CLASSIFICATION OF FATOU COMPONENTS

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1. Introduction

We build up the machinery necessary to state and prove the Fatou component classification theorem, following Dozier's 2012 senior thesis on this topic [Doz12]. Similar to Dozier's paper, we introduce the topic in the following order:

(1) **Hyperbolic geometry.** We define Poincaré metrics on $\mathbb{D}$ and $\mathbb{H}$, the essentially unique metrics on $\mathbb{D}$ and $\mathbb{H}$ that preserve angles with respect to every automorphism on $\mathbb{D}$ and $\mathbb{H}$. We also find geodesics on $\mathbb{D}$ and $\mathbb{H}$ with respect to the Poincaré metrics.

(2) **Uniformization.** Using hyperbolic geometry, we show that most open subsets of $\mathbb{C}$ have universal cover isomorphic to the disk. These subsets are named hyperbolic surfaces.

(3) **Properties of hyperbolic surfaces.** We derive several topological properties of hyperbolic surfaces from the Poincaré metrics, define normal families of holomorphic maps, and prove Montel's theorem that gives a sufficient condition for normality.

(4) **Classification of maps on hyperbolic surfaces.** We prove the Schwarz-Pick lemma for hyperbolic surfaces (an extension of the Schwarz lemma on the disk) to show holomorphic maps are distance non-increasing with respect to the Poincaré metric. Then, we show that a holomorphic self-map on a hyperbolic surface can be classified into four cases: *Attracting*, *Escape*, *Finite Order*, and *Irrational Rotation*.

(5) **Fatou and Julia sets.** We define Fatou and Julia sets with normal families and prove several key results concerning the dynamics near fixed points.

(6) **Fatou component classification theorem.** Using the "classification of maps on hyperbolic surfaces" and lemmas from "Fatou and Julia sets," we prove that Fatou components can be classified into five classes: *Superattracting*, *Geometrically Attracting*, *Parabolic*, *Siegal Disk*, and *Herman Ring*.

(7) **Further extensions.** Finally, we conclude the paper with a discussion on other results on the topic of Fatou components that were too technical or advanced to be covered in this paper.

The tools we develop from hyperbolic geometry are not only useful in proving the uniformization theorem but imperative for the Schwarz-Pick lemma, which states that holomorphic self-maps on hyperbolic surfaces are *distance non-increasing*. This structure on holomorphic maps on $\hat{\mathbb{C}}$ gives rise to a precise description of the possible dynamics on hyperbolic surfaces. This result translates to our theorem on Fatou components, after developing the appropriate theory on Fatou and Julia sets and the dynamics related to fixed points of holomorphic maps.

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2. Notation

- $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere
- $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ is the upper half-plane
- $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ is the open disk of radius r
- $\mathbb{D} = \mathbb{D}_1 = \{z \in \mathbb{C} : |z| < 1\}$ is the open disk of radius 1
- $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is the punctured complex plane
- $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ is the punctured unit disk
- $f^{\circ k} = f \circ \cdots \circ f$ is the kth iterate of f where $f : S \to S$ is a function from some set S to itself. When $k = 0$, $f^{\circ k}$ is the identity function on S
- $\operatorname{Aut} U$ is the (conformal) automorphism group of U. For this paper, U is always an open subset of $\hat{\mathbb{C}}$.

3. Hyperbolic Geometry

3.1. **Metrics on Manifolds.** To specify a geometry (some additional structure like curvature) on a smooth manifold M, we need a notion of distance. We do this by specifying a *Riemannian metric* which consists of a positive, definite inner product on each tangent space so that the inner products vary smoothly. The inner product induces a metric in the following way:

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y\rangle}.$$

The manifolds that we will be working with have two real dimensions, and so we can more simply write the metric at a point $z = x + iy$ as

$$ds^2 = a_{11}\, dx^2 + 2a_{12}\, dx\, dy + a_{22}\, dy^2 \tag{3.1}$$

where $[a_{ij}]$ is a positive-definite matrix that depends smoothly on the point z. We say that the metric is *conformal* if it is invariant under local rotations i.e., the length of the tangent vector is unchanged by a rotation about that point.

In particular, a conformal metric preserves length under a $\pi/2$ rotation. Thus, the metric on $\langle dx, dy\rangle$ equals the metric on $\langle -\, dy, dx\rangle$, the second of which is equal to

$$\left(ds_{\pi/2}\right)^2 = a_{11}\, dy^2 - 2a_{12}\, dx\, dy + a_{22}\, dx^2. \tag{3.2}$$

Comparing (3.1) and (3.2), we find $a_{11} = a_{22}$ and $a_{12} = 0$. Then, we may write

$$ds = \gamma(z)|dz|$$

where $\gamma(z)$ is a smoothly varying function of z (note that $\gamma^2 = a_{11} = a_{12}$). Since ds^2 is independent of the direction of dz, we have obtained the general form for a conformal metric on a smooth manifold with 2 real dimensions.

Pullbacks are another another way to obtain a (not-necessarily conformal) metric given a metric and a holomorphic map.

Definition 3.1 (Pullback of metric)**.** A holomorphic map $f : S_1 \to S_2$ of Riemann surfaces and a conformal metric $ds = \gamma(z)|dz|$ on S_2 together induce a metric $f^*(ds)$ on S_1, the *pullback* of γ along f, by

$$f^*(ds) = \gamma(f(z))|df(z)| = \gamma(f(z)) \cdot |f'(z)||dz|.$$

Intuitively, we can measure distances between two points in S_1 by mapping them to two points S_2 using f, then reading off their distances in S_2.

We care about metrics that are invariant along conformal automorphisms.

Definition 3.2 (Conformally invariant)**.** A conformal metric $ds = \gamma(z)|dz|$ is *conformally invariant* if for every $f \in \mathrm{Aut}(S)$, the induced metric is exactly the original metric $ds = f^*(ds)$. That is,

$$\gamma(z)|dz| = \gamma(f(z)) \cdot |f'(z)||dz|.$$

In the subsequent subsections, we will find conformally invariant metrics for the disk and the half-plane and show that there is a unique conformally invariant metric (up to constant scaling) for each subset of $\mathbb{C}$.

3.2. **Disk and Half-plane.** We are primarily interested in studying metrics on the disk due to Theorems 3.3 and 4.2, which state that any nonzero open subset of $\hat{\mathbb{C}}$ is conformally isomorphic to some copies of $\mathbb{D}$, $\mathbb{C}$, or $\hat{\mathbb{C}}$. We also study the half-plane $\mathbb{H}$ since it is conformally isomorphic to $\mathbb{D}$ but sometimes more convenient to work with.

Theorem 3.3 (Riemann Mapping theorem)**.** *Any proper open subset of $\mathbb{C}$ that is simply connected is conformally isomorphic to the unit disk $\mathbb{D}$.*

Proof. See Theorem 1.3 of Chapter 12 of the Complex Analysis book. ■

Example. For the proper open subset $\mathbb{H} \subset \mathbb{C}$, the conformal isomorphism $\mathbb{D} \to \mathbb{H}$ is given by

$$z \mapsto i \cdot \frac{1+z}{1-z}.$$

The following two lemmas classify the conformal automorphisms on $\mathbb{D}$. Note that for Proposition 3.5, conformal automorphisms on $\mathbb{D}$ are exactly the holomorphic functions with nonzero derivatives on $\mathbb{D}$.

Lemma 3.4 (Schwarz)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$. Then $|f(z)| \leq |z|$ for all z and $|f'(0)| \leq 1$. If equality holds in any of these expressions, then f is a conformal automorphism given by $f(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi)$.*

Proof. See Section 2 of Chapter 14 of the Complex Analysis book. ■

Proposition 3.5. *A map f belongs to* $\mathrm{Aut}(\mathbb{D})$ *if and only if it can be written as a Blaschke factor*

$$B_{a,\theta}(z) = e^{i\theta} \frac{z-a}{\bar{a}z - 1}$$

where $a \in \mathbb{D}$ and $\theta \in [0, 2\pi)$. We adopt the notation $B_a = B_{a,0}$.

