QUATERNIONIC ANALYSIS

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1. Introduction to Quaternions

Quaternions are an extension of the complex number system; they were first developed and studied by William Hamilton in the 19th century, who, after unsuccessfully trying to extend the complex numbers to three dimensions, extended it to four. The field of quaternionic analysis, though lacking in many of the nice properties and results of complex analysis, has a variety of applications in both pure and applied mathematics. In this paper, we present results taken from [1, 2, 3].

The set of quaternions is represented by $\mathbb{H}$, for Hamilton, and is a 4-dimensional vector space over $\mathbb{R}$, with a basis of $1, i, j, k$ where *i, j, k* are the basic quaternions (behaving much in the same way as the imaginary unit i).

Definition 1.1 (Quaternions)**.** The set of quaternions $\mathbb{H}$ is a field extension over $\mathbb{R}$ and over $\mathbb{C}$. The field of quaternions is represented as

$$\mathbb{H} = \{t + xi + yj + zk \,|\, t, x, y, z \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}.$$

Note that the last part of this definition implies anticommutativity for multiplication of the basic quaternions (ie. that $ij = -ji$ and so on for any two of the three); multiplication among the quaternions can be inferred from this definition.

Moreover, a field extension is defined as follows; a field $\mathbb{K}$ extends a field $\mathbb{F}$ if the elements of $\mathbb{F}$ are a subset of the elements of $\mathbb{K}$ and if the operations of $\mathbb{E}$ are those of $\mathbb{F}$ restricted to $\mathbb{E}$. In this case, we do have $\mathbb{R} \in \mathbb{H}$ and $\mathbb{C} \in \mathbb{H}$, by setting the coefficients $x, y, z = 0$ and $y, z = 0$ respectively, and the two operations of the field (addition and multiplication) remain the same when restricted to $\mathbb{R}$ and $\mathbb{C}$; also, $\mathbb{H}$ is a vector space over $\mathbb{R}$ and $\mathbb{C}$ (this follows from it being a field extension).

Definition 1.2. For quaternions, we have the following definitions (many similar to those found in complex analysis):

i. Quaternionic multiplication can be inferred from the multiplication of the basic quaternions.
ii. The conjugate of a quaternion q is $\bar{q} = t - xi - yj - zk$. Note that, because of anticommutativity, $\overline{pq} = \bar{q}\bar{p}$. Like in complex analysis, the conjugate can be used to isolate the real and imaginary parts of a quaternion ($\frac{1}{2}(q + \bar{q})$ and $\frac{1}{2}(q - \bar{q})$ respectively).
iii. The norm of a quaternion $q = t + xi + yj + zk$, $|q|$, is defined as $\sqrt{q\bar{q}} = \sqrt{t^2 + x^2 + y^2 + z^2}$.
iv. The multiplicative inverse of a quaternion $q = t + xi + yj + zk$ is $\frac{\bar{q}}{|q|^2}$.
v. The distance $d(p, q)$ between two quaternions p, q is $|p - q|$, making $\mathbb{H}$ a metric space.

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Remark 1.3 (Other properties of quaternions)**.** There are several other notable or useful properties of quaternions; For example, we have

i. The center of $\mathbb{H}$ is the real quaternions; this is to say that the elements of $\mathbb{R}$ commute with all other elements quaternions (or that $aq = qa$ for $a \in \mathbb{R}$, $q \in \mathbb{H}$). No other quaternions are in the center of $\mathbb{H}$ due to the anticommutativity of the basic quaternions.
ii. The vector space spanned by 1 and any $q \in \mathbb{H}$ is a subfield of the quaternions.
iii. If 1 and $q \in \mathbb{H}$ are linearly independent, the aforementioned subfield is isomorphic to $\mathbb{C}$.

The proofs of these facts are all relatively straightforward from the definition of $\mathbb{H}$ and from the definition of quaternion multiplication.

Remark 1.4. Quaternions can also be represented in various other forms, such as as an ordered pair of complex numbers (a, b), as a scalar and 3-vector, a 2×2 matrix of complex numbers and a 4×4 matrix of real numbers. In this paper, the most important alternative representation shall be the last.

