HARMONIC FUNCTIONS

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Abstract. Harmonic functions are functions $f : \mathbb{R}^n \to \mathbb{R}$ satisfying the Laplace equation

$$\frac{\partial^2 f}{\partial^2 x_1} + \cdots + \frac{\partial^2 f}{\partial^2 x_n} = 0.$$

In this paper, we will focus on the case where $n = 2$. By taking $n = 2$, we can compose the natural map $\theta : \mathbb{C} \to \mathbb{R}^2$ given by $\theta(x + yi) = (x, y)$ with f to obtain the composition $\theta \circ f : \mathbb{C} \to \mathbb{R}$. This allows us to apply methods in complex analysis in order to characterize harmonic functions, and determine several properties that they satisfy. For instance, the Gauss Mean Value theorem for holomorphic functions

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta,$$

also holds for harmonic functions.

1. Introduction and Relation to Holomorphic Functions

Denote by Ω an open set in the complex plane and let $f : \Omega \to \mathbb{R}$ be a continuous function with continuous partial second derivatives. That is, f is a C^2-function, and the following derivatives,

$$\frac{\partial^2 f}{\partial^2 x}, \quad \frac{\partial^2 f}{\partial^2 y}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial x \partial y}$$

are continuous. We say f is *harmonic* if the Laplace operator

$$\Delta f = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y},$$

vanishes. For the remainder of this paper, Ω will always be used to define an open set and for any set $S \subseteq \mathbb{C}$, define S^C to be the complement of S.

Example. The function $f : \mathbb{C} \to \mathbb{R}$ defined via the map $x + yi \to x^2 - y^2$ is a harmonic function. Note that $x^2 - y^2$ is the real part of the function $g(z) = z^2$. Indeed, we have $g(x + yi) = x^2 - y^2 + 2xyi$.

Harmonic functions arise naturally from holomorphic functions. To see this, let $f : \Omega \to \mathbb{C}$ be holomprphic and write $f(x + yi) = u(x, y) + iv(x, y)$ for functions $u, v : \mathbb{R}^2 \to \mathbb{R}$. Then, by the Cauchy Riemann equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Taking second derivatives, we obtain

$$\frac{\partial^2 u}{\partial^2 x} = \frac{\partial^2 v}{\partial y \partial x}, \text{ and } \frac{\partial^2 u}{\partial^2 y} = -\frac{\partial^2 v}{\partial x \partial y},$$

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and so it follows that

$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = 0.$$

Hence u is harmonic, and similarly, so is v.

Theorem 1.1. *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function for an open set $U \subseteq \mathbb{C}$. Then, the real and imaginary parts of f are harmonic.*

Example. Let $f(z) = \ln z$ be a holomorphic function on a well defined region of the complex plane, and write $z = re^{i\theta}$. Then, $f(z) = \ln(re^{i\theta}) = \ln r + i\theta$. Now, we let $z = x + yi$, and so $r = \sqrt{x^2 + y^2}$. Therefore, $\Re f(z) = \ln\left(\sqrt{x^2+y^2}\right) = \frac{\ln(x^2+y^2)}{2}$, which is a harmonic function.

A natural question to ask is whether or not given a harmonic function $u : \Omega \to \mathbb{C}$ we can construct a holomorphic function $f : \Omega \to \mathbb{C}$ such that $\Re f(z) = \Omega(z)$. It turns out that this is true, but only up to a constant.

Theorem 1.2. *Let Ω be a finitely connected region, and let $K_1, \ldots, K_N$ denote the bounded components of Ω^C. For each K_i, pick a point a_i in K_i. If $u : \Omega \to \mathbb{R}$ is a harmonic function, then there is a holomorphic function $f : \Omega \to \mathbb{C}$ for which*

$$u(z) = \Re f(z) + \sum_{j=1}^{N} c_j \ln|z - a_j|,$$

for certain real numbers $c_1, \ldots, c_N$.

Remark 1.3. The term finitely connected essentially means that Ω doesn't have infinitely many holes. If it did, there could be infinitely many bounded components in Ω^C.

Before proving the theorem, let us take a moment to understand its significance. Suppose we had a harmonic function $u(z)$ defined on all of $\mathbb{C}$. Since $\mathbb{C}$ has no bounded components in its complement, this means that there is an analytic function $f(z)$ such that $u = \Re f$. If it happens that u is defined on an open set with holes in it, then we may not necessarily end up with an analytic function f for which $u = \Re f$.

Proof. We follow the proof given in [1]. Start by considering the function

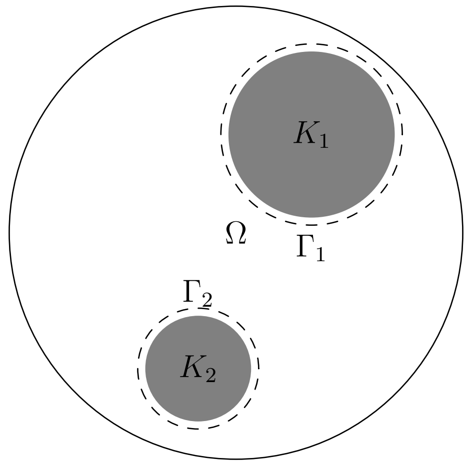
$$h(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.$$

The key idea is that this function is actually holomorphic. To see this, we need only check the Cauchy Riemann equations. But this is equivalent to checking that

$$\frac{\partial^2 u}{\partial^2 x} = -\frac{\partial^2 u}{\partial^2 y}, \text{ and}$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\left(-\frac{\partial^2 u}{\partial y \partial x}\right),$$

the first of which is tautological and the second of which is just Clairut's theorem. For each K_j, let Γ_j denote a curve contained in Ω containing K_j.