Proof. In Problem 24 of Chapter 1 of the Complex Analysis book, we showed that the Blaschke factor is a bijective function from $\mathbb{D}$ to itself. To show that the map is conformal, we check that the derivative is never zero.

$$\begin{aligned} \frac{d}{dz} B_{a,\theta}(z) &= \frac{d}{dz} e^{i\theta} \frac{z-a}{\bar{a}z-1} \\ &= e^{i\theta} \frac{(\bar{a}z-1) - (z-a)\bar{a}}{(\bar{a}z-1)^2} \\ &= e^{i\theta} \frac{a\bar{a}-1}{(\bar{a}z-1)^2}. \end{aligned}$$

Since we chose a to have $a\bar{a} < 1$, the numerator is nonzero, and thus the Blaschke factor is a conformal automorphism.

Conversely, suppose $f \in \mathrm{Aut}(\mathbb{D})$. There exists some $a \in \mathbb{D}$ such that $f(a) = 0$. Then $g = f \circ B_a$ is an automorphism with $g(0) = 0$. Let $h = g^{-1}$, and taking derivatives of $g(h(z)) = z$ and plugging in $z = 0$ gives $g'(0)h'(0) = 1$. By the Schwarz lemma, $|g'(0)| \leq 1$ and $|h'(0)| \leq 1$, which implies $|g'(0)| = |h'(0)| = 1$; hence, $g(z) = e^{i\theta} z$ for some θ. So $f = g \circ B_a^{-1} = e^{i\theta} B_a^{-1}$, and f is a Blaschke factor as desired. ■

Now we classify the automorphisms on $\mathbb{H}$.

Proposition 3.6. *A map f belongs to* $\mathrm{Aut}(\mathbb{H})$ *if and only it can be written as* $f(z) = \frac{az+b}{cz+d}$ *with* $ad - bc = 1$ *and* $a, b, c, d \in \mathbb{R}$. *It follows that* $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}_2(\mathbb{R})$, *the* projective linear group *of* 2×2 *matrices with real entries and determinant* 1.

Proof. This can be computed directly from Proposition 3.5 and the conformal map $f : \mathbb{D} \to \mathbb{H}$ given by

$$f(z) = i \cdot \frac{1+z}{1-z}, \quad f^{-1}(z) = \frac{z-i}{z+i}.$$

For a clean proof using matrices, see Section 6 of Shurman's lecture notes for MATH 311 [Shu]. ■

Now with our classification of $\mathrm{Aut}(\mathbb{D})$ and $\mathrm{Aut}(\mathbb{H})$, we can find the conformally invariant metrics on $\mathbb{D}$ and $\mathbb{H}$. The derivation is simplest for $\mathbb{H}$.

Proposition 3.7. *There is a unique (up to multiplication by a positive constant) conformally invariant metric on* $\mathbb{H}$, *given by*

$$ds = \frac{|dz|}{\mathrm{Im}(z)}.$$

Proof. Consider the pullback of $ds = \gamma(z)|dz|$ through automorphisms of the form $g(z) = az + b$ where $a, b \in \mathbb{R}$ and $a > 0$. Plugging into the equation from Definition 3.2, we have

$$\gamma(z)|dz| = \gamma(az+b) \cdot a|dz|.$$

Setting $z = i$ gives $\gamma(ai + b) = \frac{\gamma(i)}{a}$. We may let $\gamma(i) = 1$ since we will regard metrics equivalent up to scaling by a positive factor. Thus, for $z = ai + b$, we get

$$ds = \gamma(z)|dz| = \frac{|dz|}{\mathrm{Im}(z)}.$$

Now, we must show that this metric is actually invariant over all $f \in \mathrm{Aut}(\mathbb{H})$. We show that for a given point $w \in \mathbb{H}$, the metric and its pullback along f agree at w. Because the group of maps $z \mapsto az + b$ with $a, b \in \mathbb{R}$ acts transitively on $\mathbb{H}$ (or by straight computation), there is a map $g(z) = az + b$ such that $h = g^{-1} \circ f$ fixes w. By the previous part of the proof, the metric is invariant over the pullback along g.

To show that the metric is invariant over the pullback along h, transform $h : \mathbb{H} \to \mathbb{H}$ to the disk to obtain the corresponding $\tilde{h} : \mathbb{D} \to \mathbb{D}$. Since $\tilde{h}$ is holomorphic with a fixed point $\tilde{w}$, the Schwarz lemma $|\tilde{h}'(\tilde{w})| = 1$. By the chain rule, it follows that $|h(w)| = 1$, so by Definition 3.2, the pullback along h preserves the metric. Finally, by the chain rule, the pullback along $f = g \circ h$ preserves the metric as desired. ■

Similarly there is a unique conformally invariant metric on $\mathbb{D}$.

Proposition 3.8. *There is a unique (up to multiplication by a positive constant) conformally invariant metric on* $\mathbb{D}$, *given by*

$$ds = \frac{|dz|}{1 - |z|^2}.$$

Proof. We leave the proof as an exercise. It may be helpful to remember that the conformal map $\mathbb{D} \to \mathbb{H}$ is

$$z \mapsto i \circ \frac{1+z}{1-z}.$$

■

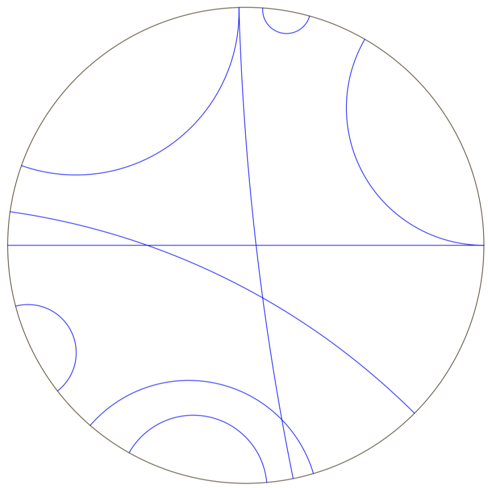


Figure 1. Assorted Geodesics of $\mathbb{D}$

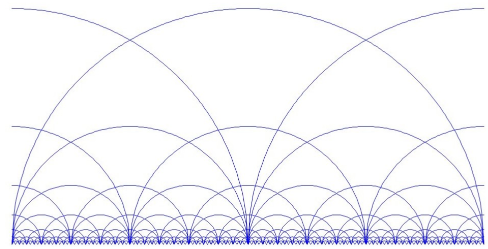


Figure 2. Geodesics of $\mathbb{H}$[1]

These metrics on simply connected domains are called *Poincaré metrics*, and for the remainder of the paper, we will assume that the Poincaré metric is being used unless otherwise stated.

3.3. **Geodesics.** Any metric ds on U naturally gives rise to a distance function d_U defined by

$$d_U(x, y) = \inf_{\gamma} \int_{\gamma} ds$$

where we take the infimum over all piecewise-smooth paths γ with endpoints at x and y.

Definition 3.9 (Isometry)**.** A map that preserves this notion of distance is said to be an *isometry* for the metric.

Definition 3.10 (Geodesic)**.** A *geodesic segment* in U is a path that is locally the shortest path between two points. This is the path γ in the definition of the distance function that gives the minimum path integral $\int_\gamma ds$ between the two endpoints. We call a geodesic a *geodesic line* if it is maximal i.e., there is no other geodesic of which it is a proper subset.

Intuitively, if an ant were to walk along a "straight line" according to the geometry defined by the metric, then its path would be a geodesic line. Some geodesics of $\mathbb{D}$ and $\mathbb{H}$ are shown in Figures 1 and 2.