A quaternion $q = t + xi + yj + zk$ may also be represented as the 4×4 matrix

$$M = \begin{bmatrix} t & x & y & z \\ -x & t & -z & y \\ -y & z & t & -x \\ -z & -y & x & t \end{bmatrix}$$

Remark 1.5. In some instances throughout this paper, we will also represent quaternions in the form

$$q = t + x_1 e_1 + x_2 e_2 + x_3 e_3,$$

where $e_1, e_2, e_3 = i, j, k$ and $x_1, x_2, x_3 = x, y, z$, for the purpose of easier indexing in summations.

2. Differential Forms for Quaternions

To make precise the several different notions of differentiability in quaternionic analysis, in this section, we introduce several differential forms.

Definition 2.1. The gradient operator for a function $f : \mathbb{H} \to \mathbb{H}$, $\Box$, is defined by

$$\Box = \frac{\partial}{\partial t} + i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

Definition 2.2. The Laplace operator for a function $f : \mathbb{H} \to \mathbb{H}$, Δ, is defined by

$$\Delta f = \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Definition 2.3 (Real-differentiability)**.** If a quaternionic function $f : \mathbb{H} \to \mathbb{H}$ is differentiable with respect to its individual real and imaginary components, we consider it to be real-differentiable. Its differential at a point q is then denoted $df_q : \mathbb{H} \to \mathbb{H}$ and can be written as the 1-form

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

Then, the differential of the identity function, dq, is defined as

$$dq = dt + idx + jdy + kdz,$$

which we will use throughout the rest of this paper.

Definition 2.4. We also define the wedge product, $\theta \wedge \phi$, on quaternion n-forms. If θ is an r-form and ϕ is an s-form, their wedge product is defined as

$$\theta \wedge \phi \left(h_1, \ldots, h_{r+s}\right) = \frac{1}{r!s!} \sum_{\rho} \epsilon(\rho) \theta \left(h_{\rho(1)}, \ldots, h_{\rho(r)}\right) \phi \left(h_{\rho(r+1)}, \ldots, h_{\rho(r+s)}\right)$$

with ρ iterating over the set of permutations of $r+s$ objects and $\epsilon(\rho)$ being the permutation's sign ($+1$ if ρ is even and -1 if it is odd).

Remark 2.5. Note the following properties of the wedge, or exterior, product of quaternionic n-forms:

$$\begin{aligned} a(\theta \wedge \phi) &= (a\theta) \wedge \phi, \\ (\theta \wedge \phi)a &= \theta \wedge (\phi a), \\ (\theta a) \wedge \phi &= \theta \wedge (a\phi). \end{aligned}$$

We also define the operators v and $d\mathbf{Q}$ through the usage of these.

Definition 2.6. The alternating symbol for 3 dimensions, or ϵ_{ijk}, where $\{i, j, k\} \in \{1, 2, 3\}$, is defined as follows:

$$\epsilon_{ijk} = \begin{cases} 1 & (i, j, k) \text{ is an even permutation of } (1,\ 2,\ 3) \\ -1 & (i, j, k) \text{ is an odd permutation of } (1,\ 2,\ 3) \\ 0 & i = j \text{ or } j = k \text{ or } k = i. \end{cases}$$

Remark 2.7. The wedge product of dq with itself is as follows:

$$dq \wedge dq = idy \wedge dz + jdz \wedge dx + kdx \wedge dy.$$

Definition 2.8. We also define the operator v, where

$$v = dt \wedge dx \wedge dy \wedge dz$$

and $v(1, i, j, k) = 1$.

With all these pieces, we are now also able to define the differential operator $d\mathbf{Q}$, which is used in section 5 in the quaternionic analogue to Cauchy's theorem.

Definition 2.9. The differential operator $d\mathbf{Q}$ is defined by Deavours in the following somewhat abstract way:

$$d\mathbf{Q} = d\mathbf{Q}_0 + d\mathbf{Q}_1 i + d\mathbf{Q}_2 j + d\mathbf{Q}_3 k,$$

or the outwardly directed surface elements of $\partial\sigma$, the boundary hypersurface of a simply connected domain D in 4-dimensional Euclidean space.

