligned}$$

we have

$$\begin{aligned}
e^{h(\zeta_n)} = e^{\zeta_n+\zeta_n^2/2}\,\zeta_n^{-(n+1)} &= \frac{\exp\big(n/2 + \sqrt{n}/2 + 1/4 + O(n^{-1/2})\big)}{n^{(n+1)/2}\exp\big(-\sqrt{n}/2 + 1/2 + O(n^{-1/2})\big)} \\
&= \frac{1}{\sqrt{n}^{n+1}}\exp\left(\frac{n}{2} + \sqrt{n} - \frac{1}{4}\right)(1+O(n^{-1/2})).
\end{aligned}$$

Combining this with the above result gives us

$$\frac{F_n}{n!} \sim \frac{e^{h(\zeta_n)}}{\sqrt{2\pi h''(\zeta_n)}} \sim \frac{1}{2\sqrt{\pi n^{n+1}}}\exp\left(\frac{n}{2} + \sqrt{n} - \frac{1}{4}\right).$$

Finally, using Stirling's formula $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, we obtain

$$F_n \sim \frac{n^{n/2}}{\sqrt{2}}\exp\left(\frac{n}{2} + \sqrt{n} - \frac{1}{4}\right) \qquad \text{as } n \to \infty.$$

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