We specialize to the case of $U = \mathbb{H}$ and $\mathbb{D}$.

Proposition 3.11 (Geodesics in $\mathbb{H}$)**.** *The geodesic lines of $\mathbb{H}$ in its Poincaré metric consists of circles/lines that are orthogonal to $\partial\mathbb{H}$. That is, they are circular arcs that intersect the real axis orthogonally at two points, as well lines orthogonal to the real axis.*

Proposition 3.12 (Geodesics in $\mathbb{D}$)**.** *The geodesic lines of $\mathbb{D}$ in its Poincaré metric consists of circles/lines that are orthogonal to $\partial\mathbb{D}$. That is, they are circular arcs that intersect $\partial\mathbb{D}$ orthogonally at two points, as well diameters of $\partial\mathbb{D}$.*

[1]`https://thatsmaths.com/2013/10/11/poincares-half-plane-model/`

Proof. We sketch the proof for $\mathbb{H}$; the full proof can be found in Dozier's paper [Doz12].

Given two points $w_1, w_2 \in \mathbb{H}$, we may transform them to i and ki for some real $k > 0$ by an automorphism $f \in \mathrm{Aut}(\mathbb{H})$. Then, the geodesic line through i and ki is the vertical ray $i\mathbb{R}^+$ and the geodesic line through w_1 and w_2 is $f^{-1}(i\mathbb{R}^+)$ since f and f^{-1} are isometries with respect to the Poincaré metric. Moreover, f is a Mobius transformation which takes circles/lines to circles/lines, so the desired geodesic through w_1 and w_2 is part of a circle or line. Since angles are preserved by conformal isometries, the geodesic remains orthogonal to the real axis during the transformation.

For $\mathbb{D}$, we transform the circular/linear geodesics of $\mathbb{H}$ to circles/lines on $\mathbb{D}$, which remain orthogonal to the boundary $\partial\mathbb{D}$. ■

Example. We can explicitly find the distance along a circular geodesic centered at 0 for $\mathbb{H}$ without much trouble. Our circle is parameterized by $z = re^{it}$ where $r > 0$, $t \in [a, b]$, and $0 < a < b < \pi$.

We directly compute:

$$\begin{aligned} L &= \int ds \\ &= \int \frac{1}{\mathrm{Im}(z)} |dz| \\ &= \int_a^b \frac{1}{r \sin t} \cdot r\, dt \\ &= \int_a^b \csc t\, dt \\ &= \left[-\log|\csc t + \cot t|\right]_a^b . \end{aligned}$$

For instance, the Poincaré distance between $-1 + i$ and $1 + i$ is

$$\begin{aligned} d_{\mathbb{H}}(-1 + i, 1 + i) &= -\log\left|\csc \frac{3\pi}{4} + \cot \frac{3\pi}{4}\right| + \log\left|\csc \frac{\pi}{4} + \cot \frac{\pi}{4}\right| \\ &\approx 1.763. \end{aligned}$$

Example. We can integrate the metric on $\mathbb{D}$ from 0 to a boundary point of $\mathbb{D}$ to obtain the distance function

$$d_{\mathbb{D}}(0, z) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|}.$$

Notice that when $z = 0$, the distance is 0, and as $|z| \to 1$, the distance goes to ∞.

4. Uniformization

Uniformization allows us to adapt the hyperbolic geometry of $\mathbb{D}$ to a variety of spaces. First, we define covering spaces using the wording given in the Complex Analysis book to avoid introducing new terminology from topology.

Definition 4.1 (Covering space)**.** Let $U, V \subset \hat{\mathbb{C}}$ be two open subsets. Suppose that $p : U \to V$ is a surjective continuous function. We say p is a covering map, and U is a covering space

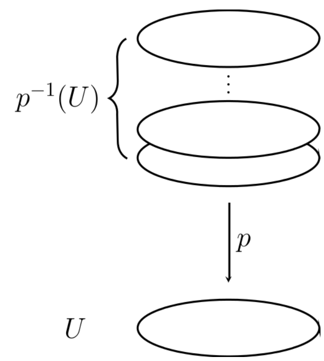


Figure 3. A covering map from n disks onto one disk.[2]

of V if for every $z \in V$, there is an open neighborhood W containing z such that $p^{-1}(W)$ consists of a union of sets $A_1, A_2, \ldots$ such that p restricted to each A_n is a bijective function from A_n to W.

Example. The simplest picture of a covering map is n open disks projecting down onto a single open disk as shown in Figure 3. Each $p^{-1}(W)$ consists of n copies of W.

Theorem 4.2 (Uniformization of plane domains)**.** *Let U be a connected open subset of $\hat{\mathbb{C}}$. Then the universal covering space of U is conformally isomorphic to either $\hat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{D}$.*

Proof. For the sake of brevity, we leave out certain technical details, but the full proof can be found in Dozier's paper [Doz12]. First, we note that if $\mathbb{D}$, $\mathbb{C}$, or $\hat{\mathbb{C}}$ were a covering space of U, then it follows from the simply connectedness of $\mathbb{D}$, $\mathbb{C}$, and $\hat{\mathbb{C}}$ that it is also a universal cover.

Now, we prove the theorem in three parts:

(1) Classify the open subsets of $\hat{\mathbb{C}}$ with universal cover isomorphic to $\mathbb{C}$ and $\hat{\mathbb{C}}$,
(2) Show that the universal cover of $\mathbb{C} \setminus \{0, 1\}$ is isomorphic to $\mathbb{D}$,
(3) Show that the universal cover of any open subset of $\mathbb{C} \setminus \{0, 1\}$ is isomorphic to $\mathbb{D}$.

Part 1. We claim that there are no subsets of $\mathbb{C}$ whose universal cover is isomorphic to $\hat{\mathbb{C}}$, and the only subsets of $\mathbb{C}$ covered by $\mathbb{C}$ are $\mathbb{C}$ and the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. To start, we note that any space covered by $\hat{\mathbb{C}}$ or $\mathbb{C}$ can be obtained by taking the quotient of the spaces by a group of analytic automorphisms *acting freely* and *properly discontinuously* on that space. As a refresher, we define these terms:

- A group G *acts freely* on a set X, if for all $x \in X$, $g \cdot x = x$ implies $g = I \in G$. That is, g sending *any* x to itself implies g is the identity. Only the identity element fixes any x.
- A group action G on a topological space X is *properly discontinuous* if X is a locally compact space (each small portion of X looks like a compact space) and for every compact subset $K \subset X$, the set $\{g \in G : gK \cap K \neq \emptyset\}$ is finite. For instance, if G consists of the maps $x \to x + a$ for all $a \in \mathbb{R}$, then G acts freely on $\mathbb{C}$, but is not properly discontinuous.

[2] https://en.wikipedia.org/wiki/Covering_space

There are no non-trivial subgroups of the automorphism group of $\hat{\mathbb{C}}$ with the above two properties, since every analytic automorphism is a Möbius transformation, which has a fixed point.

For $\mathbb{C}$, the analytic automorphisms are again the Möbius transformations, but we may force the fixed point to be ∞ by taking transformations of the form $z \to az + b$ for complex number a, b. Thus, the non-trivial subgroups that act freely and properly discontinuously are isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$, which when taken the quotient with $\mathbb{C}$ give $\mathbb{C}^*$ and the torus, respectively. We omit the torus from our discussion since it cannot be embedded in $\mathbb{C}$ and thus is not a subset of $\mathbb{C}$.

Part 2. Now, we show that there exists a covering map $p : \mathbb{D} \to \mathbb{C} \setminus \{0, 1\}$ from the unit disk to the triply punctured sphere. We do this geometrically by tiling $\mathbb{D}$ with hyperbolic triangles then using the union of the central two triangles as our fundamental domain, which turns out to be topologically equivalent to $\mathbb{C} \setminus \{0, 1\}$.