Sudbery defines $d\mathbf{Q}$ in an equivalent but significantly more involved fashion, which we include here for the sake of some of the more technical proofs.

Definition 2.10. $d\mathbf{Q}$ is also defined as the following 3-form:

$$\langle h_1, Dq\left(h_2, h_3, h_4\right)\rangle = v\left(h_1, h_2, h_3, h_4\right),$$

which simplifies to the wedge product

$$d\mathbf{Q} = dx \wedge dy \wedge dz - idt \wedge dy \wedge dz - jdt \wedge dz \wedge dx - kdt \wedge dx \wedge dy$$

Theorem 2.11. $d\boldsymbol{Q}(a,b,c) = \frac{1}{2}(c\bar{a}b - b\bar{a}c)$.

Proof. Because $q \mapsto uq$ is an orthogonal transformation (i.e., one that preserves symmetric inner products), we have

$$d\mathbf{Q}(ua, ub, uc) = ud\mathbf{Q}(a,b,c).$$

Therefore, we set $u = |a|^{-1}a$, and therefore

$$|a|^{-2}d\mathbf{Q}(a,b,c) = ad\mathbf{Q}(1, a^{-1}b, a^{-1}c).$$

We also have $d\mathbf{Q}(1,h_1,h_2) = \frac{1}{2}(h_2h_1 - h_1h_2)$, by applying linearity to the equation $dQ(1,e_i,e_j) = -\epsilon_{ijk}e_k = \frac{1}{2}(e_je_i - e_ie_j)$. Therefore we have

$$d\mathbf{Q}(a,b,c) = \frac{1}{2}|a|^2 a\left(a^{-1}ca^{-1}b - a^{-1}ba^{-1}c\right)$$

and therefore $d\mathbf{Q} = \frac{1}{2}(c\bar{a}b - b\bar{a}c)$. □

There are a few important consequences of this generalization; for our purposes, we see that the argument above can be generalized to show that

$$d\mathbf{Q}(ah_1b, ah_2b, ah_3b) = |a|^2|b|^2 a\, d\mathbf{Q}(h_1,h_2,h_3)b. \tag{2.1}$$

Definition 2.12. The differential operators $\partial_\ell f$ and $\bar{\partial}_\ell f$ are defined as

$$\partial_\ell f = \frac{1}{2}\bar{\Gamma}_r(df) = \frac{1}{2}\left(\frac{\partial f}{\partial t} - i\frac{\partial f}{\partial x} - j\frac{\partial f}{\partial y} - k\frac{\partial f}{\partial z}\right)$$

$$\bar{\partial}_\ell f = \frac{1}{2}\Gamma_r(df) = \frac{1}{2}\left(\frac{\partial f}{\partial t} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right)$$

where Γ_r is the map

$$\Gamma_r(df) = \frac{\partial f}{\partial t} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}.$$

3. Regular Functions

In quaternionic analysis, unlike complex analysis, the notions of holomorphicity, analyticity, harmonicity, and conformality do not coincide and do not provide a suitable analogue of the holomorphic functions that we study in complex analysis. In this section, however, we search for an analogue to holomorphic functions from complex analysis, namely regular functions, the main subjects of our analysis. We find that although the four concepts of holomorphicity, analyticity, harmonicity, and conformality are not equivalent in complex analysis, each of these classes of functions has a unique role to play in the construction of regular functions.

3.1. **Quaternionic Differentiability.**

Definition 3.1 (Differentiability)**.** A quaternionic function $f : \mathbb{H} \to \mathbb{H}$ is differentiable on the left at a point q if the limit

$$\lim_{h\to 0} h^{-1}(f(q+h) - f(q))$$

exists as $h \to 0$ from any direction in $\mathbb{H}$.

Evidently, an immediate issue arises with commutativity; there exists both a left difference quotient and a right difference quotient, depending on if the h^{-1} term occurs to the right or to the left of the limit term.

Also, however, as we will see, this definition is not very useful, as the only quaternion-differentiable functions are linear.

Theorem 3.2. *If a function is quaternion-differentiable on the left on a connected open set U, then on U it must be of the form $f(q) = a + qb$ where $a, b \in \mathbb{H}$, and if it is differentiable on the right, it must be of the form $f(q) = a + bq$.*

Proof. See [3]. □

3.2. **Quaternionic analyticity.** We turn to analytic quaternionic functions, which are, obviously, constructed from quaternionic monomials, which must be defined in a special way due to the anticommutativity of quaternions.

Definition 3.3 (Quaternionic monomials)**.** A quaternionic monomial $f(q)$ is of the form $\prod_{i=1}^{r} a_i q$, where $a_i \in \mathbb{H}$ and r is a nonnegative integer. Note that because of the anticommutativity of quaternions, it is impossible to move the a_i's to the left of the power of q.

Quaternionic polynomials are finite sums of quaternionic monomials.

Remark 3.4. The notion of a function being analytic in a neighbourhood of the origin in $\mathbb{H}$ are equivalent to the notion of a function being real-analytic in 4 real variables in a neighbourhood of the origin.

Proof. This is due to a key difference between complex numbers and quaternions; unlike complex numbers, each of the components (t, x, y, z) of a quaternion can be written as a polynomial in the quaternion; namely, we have

$$\begin{aligned} t &= \frac{1}{4}(q - iqi - jqj - kqk), \\ x &= \frac{1}{4i}(q - iqi + jqj + kqk), \\ y &= \frac{1}{4j}(q + iqi - jqj + kqk), \\ z &= \frac{1}{4k}(q + iqi + jqj - kqk). \end{aligned}$$

Therefore, *any* polynomial - and thus real-analytic in 4 dimensions - function of $q = t + xi + yj + zk$ can be expressed as a quaternionic polynomial, and is analytic under our established condition. □

As complex analyticity is a much stricter condition than analyticity in 2 variables, this idea of quaternionic analyticity is evidently not restrictive enough, so analyticity is not a useful measure for quaternionic functions either.

After exhausting these two options, we turn to a new class of functions, developed by Fueter over a century after Hamilton's discovery of the quaternions.

3.3. Regular functions.

Definition 3.5. We define the left and right Cauchy–Fueter equations as follows:

$$\frac{\partial_l f}{\partial \bar{q}} = \frac{\partial f}{\partial t} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

and

$$\frac{\partial_r f}{\partial \bar{q}} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$$

Definition 3.6 (Regular function)**.** The main functions of interest in quaternionic analysis are regular functions. A quaternionic function is left- or right-regular if

$$\frac{\partial_l f}{\partial \bar{q}} = 0 \text{ or } \frac{\partial_r f}{\partial \bar{q}} = 0,$$

respectively.

The concept of regular functions, though useful in some senses that we will see in the coming sections, is still quite bizarre. Note the following examples:

Example. The identity function $f(q) = q$ is neither right- nor left-regular function, as

$$\frac{\partial_l f}{\partial \bar{q}} = \frac{\partial f}{\partial t} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = 1 - 1 - 1 - 1 = -2 \neq 0,$$

and similarly $\frac{\partial_r f}{\partial \bar{q}} = -2$.

Moreover, no polynomials are regular.

Quaternionic analysis brings with it from complex analysis the concepts of conformality and harmonicity, however; this shall be useful in section 4, but we redefine them here.

Definition 3.7 (Harmonicity)**.** Harmonic functions are those which satisfy Laplace's equation, which for a function on $\mathbb{R}^n$ is

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0.$$

In the context of quaternions, this means that a function $f(q)$ satisfies the equation

$$\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f = 0.$$

In the world of complex analysis, typical harmonic functions are the real and imaginary parts of holomorphic functions. This correspondence does not hold for higher dimensions, but in the case of quaternions, harmonic functions can be used to construct regular functions in several ways.

Definition 3.8 (Conformality)**.** Conformal mappings in quaternionic analysis are the same as those in complex analysis; they are mappings which are locally angle-preserving.

Conformality is useful for constructing regular functions, as conformal transformations of regular functions, as we will see, produce regular functions.