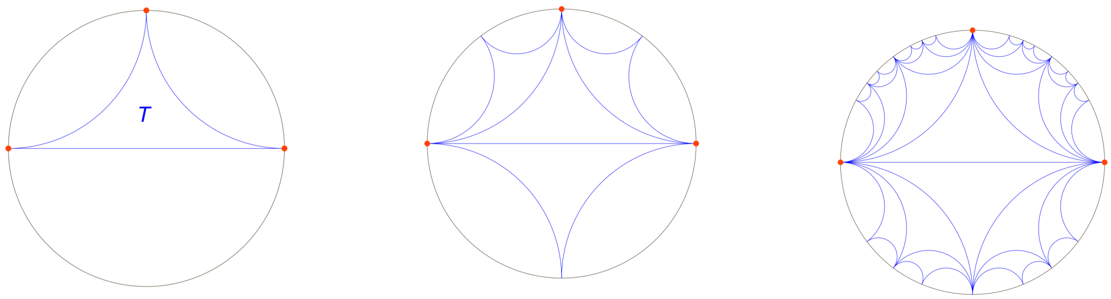
The first triangle has vertices $-1, 1, i$, and we generate the tiling by reflecting the vertices over the opposite sides (draw a geodesic segment that is perpendicular to the side and passes through the vertex we want to reflect). This iterative process is shown in Figure 4. To verify that the tiling indeed covers the disk, draw a geodesic segment from $p = 0$ to any point $z \in \mathbb{D}$. We reflect this geodesic segment over the sides of the original triangle so that we reach a new point z' contained entirely within the original triangle. See Figure 5. The path traced out to reach z and z' are the same and equal the Poincaré distance $d_U(p, z)$. Since the length of the path from p to z' is finite, there must have been a finite sequence of reflections of the path on the sides of the triangle to reach z'; we can now convert this into a finite sequence of reflections of z' over the sides of the tiling to reach z. This shows that every point in $\mathbb{D}$ is eventually covered by the tiling.

Let Γ_0 be the group of compositions of two reflections over the sides of the tiling, and let F be the union of the two central triangles with vertices $-1, 1, i$ and $-1, 1, -i$. Since no two points in the interior of F are in the same orbit of Γ_0, it is clear that F is a fundamental domain for Γ_0. That is, F contains exactly one point from each coset of the quotient $\mathbb{D}/\Gamma_0$.

The side $S_{-1,i}$ from -1 to i is identified with the side $S_{-1,-i}$ from -1 to $-i$, and similar for the sides $S_{1,i}$ and $S_{1,-i}$. See Figure 6 for the identification of edges of F. Since $F \subset \mathbb{D}$ does not contain the vertices $\pm 1, \pm i$, it turns out that F is topologically equivalent to the triply punctured sphere.

Part 3. We generalize our result from $U = \mathbb{C} \setminus \{0, 1\}$ to a open subset $U \subset \mathbb{C} \setminus \{0, 1\}$ by using the above constructed covering map $p : \mathbb{D} \to \mathbb{C} \setminus \{0, 1\}$. Consider a connected component V of $p^{-1}(U) \subset \mathbb{D}$. We can restrict p to a covering map $V \to U$, and hence, the universal covers of U and V are isomorphic. Now, to show that $\mathbb{D}$ is a universal cover of V, we construct a sequence of subsets $V = V_0, V_1, V_2, \ldots$ of the unit disk and covering maps $p_n : V_n \to V_{n-1}$ such that $\{V_n\}_{n \geq 0}$ are increasing larger spaces that approach the whole disk. We do this by taking p_n as the squaring function z^2 composed with Blaschke factors to avoid the branch point at 0.

Define $c_n = p_1 \circ \cdots \circ p_n$ as covering maps from V_n to $V_0 = V$. Let $f_0 : \tilde{U} \to V$ be the universal covering map, then from the property of universal covers, there exists a covering map $f_n : \tilde{U} \to V_n$ such that $c_n \circ f_n = f_0$. By a compactness argument, some subsequence of $c_1, c_2, \ldots$ converges to a holomorphic map $c : \mathbb{D} \to V_0 = V$. The corresponding subsequence of $f_1, f_2, \ldots$ converges to a holomorphic function $f : \tilde{U} \to \mathbb{C}$. See Figure 7 The rest of the proof consists of showing that $\mathrm{Im}(f) = \mathbb{D}$ and that f is an isomorphism between $\tilde{U}$ and $\mathbb{D}$,



(a) Hyperbolic triangle with vertices at 1, i, and -1.

(b) Reflection of the vertices over the sides

(c) Disk after many iterations

Figure 4. The tiling of $\mathbb{D}$ with hyperbolic triangles is defined iteratively through reflections.

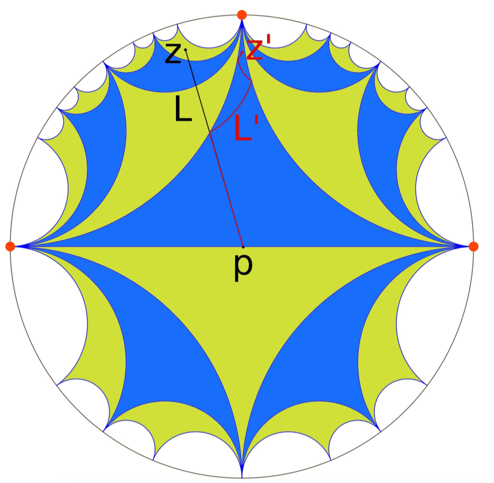


Figure 5. z' is a copy of z in the original triangle T.

by which we have shown that the universal cover of U is conformally isomorphic to the unit disk. ■

By uniformization, whenever U is a connected proper subset of $\mathbb{C}$, there exists a universal covering map $p : \mathbb{D} \to U$. By a pullback along p, the Poincaré metric on $\mathbb{D}$ passes to local metrics on U. More specifically, given a point $z \in U$, choose a $z' \in \mathbb{D}$ in the fiber of z (i.e., $z' \in p^{-1}(z)$) then we can "pushforward" the metric ds on $\mathbb{D}$ to ds_U on U by noting that p restricted to a neighborhood of z' is bijective. The choice of z' does not matter since the deck transformation that takes z' to another $z'' \in p^{-1}(z)$ is a conformal automorphism of $\mathbb{D}$, and hence the conformally invariant metric ds on $\mathbb{D}$ is unchanged.

Note that p is a *local isometry* that maps sufficiently small neighborhoods in $\mathbb{D}$ to its image by an isometry, which gives rise to a metric ds_U. Note that p may not be a *global isometry*. For instance, the distance between distinct $z', z'' \in p^{-1}(z)$ is positive so $ds > 0$, but we desire $ds_U = 0$ since the images of the two points are the same.

The induced metric ds_U is conformally invariant since any conformal automorphism on U can be lifted to an automorphism of $\mathbb{D}$, which must preserve ds and hence the original automorphism on U preserves ds_U. We also refer to the metric ds_U as Poincaré metrics and

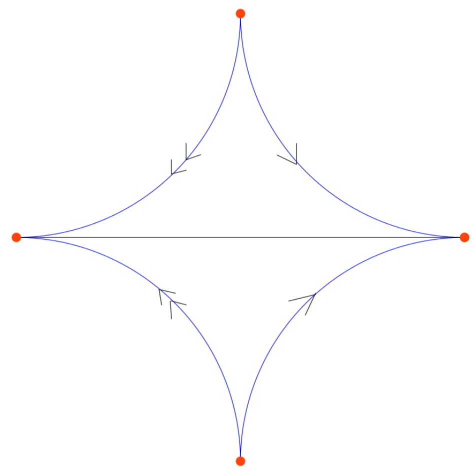


Figure 6. Identifying the sides of F forms a triply punctured sphere.