Theorem 3.9. *The conformal group* $\mathrm{SL}(2,\mathbb{H})$ *(or group of all conformal mappings* $\mathbb{H}^* \to \mathbb{H}^*$ *where* $\mathbb{H}^* = \mathbb{H} \cup \{\infty\}$*) of* $\mathbb{H}$ *is useful for the study of regular functions, and consists of functions of the form*

$$f(q) = (aq+b)(cq+d)^{-1}.$$

Proof. We denote the group of conformal mappings of $\mathbb{H}^*$ as C, and the group of mappings of the form $f(q) = (aq+b)(cq+d)^{-1}$ as D. Then the differential of functions $f \in D$ is

$$df_q = \left(ac^{-1}d - b\right)(cq+d)^{-1}cdq(cq+d)^{-1}.$$

df_q is of the form $\alpha dq\beta$, signifying a combination of a rotation and a dilation, and thus is a conformal mapping and is in D. Therefore $D \subset C$. Furthermore, conformal mappings are constructed from rotations, dilations, translations, and inversions followed by reflections in the unit sphere. These mappings are of the form $q \mapsto \alpha q \beta$, $q \mapsto q + \gamma$, and $q \mapsto q^{-1}$, and if such a mapping is applied to a function in D, said function remains in D. Therefore, $CD \subseteq D$, and it follows that $C = D$. □

4. Constructing Regular Functions

There are two ways to construct regular functions from real harmonic functions, the first being through the differential operator $\partial_l f$ and the second by considering it as, locally, the real part of a regular function.

Proposition 4.1. *If* f *is a harmonic real-valued function, then* $\partial_l f$ *is regular.*

Proof. See [3] for an explanation using the fact that that $\Delta = 4\bar{\partial}_\ell \partial_\ell$. □

Theorem 4.2. *If* u *is a harmonic real-valued twice-differentiable function defined on an open star-shaped set* $U \in \mathbb{H}$*, there exists a regular function* f *on* U *such that* $\operatorname{Re} f = u$.

Proof. Without loss of generality, we assume that U is star-shaped with respect to the origin; our construction can be modified easily to fit other points. We show that the function

$$f(q) = u(q) + 2\ \mathrm{Pu} \int_0^1 s^2 \partial_\ell s u(sq) q\, ds$$

is regular, where Pu denotes the pure quaternion, or non-real, parts of the integral.

We have

$$\begin{aligned} \operatorname{Re} \int_0^1 s^2 \partial_\ell u(sq) q\, ds &= \frac{1}{2} \int_0^1 s^2 \left\{ t\frac{\partial u}{\partial t}(sq) + x_i \frac{\partial u}{\partial x_i}(sq) \right\} ds \\ &= \frac{1}{2} \int_0^1 s^2 \frac{d}{ds}[u(sq)]\, ds \\ &= \frac{1}{2} u(q) - \int_0^1 s u(sq)\, ds, \end{aligned}$$

so, rearranging, we have

$$f(q) = 2 \int_0^1 s^2 \partial_\ell u(sq) q\, ds + 2 \int_0^1 s u(sq)\, ds.$$

Differentiating, we have

$$\bar{\partial}_\ell f(q) = 2\int_0^1 s^2 \bar{\partial}_\ell \left[\partial_\ell u(sq)\right] q\, ds + \int_0^1 s^2 \left\{\partial_\ell u(sq) + e_i \partial_\ell u(sq) e_i\right\}\, ds + 2\int_0^1 s^2 \bar{\partial}_\ell u(sq)\, ds$$

Because u is harmonic in U, we have $\bar{\partial}_\ell \left[\partial_\ell u(sq)\right] = \frac{1}{4} s \Delta u(sq) = 0$. Also, $\partial_\ell u(sq) + e_i \partial_\ell u(sq) e_i = -2\overline{\partial_\ell u(sq)}$ (it can be checked easily that $q + e_i q e_i = -2\overline{q}$), and because u is real, this is equal to $2\bar{\partial}_\ell u(sq)$. Therefore, the second integral cancels the third, and we have $\bar{\partial}_\ell f = 0$, meaning f is regular in U. □

It is also possible to construct a regular function from an analytic function of a complex variable.