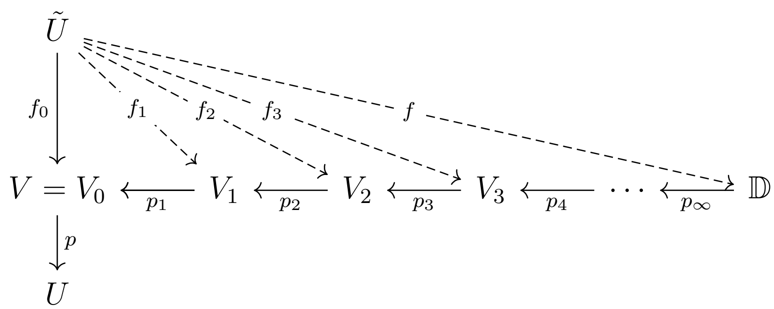


Figure 7. Commutative diagram for the proof of the uniformization theorem.

call U a *hyperbolic surface* since ds_U gives U a hyperbolic geometry similar to that of $\mathbb{D}$ and $\mathbb{H}$.

5. Properties of Hyperbolic Surfaces

In this section, we derive several key properties of hyperbolic surfaces, which are necessary to study the Julia and Fatou sets. We first define the *closed ball* of radius r about z_0 where the "radius" is measured with respect to the Poincaré metric on U:

$$B(z_0, r) = \{z \in U : d_U(z_0, z) \leq r\}.$$

Example. For $U = \mathbb{D}$, we may integrate the metric given in Proposition 3.8 to obtain a formula for the distance from the origin:

$$d_U(0, z) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|}. \tag{5.1}$$

Then the ball $B\left(0, \frac{1}{2}\right)$ is identical to the open disk $\mathbb{D}_r$ with radius $r = \frac{1}{2} \log 3 \approx 0.549$.

We derive two key topological properties of hyperbolic surfaces.

Proposition 5.1 (Closed ball is compact)**.** *Let U be a hyperbolic surface and $z_0 \in U$. The closed ball $B(z_0, r) = \{z : d_U(z_0, z) \leq r\}$ is compact for any $r > 0$.*

Proof. First, suppose $U = \mathbb{D}$. By composing with an isometry, we may assume $z = 0$. Since $B(0,r)$ is evidently closed, we only need to show it is contained in a compact set. By inspection of Equation (5.1), as $r \to \infty$, $|z| \to 1^-$. Thus, for any given $r > 0$, we may find some $R < 1$ such that $B(0,r) \subset \mathbb{D}_R$. This implies $B(0,r)$ is contained in the compact set $\overline{\mathbb{D}}_R$.

For the general case, consider the covering map $p : \mathbb{D} \to U$. We may assume $0 \in p^{-1}(z_0)$ by composing with an isometry on $\mathbb{D}$. Note that $B(z_0,r) \subset p(B(0,r))$ since p is a local isometry; note that equality may not hold since $p(B(0,r))$ may consist of several copies of $B(z_0,r)$. By the previous paragraph, $B(0,r)$ is contained a compact set K, and hence $B(z_0,r)$ is contained in the compact set $p(K)$. ■

Proposition 5.2. *Every hyperbolic surface U is contained in a union of a nested sequence of compact subsets $K_1 \subset K_2 \subset \cdots$ of U.*

Proof. By Proposition 5.1, we choose a basepoint $z_0 \in U$ and take $K_n = B(z_0,n) \subset U$. Since every point $z \in U$ has a finite Poincaré distance from z_0, it follows that $U = \bigcup_{n=1}^{\infty} K_n$. ■

Since Julia and Fatou sets are defined in the language of normal families, we must develop some related theory.

Definition 5.3 (Normal family)**.** A collection of holomorphic maps $\mathcal{F} \subset \{f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}$ is said to be a *normal family* if every sequence of maps in $\mathcal{F}$ has a subsequence that converges locally uniformly to a holomorphic map.

Intuitively, $\mathcal{F}$ is a normal family of holomorphic maps if every limit point of $\mathcal{F}$ is also a holomorphic map, where we require the convergence to be locally uniform.

The following theorem by Montel gives us a condition that lets us easily check if $\mathcal{F}$ is a normal family. Note that this Montel's theorem is the harder variant from class that we did not prove.

Theorem 5.4 (Montel)**.** *Let S be a Riemann surface and $\mathcal{F}$ a family of maps $S \to \hat{\mathbb{C}}$ with the property that there are three distinct points a, b, c such that $f(S) \subset \hat{\mathbb{C}} \setminus \{a, b, c\}$ for all $f \in \mathcal{F}$. Then the family $\mathcal{F}$ is normal.*

Proof. Since normality is a local property, we may assume S is a small open subset U of the plane. Moreover, by composing with a Möbius transformation, we can also assume that $\{a, b, c\} = \{0, 1, \infty\}$. From the uniformization theorem (Theorem 4.2), there exists a covering map $p : \mathbb{D} \to \mathbb{C} \setminus \{0, 1\}$. Each map $f \in \mathcal{F}$ lifts to a map $\tilde{f} : U \to \mathbb{D}$. The family of lifts $\{\tilde{f}\}$ is bounded and equicontinuous family of holomorphic maps, where equicontinuity comes from Cauchy's formula for the derivative. Hence, the Arzelà-Ascoli Theorem says that any sequence $\tilde{f}_1, \tilde{f}_2, \ldots$ has a uniformly convergent subsequence. Thus, $\{\tilde{f}\}$ is normal.

The limit $g = \lim_{n\to\infty} \tilde{f}_n$ of a sequence in $\{\tilde{f}\}$ may be such that $g(U)$ contains points in the boundary $\partial\mathbb{D}$. Thankfully, by the way that the covering map p was constructed, it may be easily extended to a map $P : \overline{\mathbb{D}} \to \overline{\mathbb{C}}$. Then, the family $\mathcal{F} = \{f = p \circ \tilde{f}\}$ is also normal, since if a sequence $\{\tilde{f}_n\}_{n\geq 1}$ converges to g, then the corresponding sequence $\{p \circ \tilde{f}_n\}_{n\geq 1}$ in $\mathcal{F}$ converges to $P \circ g$. ■

Montel's theorem was instrumental in the early work on complex dynamics. We may also analyze the behavior of the Poincaré metric near the boundary of an embedded surface (hyperbolic surfaces are often embedded in the complex plane).

Proposition 5.5. *Let $U \subset \hat{\mathbb{C}}$ be a hyperbolic surface, and let $z_1, z_2, \ldots$ be a sequence of points converging to a boundary point $\hat{z} \in \partial U$. Then for a given $r > 0$, the closed balls $B(z_n, r)$ converge uniformly to $\hat{z}$ as $n \to \infty$.*

Proof. The idea is to think of $B(z_n, r)$ as images of a fixed ball $B(0, r)$ by a sequence of universal covering maps, then apply Montel's theorem to obtain the limit. We leave the details of the proof to Dozier's paper [Doz12]. ■

6. Classification of Maps on Hyperbolic Surfaces

We use the Schwarz lemma (Lemma 3.4) and hyperbolic isometries to show that holomorphic maps on the disk are distance non-increasing.

Lemma 6.1. *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic. Then for any two distinct points $p_1, p_2 \in \mathbb{D}$, we have $d_{\mathbb{D}}(p_1, p_2) \geq d_{\mathbb{D}}(f(p_1), f(p_2))$, with equality if and only if f is a conformal isomorphism.*

Proof. The proof we give is merely computational; for a more elegant approach with tangent spaces, see Dozier's paper [Doz12].

Fix $z_1 \in \mathbb{D}$ and define the Möbius transformations

$$M(z) = \frac{z_1 - z}{1 - \overline{z_1} z}, \quad \phi(z) = \frac{f(z_1) - z}{1 - \overline{f(z_1)} z}.$$

Since $M(z_1) = 0$ or equivalently $M^{-1}(0) = z_1$, the composition $\phi(f(M^{-1}(z)))$ maps 0 to 0 and the unit disk to itself. Thus, we may apply Schwarz lemma and obtain

$$|\phi(f(M^{-1}(z)))| = \left| \frac{f(z_1) - f(M^{-1}(z))}{1 - \overline{f(z_1)} f(M^{-1}(z))} \right| \leq |z|.$$

Now call $z_2 = M^{-1}(z)$ and our inequality becomes

$$\left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} f(z_2)} \right| \leq \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right|.$$

The Poincaré distance between z_1 and z_2 is

$$d(z_1, z_2) = \tanh^{-1} \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right|$$

where we note that $\tanh^{-1}$ is a strictly increasing function. Thus, we have proved the inequality part of the lemma.

If equality holds for all $z_1, z_2 \in \mathbb{D}$, then f is evidently a conformal isomorphism since it preserves distance. ■

The following theorem generalizes the above result from the unit disk to hyperbolic surfaces.

Theorem 6.2 (Schwarz-Pick). *Let $f : S \to S'$ be a map between hyperbolic surfaces, considered along with their Poincaré metrics. Then one of the following possibilities holds:*

1. *f is conformal and a global isometry*
2. *f is a covering map, but is not injective. In this case, f is a distance non-increasing local isometry, but not a global isometry.*

(3) *f strictly decreases all non-zero distances.*

Proof. Consider the lift of f to $\tilde{f} : \mathbb{D} \to \mathbb{D}$ from the universal cover of S to that of S'. By Lemma 6.1, $\tilde{f}$ is either a conformal automorphism or strictly decreases all non-zero distances. In the first case, f is either a conformal global isometry or a covering map. In the second case, $\tilde{f}$ decreases the hyperbolic lengths of all paths, so f decreases all non-zero distances. ■

In the following theorem, we let $S = S'$ from Theorem 6.2 and study the iterates of the self-map $f : S \to S$. This classification will be essential to our main theorem on the classification of Fatou Components.

Theorem 6.3 (Classification)**.** *For any holomorphic map $f : S \to S$ of hyperbolic surfaces, exactly one of the following four possibilities holds:*

(1) *(Attracting) f has a fixed point p, contained in a neighborhood N such that all orbits $\{f^{\circ n}(z)\}$ of points $z \in N$ converge to p.*
(2) *(Escape) Every orbit of f eventually escapes any compact subset $K \subset S$.*
(3) *(Finite Order) Some iterate $f^{\circ n}$ is the identity and every point of S is periodic.*
(4) *(Irrational Rotation) (S, f) is a* rotation domain*. That is, S is conformally isomorphic to a disk $\mathbb{D}$, punctured disk $\mathbb{D}^*$, or annulus $A_r = \{z : 1 < |z| < r\}$; and f corresponds to an irrational rotation $z \mapsto e^{2\pi i\theta}$ with $\theta \notin \mathbb{Q}$.*

Proof. The reader should convince themselves that no two of the above cases can occur simultaneously. We apply Schwarz-Pick (Theorem 6.2) to f.

Distance-Decreasing Case. Suppose f is not a local isometry. Then by Schwarz-Pick, f strictly decreases all distances. If every orbit eventually escapes any compact subset $K \subset S$, then we are in the *Escape* case.

Thus, suppose there is some $z_0 \in S$ such that the orbit $\{z_n = f^{\circ n}(z_0)\}_{n \geq 0}$ visits some compact subset $L \subset S$ infinitely many times. Let K be a compact neighborhood of $L \cup f(L)$. Since $d_S(f(z), f(w)) < d_S(z, w)$ for all $z, w \in S$, by the compactness of K, there exists a constant $c_K < 1$ such that for any $z, w \in K$, we have $d_S(f(z), f(w)) < c_K d_S(z, w)$. Then, for any $z_m \in L$, since $z_{m+1} \in f(L) \subset K$, we have $d_S(z_{m+2}, z_{m+1}) < c_K d_S(z_{m+1}, z_m)$. Since infinitely many z_m lie within K, it follows that there exists a subsequence of $\{z_m\}$ such that the distances $d_S(z_{m+1}, z_m)$ decrease exponentially by at least a factor of c_K. Because distances between consecutive points are strictly decreasing even outside of K, the limit of the consecutive distances for the whole sequence goes to 0; that is, $\lim_{n\to\infty} d_S(z_{n+1}, z_n) = 0$. Thus, the sequence $\{z_n\}$ converges to a point p, which must be a fixed point of f by continuity.

Now, let $B_r = B(p, r)$ be any ball around p contained inside of K. For any $z \in B_r$, $d_S(p, f(z)) < c_K d_S(p, z)$ since f fixes p. Thus, $d_S(p, f^{\circ n}(z)) < c_K^n d_S(p, z) < c_K^n r$, and hence the orbit of z converges to p. It follows that f belongs to the *Attracting* case.

Distance-Preserving Case. Now, suppose f is a local isometry. First, suppose S is simply connected so that we may assume $S = \mathbb{D}$. By Schwarz-Pick, f is a covering map, so it must be a conformal automorphism of $\mathbb{D}$, i.e., a Möbius transformation. If f has a fixed point, then by the Schwarz lemma we are in the *Finite Order* or *Irrational Rotation* case depending on the rotational angle. If f does not have a fixed point in $\mathbb{D}$, then by the Brouwer fixed-point theorem, the extension of f to $\overline{\mathbb{D}}$ has one or more fixed points on the boundary $\partial\mathbb{D}$, in which case all orbits converge to the boundary, which implies we are in the *Escape* case.

Now suppose that S is not simply connected. If $f^{\circ k}$ is the identity for some k, then we are in the *Finite Order* case, so assume this does not happen. Let $\phi : (\mathbb{D}, 0) \to (S, z_0)$ where z_0 is some fixed based point and ϕ maps 0 to z_0.

Let $\mathcal{G}$ be the deck transformations of the covering i.e., all homeomorphisms of $\mathbb{D}$ such that the projection p is preserved. For the example of n disks projecting onto one disk (Example 4), the deck transformation consist of permutations of the n disks. The map f lifts via ϕ to a map $F : \mathbb{D} \to \mathbb{D}$ that must be a conformal automorphism since f was a covering map. Then, let Γ be the group generated by F and the elements of $\mathcal{G}$. Specifically, Γ consists of sequences of deck transformations and F, a conformal automorphism of the disk. It turns out that if Γ is a discrete topological group then we are in the *Escape* case, otherwise in the *Irrational Rotation* case. We omit the details of the proof since they require topological groups, which would detract us from the focus of this paper. However, advanced readers are encouraged to read the details in Dozier's paper [Doz12]. ■

7. Julia and Fatou Sets

In this section, we define and derive several key properties of Julia and Fatou sets. We restrict our study to holomorphic maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ although Fatou and Julia sets can be defined in more general settings. It turns out that any holomorphic map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is rational, i.e., can be written as a ratio of two polynomials p/q. We define the *degree* of f to be the maximum degrees of p and q where p and q do not share any common factors. By the fundamental theorem of algebra, any rational map of degree at least 1 is surjective.

Although f is defined over $\hat{\mathbb{C}}$, which is not a hyperbolic surface, we often consider f restricted to a subset of $\mathbb{C}$, which is a hyperbolic surface and thus admits the Poincaré metric.

Intuitively, Julia sets consist of points whose orbits are sensitive to small deviations in the starting point, while points in the Fatou set are not. More formally, they are defined with normal families.