Definition 4.3. We define the embedding of the complex numbers $\eta_q : \mathbb{C} \to \mathbb{H}$ into the quaternions as follows:

$$\eta_q(x + iy) = x + \frac{\text{Pu } q}{|\text{Pu } q|} y,$$

where q is the image of a complex number $\zeta(q)$ lying in the upper half-plane such that

$$\zeta(q) = \text{Re } + i\, |\text{Pu } q|.$$

Theorem 4.4. *Given a complex function $f : \mathbb{C} \to \mathbb{C}$ that is analytic on an open set $U \in \mathbb{C}$, we define $\tilde{f} : \mathbb{H} \to \mathbb{H}$ as*

$$\tilde{f}(q) = \eta_q \circ f \circ \zeta(q),$$

and $\Delta \tilde{f}$ is regular on $\zeta^{-1}(U)$.

Proof. For ease of notation, we write $t = \text{Re } q$, $r = \text{Pu } q$, and $u(x, y)$ and $v(x, y)$ for the real and imaginary parts of $f(x + iy)$ respectively. The subscripts $1, 2$ on u and v signify differentiation with respect to x and y respectively. Then we have

$$\tilde{f} = u(t, |r|), + \frac{r}{|r|} v(t, |r|).$$

This gives

$$\text{Re}\left[\bar{\partial}_\ell \tilde{f}(q)\right] = \frac{1}{2}\left\{\frac{\partial}{\partial t}[u(t, |r|)] - \Box \cdot \left[\frac{r}{|r|} v(t, |r|)\right]\right\} - \frac{1}{2} u_1(t, |r|) - \frac{v(t, |r|)}{|r|} - \frac{1}{2} v_2(t, |r|)$$

$$\begin{aligned}\text{Pu}\left[\bar{\partial}_\ell \tilde{f}(q)\right] &= \frac{1}{2}\left\{\Box[u(t, |r|)] + \frac{\partial}{\partial t}\left[\frac{r}{|r|} v(t, |r|)\right] + \Box \times \left[\frac{r}{|r|} v(t, |r|)\right]\right\} \\ &= \frac{1}{2}\left\{\frac{r}{|r|} u_2(t, |r|) + \frac{r}{|r|} v_1(t, |r|)\right\}.\end{aligned}$$

Because f is analytic, it obviously must satisfy the Cauchy-Riemann equations, meaning that $\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$, so

$$\bar{\partial}_\ell f = -\frac{v(t, |r|)}{|r|}.$$

Even so,

$$\begin{aligned}\bar{\partial}_\ell \Delta \tilde{f}(q) = \Delta \bar{\partial}_\ell f(q) &= -\left(\frac{\partial^2}{\partial t^2} + \frac{1}{r}\frac{\partial^2}{\partial r^2} r\right) \frac{v(t, r)}{r} \\ &= -\frac{v_{11}(t, r) + v_{22}(t, r)}{r} \\ &= 0,\end{aligned}$$

meaning that $\Delta\tilde{f}$ is regular on $\zeta^{-1}(U)$. □

Theorem 4.5. *Conformal mappings applied to regular functions produce regular functions.*

Before we begin this proof, we must first establish two important lemmas.

Lemma 4.6. *If $f : \mathbb{H} \to \mathbb{H}$ is regular at q^{-1}, then we define $If = \frac{q^{-1}}{|q|^2} f(q^{-1})$, and If is regular at q.*

Proof. By Theorem 8 of [3], it suffices to show that $d\mathbf{Q} \wedge d(If)_q = 0$. We split If as follows: $If = G \circ f \circ \iota$, where $G(q) = \frac{q^{-1}}{|q|^2}$ and $\iota(q) = q^{-1}$. Hence

$$\begin{aligned} Dq \wedge d(If)_q &= Dq \wedge dG_q f\left(q^{-1}\right) + Dq \wedge G(q) d(f \circ \imath)_q \\ &= DqG(q) \wedge \imath_q^* df_{q^{-1}} \end{aligned}$$

since G is regular at $q \neq 0$. But

$$\begin{aligned} \imath_q^* Dq\left(h_1, h_2, h_3\right) &= Dq\left(-q^{-1} h_1 q^{-1}, -q^{-1} h_2 q^{-1}, -q^{-1} h_3 q^{-1}\right) \\ &= -\frac{q^{-1}}{|q|^4} Dq\left(h_1, h_2, h_3\right) q^{-1} \end{aligned}$$

as we showed in (2.1). Thus

$$DqG(q) = -|q|^2 q \imath_q^* Dq$$

and so

$$\begin{aligned} Dq \wedge d(If)_q &= -|q|^2 q \imath_q^* \left(Dq \wedge df_{q^{-1}}\right) \\ &= 0 \end{aligned}$$

since f is regular at q^{-1}. □

Lemma 4.7. *If a function $f : \mathbb{H} \to \mathbb{H}$ is regular at aqb, the function $[M(a,b)f](q) = bf(a^{-1}qb)$ is regular at q.*

Proof. Define $\mu(q) = aqb$. Then, as we showed in 2.1 and referenced above, we have

$$\mu^* d\mathbf{Q} = |a|^2 |b|^2 a d\mathbf{Q} b.$$

Then,

$$\begin{aligned} Dq \wedge d[M(a,b)f]_q &= Dq \wedge b\mu_q^* df_{\mu(q)} \\ &= |a|^{-2}|b|^{-2} a^{-1}\left(\mu_q^* Dq\right) b^{-1} \wedge b\mu_q^* df_{\mu(q)} \\ &= |a|^{-2}|b|^{-2} a^{-1} \mu_q^*\left(Dq \wedge df_{\mu(q)}\right) \\ &= 0, \end{aligned}$$

which, by Theorem 8 of [3], means that $[M(a,b)f]$ is regular at q. □

We now prove Theorem 4.5.

Proof of Theorem 4.5. In order to obtain the map $q \mapsto \nu(q) = (aq+b)(cq+d)^{-1}$, we compose the following series of maps:

$$\begin{aligned} q &\to q' = cq\left(b - ac^{-1}d\right)^{-1} \\ q' &\to q'' = q' + d\left(b - ac^{-1}d\right)^{-1} \\ q'' &\to q''' = q''^{-1} \\ q''' &\to \nu(q) = q''' + ac^{-1} \end{aligned}$$

Obviously, translation preserves regularity, so steps (1) and (3) must preserve regularity. By Lemmas 4.6 and 4.7 respectively, steps (4) and (2) must preserve regularity, meaning that conformal maps preserve regularity. □

In the previous two sections, we have motivated the definition of regular functions and provided several examples of ways to construct them. In our final section, we will prove important results in quaternionic analysis related to these regular functions.

5. Quaternionic Analogues to Integral and Series Results

In this section, we will share the main results of our paper, by stating three important theorems of quaternionic analysis, which are rich analogues to the divergence theorem, Cauchy's theorem, and the Cauchy integral formula, and proving two of them.

Theorem 5.1 (Quaternion analogue to the divergence theorem)**.** *For a quaternionic function $f : \mathbb{H} \to \mathbb{H}$ of a quaternionic variable q, we have*

$$\int_{\partial\sigma} (d\boldsymbol{Q})f = \int_{\sigma} \Box f \, dV.$$

Proof. Recall from Section 2 Definition 2.9 that the definition of $d\mathbf{Q}$ is

$$d\mathbf{Q} = d\mathbf{Q}_0 + d\mathbf{Q}_1 i + d\mathbf{Q}_2 j + d\mathbf{Q}_3 k,$$

or the outwardly directed surface elements of $\partial\sigma$. We define $[d\mathbf{Q}] = (dQ_0, dQ_1, dQ_2, dQ_3)$, a row vector with the components of $d\mathbf{Q}$. Also, let M be the 4×4 matrix representation of f. Then, by the divergence theorem, we have

$$\int_{\partial\sigma} [d\mathbf{Q}]M = \int_{\sigma} \operatorname{div} M \, dV.$$