Definition 7.1. The *Fatou set* of a rational map f consists of all points $z \in \hat{\mathbb{C}}$ for which there exists a neighborhood $N \ni z$ such that the family $\{(f|_N)^{\circ n} : n \in \mathbb{Z}^+\}$ is normal. We refer to such N as a *neighborhood of normality*. The *Julia set* $J(f)$ is the complement of the Fatou set in $\hat{\mathbb{C}}$.

Note that by definition, the Fatou set is open and the Julia set is closed.

Example. Let $f(z) = z^2$ be the squaring map. If $|z_0| < 1$, then $f^{\circ k}(z_0)$ tends towards 0 as $k \to \infty$, and moreover, we can find an open neighborhood around z_0 contained in $\mathbb{D}_{1-\epsilon}$ whose orbit also tends towards 0. We are careful to pick our neighborhood of z_0 with some distance away from the boundary $\partial\mathbb{D}$ since otherwise, the orbit of the neighborhood does not go to 0 uniformly. If $|z| > 1$, the orbit goes to ∞. Thus, we conclude the Fatou set of z^2 is $\mathbb{D}$ and the Julia set is $\hat{\mathbb{C}} \setminus \mathbb{D}$.

The Fatou and Julia sets are robust in the following sense.

Lemma 7.2 (Invariance)**.** *We say that a set S is fully invariant under f if $f(S) \subset S$ and $f^{-1}(S) \subset S$ i.e., z belongs to the image of S if and only if a belongs to S. Both Julia and Fatou sets are fully invariant.*

Proof. Suppose z is in the Fatou set with $N \ni z$ as its neighborhood of normality. Then we claim $f(z)$ is in the Fatou set with $f(N)$ as its neighborhood of normality. The iterates $\{f^{\circ k}|_{f(N)}\}$ is exactly $\{f^{\circ k+1}|_N\}$, which is a right-shifted version of the iterates for z, i.e., $\{f^{\circ k}|_N\}$. Since normality only depends on the long-term behavior of the family of functions, the result follows. The proof is similar for $f^{-1}(z)$ and its neighborhood of normality $f^{-1}(N)$. ■

When analyzing dynamical systems, the study of fixed points is usually a good starting point because fixed points are easier to analyze than arbitrary points and the neighborhoods of fixed points typically behave predictably based on characteristics of the fixed point.

Note that if a rational map has infinitely many fixed points, then its fixed points have a limit point since $\hat{\mathbb{C}}$ is compact. Then, f must be the identity for which the dynamics are trivial. Thus, unless f is the identity, it must have finite and isolated fixed points. We classify these fixed points based on the derivatives of f at that point.

Definition 7.3 (Multiplier of Fixed Points)**.** Suppose that $z_0 \in \hat{\mathbb{C}}$ is a fixed point of f. We can choose a local coordinate chart so that z_0 corresponds to the origin. We define the *multiplier* λ of z_0 (with respect to f) to be the derivative $f'(0)$ using the chosen coordinates.

To check that the multiplier is well-defined, one can verify that the derivative does not depend on the choice of local coordinates using the chain rule and the formula for the derivative of an inverse function. However, the multiplier is not generally well-defined for points not fixed by f. We can salvage the definition for periodic points:

Definition 7.4 (Multiplier of Periodic Points)**.** A point $z_0 \in \hat{\mathbb{C}}$ is a *periodic point* of f if there exists some k for which z_0 is a fixed point of $f^{\circ k}$. The *multiplier* of such a periodic point of f is defined to be the multiplier of $f^{\circ k}$ at its fixed point z_0. If k is chosen to be minimal, then k is said to be the *period* of z_0.

Next, we prove a lemma on the Fatou and Julia sets of $f^{\circ k}$.

Lemma 7.5 (Iterate Invariance)**.** *For any integer $k > 0$, the Fatou set of $f^{\circ k}$ coincides with the Fatou set of f, and $J(f^{\circ k}) = J(f)$.*

Proof. It is clear that if $\{f^{\circ n}|_N\}$ is normal, then $\{f^{\circ nk}|_N\}$ is normal; hence the Fatou set of f is contained in that of $f^{\circ k}$. For the converse, let z be a point in the Fatou set of $f^{\circ k}$ and $N \ni z$ be a neighborhood of normality. Let $\{f^{\circ n_j}\}$ be some sequence of iterates of f for which we desire to find a locally uniformly convergent subsequence. Note that any sequence in the family $\{f^{\circ nk}|_N\}_{n\geq 1}$ has a locally uniformly convergent sequence, and same holds for each of the families $\{f^{\circ nk+1}|_N\}, \ldots, \{f^{\circ nk+(k-1)}|_N\}$. At least one of these k families shares infinitely many elements with $\{f^{\circ n_j}\}$, and since that common sequence has a locally uniformly convergent subsequence, so does $\{f^{\circ n_j}\}$. Thus, z is in the Fatou set as desired. The result for the Julia set follows immediately. ■

Similarly, results on fixed points can often be extended to periodic points. For our purposes, however, we do not require discussion of periodic points.

We return to our discussion of fixed points of f. The dynamics near a fixed point z_0 is strongly influenced by the multiplier of that point.

Definition 7.6. Suppose f is a rational map and that z_0 is a periodic point with multiplier λ.

- If $|\lambda| < 1$, then z_0 is said to be *attracting.* If $\lambda = 0$, then z_0 is *superattracting.*
- If $|\lambda| > 1$, then z_0 is *repelling.*
- If $\lambda^n = 1$ for some n, and f is not the identity, then z_0 is *parabolic.*
- If $|\lambda| = 1$ and $\lambda^n \neq 1$ for any n, then z_0 is *indifferent.*

In the following several propositions, we derive dynamic properties for these fixed points and their neighborhoods, which explain the characteristic nicknames from Definition 7.6.

Proposition 7.7. *Every attracting fixed point z_0 of f is in the Fatou set. Furthermore, the set A of all $z \in \hat{\mathbb{C}}$ whose orbits converge to z_0 is an open subset of the Fatou set. This set is called the* basin of attraction *of z_0. The connected component of A containing z_0 is called the* immediate basin of attraction*.*

Proof. Choosing local coordinates, we may assume $z_0 = 0$. Choose μ such that $|\lambda| < \mu < 1$. By a Taylor series expansion centered at z_0, there is some small ball (in the Euclidean metric) B around z_0 such that $|f(z)| < \mu|z|$ for all $z \in B$. Then it follows that the iterates of $f|_B$ converge uniformly to the constant map $z \to z_0$, and hence B is a neighborhood of normality.

Now suppose $z \in A$. Then for some k, $f^{\circ k} \in B$. It follows that $N = (f^{\circ k})^{-1}(B)$ is a neighborhood of z contained in A. Thus, A is open. The iterates of $f|_N$ converge uniformly to z_0, so A is contained in the Fatou set. ■

Proposition 7.8. *Every repelling fixed point of z_0 of f is in the Julia set.*

Proof. Choosing local coordinates, we can assume that $z_0 = 0$. Through Taylor series expansion and induction, one can show that the derivative of $f^{\circ k}$ at 0 is equal to λ^k. Since $|\lambda| > 1$, no subsequence of these derivatives will converge to a finite value.

Analysis tells us that the derivatives of a sequence of analytic functions converge to the derivative of the locally uniform limit of the functions, assuming such a limit exists. Hence, the iterates of f cannot form a normal family on any neighborhood of z_0, and so z_0 is in the Julia set. ■

Proposition 7.9. *If f has degree at least two, then any parabolic fixed point z_0 is in the Julia set.*

Proof. Again, choose local coordinates so that $z_0 = 0$. By taking an appropriate iterate of f, assume that the multiplier of the fixed point is 1. By Lemma 7.5, if z_0 belongs to the Julia set of an iterate of f, then it belongs to the Julia set of f. On a small neighborhood of z_0, we may describe f with the Taylor series

$$f(z) = z + a_n z^n + \cdots$$

for some nonzero a_n. Then taking iterates of f, we get

$$f^{\circ k}(z) = z + k a_n a^n + \cdots .$$

Hence, the nth derivative of $f^{\circ k}$ is $n! \cdot k \cdot a_n$, which goes to ∞ as $k \to \infty$. As in the proof of Proposition 7.8, the iterates of f do not constitute a normal family on any neighborhood of z_0, so z_0 is in the Julia set. ■

8. Fatou Component Classification Theorem

Our main result concerns the Fatou components of f.