Because of the definition of divergence, we also have

$$\operatorname{div} M = \Box \cdot M = \Box F,$$

proving the theorem. □

Theorem 5.2 (Cauchy's theorem)**.** *If f is regular on every point of a 4-parallelepiped C,*

$$\int_{\partial C} d\boldsymbol{Q} \, f = 0.$$

Proof. This proof is long, technical, and boring, and can be found in [3]. □

Corollary 5.3. *If f is a right-regular function and g is a left-regular function, we have that*

$$\int_{\partial\sigma} f d\boldsymbol{Q} g = 0.$$

Now that we have developed enough analysis and prerequisites, we come to the last main theorem of our paper, the Cauchy–Fueter integral formula, providing an analogous (but more complex) quaternionic version of the Cauchy integral formula.

Theorem 5.4 (Cauchy–Fueter integral formula)**.** *If F is regular and sufficiently differentiable on every point of the hypersurface $\partial\sigma$ and q_0 is a point within $\partial\sigma$,*

$$F\left(q_0\right) = \frac{1}{8\pi^2} \int_{\partial\sigma} F d\boldsymbol{Q} \Delta \left(q - q_0\right)^{-1}.$$

To prove this, we first establish another result:

Theorem 5.5. *For a hypersurface $\partial\sigma$ in $\mathbb{E}^4$ with $q_0 \in \partial\sigma$, we have*

$$\int_{\partial\sigma} \Delta\,(q-q_0)^{-1}\,dQ = 8\pi^2.$$

Proof. Note that since translation preserves regularity, we need only prove this result for $q_0 = 0$ and $\partial\sigma$ containing 0. Because Δq^{-1} is regular except at 0, by 5.3, we have $\int_{\partial\sigma} q^{-1}\,d\mathbf{Q} = -\int_{|q|=1} q^{-1}\,d\mathbf{Q}$.
Δq^{-1} can be easily calculated to be $-4q^{-1}$ using the fact that $q^{-1} = \frac{\bar{q}}{|q|^2} = \bar{q}$, since $|q| = 1$ in our integral.

For the unit sphere in $\mathbb{E}^4$ (4-dimensional Euclidean space), we have

$$d\mathbf{Q} = |q|^2 q\,dS.$$

See [2] for details. Putting this all together, we have

$$-\int_{|q|=1} q^{-1}\,d\mathbf{Q} = 4\int_{|q|=1} q^{-1}q\,dS = 8\pi^2.$$

□

With this step proven, we proceed to the proof of the Cauchy–Fueter integral formula (5.4).

Proof. Using 5.3 again, we see that because F and $\Delta(q-q_0)^{-1}$ are regular in the region between $|q-q_0| = \varepsilon$ (for sufficiently small epsilon) and $\partial\sigma$,

$$\int_{\partial\sigma} F d\mathbf{Q}\Delta\,(q-q_0)^{-1} = \int_{|q-q_0|=\varepsilon} F d\mathbf{Q}\Delta\,(q-q_0)^{-1}.$$

Then, $d\mathbf{Q}$ can be found by simply modifying the expression in 5.5:

$$d\mathbf{Q} = |q-q_0|^2(q-q_0)^{-1}\,dS.$$

Our function F is to be sufficiently differentiable [2] such that as $|q-q_0| \to 0$,

$$F(q) = F(q_0) + O(|q-q_0|).$$

Then, we evaluate

$$\begin{aligned}
\frac{1}{8\pi^2}\lim_{\varepsilon\to 0}\int_{|q-q_0|=\varepsilon} F d\mathbf{Q}\Delta\,(q-q_0)^{-1} &= \lim_{\varepsilon\to 0}\frac{4}{8\pi^2}\int_{|q-q_0|=\varepsilon} F(q)\varepsilon^2(q-q_0)\varepsilon^{-2}(q-q_0)^{-1}\,dS \\
&= \lim_{\varepsilon\to 0}\frac{1}{2\pi^2}\int_{|q-q_0|=\varepsilon} (F(q_0) + O(\varepsilon))\,dS \\
&= F(q_0).
\end{aligned}$$

This completes the proof. □

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