Definition 8.1 (Fatou Components)**.** The connected components of the Fatou set of f are called the *Fatou components* of f.

Proposition 8.2. *The image $f(U)$ of any Fatou component U is itself a Fatou component.*

Proof. The continuous image of any connected set is connected, so $f(U)$ is connected. Since it is also part of the Fatou set, it must be contained in some Fatou component U'. Our goal is to prove $f(U) = U'$, particularly by proving $f(U)$ is closed relative to the Fatou set.

Consider the closure $\overline{U}$ of U in $\hat{\mathbb{C}}$. As a closed subset of a compact set, $\overline{U}$ is compact and hence its image $f(\overline{U})$ is also compact, hence closed. Note that the boundary of U is contained in the Julia set; otherwise U would not be a Fatou component. Thus by Proposition 7.2, the image of the boundary is also in the Julia set. Hence, $f(\overline{U})$ consists of $f(U)$ along with some points in the Julia set $f(\partial U)$. In other words, $f(U)$ is the intersection of the closed set $f(\overline{U})$ with the Fatou set of f. This is enough to prove that $f(U)$ is closed in the Fatou set and hence, that $f(U) = U'$. ■

Now, we state two somewhat technical lemmas. The proofs can be found in Dozier's paper [Doz12].

Lemma 8.3 (Convergence to Boundary Fixed Points)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be rational of degree at least two, and $U \subset \hat{\mathbb{C}}$ a hyperbolic surface with $f(U) \subset U$. Suppose that some orbit of f in U has no accumulation point in U. Then there is some boundary point $p \in \partial U$ such that all orbits in U converge (in $\overline{U}$) to p.*

Definition 8.4. Let f be a holomorphic function defined on some neighborhood of V of the origin, such that 0 is a fixed point of f with multiplier λ. We say that a path $\gamma : [0, \infty) \to V \setminus \{0\}$ converges to the origin if $\lim_{t\to\infty} \gamma(t) = 0$.

Lemma 8.5 (Snail)**.** *With the set-up in Definition 8.4, if the path γ satisfies $f(\gamma(t)) = \gamma(t + 1)$, then either $|\lambda| < 1$, or $\lambda = 1$.*

The snail lemma is named for the snail-like path drawn in the proof if one assumes (for the sake of contradiction) that $|\lambda| = 1$ and $\lambda \neq 1$.

Finally, we state two results on the Julia sets, the proofs of which we omit for brevity. The proofs can be found in Dozier's paper [Doz12].

Proposition 8.6. *If f is a rational map of degree at least 2, then $J(f)$ is an infinite set.*

Proposition 8.7. *If f is a rational map of degree at least 2, then the Julia set $J(f)$ contains no isolated points.*

We are now ready to state and prove the main result of this paper, a precise classification of the five types of Fatou components that rational maps on $\hat{\mathbb{C}}$ can exhibit.

Theorem 8.8 (Five Possibilities)**.** *Suppose $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map of degree at least 2, and U is a connected component of the Fatou set such that $f(U) = U$. Then exactly one of the following holds:*

(1) *(Superattracting): U is an immediate basin of attraction for a superattracting fixed point.*
(2) *(Geometrically Attracting): U is an immediate basin of attraction for a geometrically attracting fixed point.*

(3) *(Parabolic) U has a parabolic fixed point on its boundary to which all orbits in U converge.*
(4) *(Siegal Disk) U is a* Siegal disk*, i.e., there is a conformal isomorphism $U \to \mathbb{D}$ that conjugates f to a rotation $z \mapsto e^{2\pi i\theta z}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$.*
(5) *(Herman Ring) U is a* Herman ring*, i.e., there is a conformal automorphism $U \to A_r$, where $A_r = \{a : 1 < |z| < r\}$ is some annulus, conjugating f to a rotation $z \mapsto e^{2\pi i\theta z}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$.*

Proof. Since the Julia set is infinite (Proposition 8.6), we can assume that $U \subset \mathbb{C} \setminus \{0, 1\}$ and so by the uniformization theorem, U is a hyperbolic surface. We consider the four cases of Theorem 6.3 on $f|_U$:

Attracting. In this case, $f|_U$ must have either a superattracting or geometrically attracting fixed point. Since an immediate basin of attraction is the connected component containing the fixed point, it follows that by definition, U must be *Superattracting* or *Geometrically Attracting*.

Escape. Since there is no accumulation point in U for any orbit, we can apply Lemma 8.3 to conclude that all orbits in U converge to some boundary point $p \in \partial U$. We claim that p is a parabolic fixed point, i.e., the multiplier of p is a root of unity and f is not the identity. Skipping some details for brevity, we can construct a path γ that follows an orbit in U, both of which converge to p. The snail lemma (Lemma 8.5) implies that the multiplier λ of p is either $0 < |\lambda| < 1$ or $\lambda = 1$. The first case cannot happen since the boundary point p is in the Julia set, and all fixed points with $|\lambda| < 1$ are Fatou set by Proposition 7.7. In the second case, we get that p is a parabolic fixed point as desired. Since the orbits of all points in U converge to p, we see that U is *Parabolic*.

Finite Order. In this case, some iterate of f is the identity. However, this is impossible since f is rational and its degree is assumed to be at least 2.

Irrational Rotation. In this case, we have three cases for U as stated in Theorem 6.3. If U is isomorphic to the disk, then U is a *Siegal Disk*, and if U is isomorphic to the annulus A_r, then U is a *Herman Ring*. The only remaining case is when U is isomorphic to the punctured disk $\mathbb{D}^*$, which we claim is impossible.

If U were isomorphic to $\mathbb{D}^*$. Then one component of the complement of U would consist of a single point, which would have to belong to the Fatou set since the Julia set does not contain isolated points (Proposition 8.7). But U is a Fatou component, so this cannot happen. ∎

9. Further Extensions

In this section, we discuss further results on this topic of Fatou components.

Our main result Theorem 8.8 showed that there were five possibilities for Fatou components, but do all of them actually occur? For instance, can we find a holomorphic map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that has a Siegal disk as a Fatou component?

The answer is yes, for all five types of Fatou components, but the proof is difficult for Siegal disks and Herman rings. The superattracting, geometrically attracting, and parabolic cases are more readily observed as bulbs and petals protruding from a single (fixed) point, respectively. More information on this topic can be found in the third section of Dozier's paper [Doz12].

In this paper, we studied holomorphic maps on $\hat{\mathbb{C}}$, which is equivalent to rational maps p/q where p and q are polynomials in z. The dynamics of rational maps are easier than those of transcendental maps.

Definition 9.1. Fatou components that are not eventually periodic are called *wandering domains.*

Transcendental maps may have wandering domains; however, rational maps do not.

Theorem 9.2 (No Wandering Domain)**.** *Fatou components of rational maps are eventually periodic.*

Moreover, iterates of transcendental maps can tend towards essential singularities, similar to how the iterates in a parabolic Fatou component tend towards a boundary point of the Fatou component, i.e., a singularity in the Julia set.

Definition 9.3. Transcendental maps may have *Baker domains*, a type of Fatou component in which the iterates tend to an essential singularity.

Baker domains exhibit more complex dynamics than any other types of Fatou components. See Rippon [Rip08] for more exposition on Baker domains.

References

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