We now define

$$c_j = \frac{1}{2\pi i} \int_{\Gamma_j} h(z) dz.$$

First, we must check that c_j is real. To do this, note that

$$\begin{aligned}
\Im c_j &= -\frac{1}{2\pi} \Re \int_{\Gamma_j} h(z) dz \\
&= -\frac{1}{2\pi} \Re \int_{\Gamma_j} \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) dz \\
&= -\frac{1}{2\pi} \Re \int_{\Gamma_j} \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) (dx + i dy) \\
&= -\frac{1}{2\pi} \Re \int_{\Gamma_j} \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.
\end{aligned}$$

But the last integral equals 0 by Green's theorem. Now it's time to define $f(z)$. Fix a point b in Ω. The claim is that the function

$$f(z) = \int_b^z h(z') - \frac{c_1}{z' - a_1} - \cdots - \frac{c_N}{z' - a_N} dz',$$

is well defined, in the sense that it does not depend on the path from b to z. To prove this, it suffices to show that for any simple closed curve γ in Ω, we have

$$\int_\gamma f(z) dz = 0.$$

To prove this we only need the Cauchy integral theorem. Let m_i denote the winding number of γ about a_i. Then, by the integral formula,

$$\int_\gamma h(z) dz = \sum_{j=1}^N 2\pi i m_j \int_{\Gamma_j} h(z) dz = \sum_{j=1}^N 2\pi i m_j c_j.$$

But by the definition of winding number, we also have

$$\int_\gamma \frac{c_1}{z - a_1} + \cdots + \frac{c_N}{z - a_N} dz = \sum_{j=1}^N 2\pi i m_j c_j.$$

Thus, the integral over the closed curve is 0, and so f is well defined. Now we must check that this choice of f satisfies the equation listed above. We start by noting that

$$f'(z) = h(z) - \frac{c_1}{z - a_1} - \cdots - \frac{c_N}{z - a_N}.$$

Next, define

$$q(z) = \Re f(z) + \sum_{j=1}^{N} c_j \log |z - a_j|.$$

The crux of the proof rests on adding a suitable constant to $f(z)$ such that $q(z)$ and $u(z)$ agree on b. For instance if this constant is c, then we redefine f to be $f + c$, which is still holomorphic. To prove that $q(z) = u(z)$ it suffices to show that

$$\frac{\partial q}{\partial x} = \frac{\partial u}{\partial x}, \text{ and } \frac{\partial q}{\partial y} = \frac{\partial u}{\partial y}.$$

We start by computing $\frac{\partial q}{\partial x}$. Note that since $z = x + yi$, we have $dz = dx$, and so

$$\begin{aligned}
\frac{\partial q}{\partial x} &= \frac{\partial}{\partial x}\left(\Re f(z) + \sum_{j=1}^{N} c_j \log |z - a_j|\right) \\
&= \frac{\partial}{\partial x}\Re\left(f(z) + \sum_{j=1}^{N} c_j \log(z - a_j)\right) \\
&= \frac{d}{dz}\Re\left(f(z) + \sum_{j=1}^{N} c_j \log(z - a_j)\right) \\
&= \Re\frac{d}{dz}\left(f(z) + \sum_{j=1}^{N} c_j \log(z - a_j)\right) \\
&= \Re\left(f'(z) + \sum_{j=1}^{N} \frac{c_j}{z - a_j}\right) \\
&= \Re h(z),
\end{aligned}$$

by the formula for $f'(z)$ above. But, recalling that $h(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$, we obtain $\frac{\partial q}{\partial x} = \frac{\partial u}{\partial x}$, and a similar method gives $\frac{\partial q}{\partial y} = \frac{\partial u}{\partial y}$. Hence, we have $q = u$, and we are done. ■

Harmonic functions enjoy many of the same properties as analytic functions. For example, the concept of analytic continuation can be extended to harmonic functions as well.

Theorem 1.4. *Let U and V be open sets such that $V \subseteq U$ and V is simply connected. If f and g are harmonic functions that agree on V, then they agree on U as well.*

Proof. By Theorem 1.2, since V is simply connected, we can find analytic functions F and G such that $f = \Re F$ and $g = \Re G$. Let $f_1 = f$ and $g_1 = g$. Then, we can write $F = f_1 + if_2$ and $G = g_1 + ig_2$ where we split F and G based on real and imaginary parts. The key idea is that f_2 and g_2 must differ by a constant. Indeed, since $f_1 = g_1$ on V, we have

$F - G = 0 + i(f_2 - g_2)$. But now we can apply the Cauchy Riemann equations. This gives

$$\frac{\partial(f_2 - g_2)}{\partial x} = \frac{\partial(f_2 - g_2)}{\partial y} = 0,$$

and so f_2 and g_2 differ by a constant as claimed. Suppose $f_2 - g_2 = k$ for some $k \in \mathbb{R}$. Then, $F = G + ik$.

Now, since F and $G + ik$ agree on V, they must agree on U (since they're holomorphic). But this means $\Re F = \Re(G + ik)$ over U, and so $f_1 = g_1$, as desired. ■

An important property enjoyed by holomorphic functions is the Gauss Mean Value theorem (c.f. [2]):

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta.$$

Taking real parts, we find that

$$\Re f(z_0) = \Re \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \Re f(z_0 + \rho e^{i\theta}) d\theta,$$

and so harmonic functions also satisfy the Gauss Mean Value theorem.

References

[1] Sheldon Axler. "Harmonic Functions from a Complex Analysis Viewpoint". In: *The American Mathematical Monthly* 93.4 (1986), pp. 246–258. ISSN: 00029890, 19300972. URL: `http://www.jstor.org/stable/2323672`.

[2] Simon Rubenstein Salzedo. *Complex Analysis*. 2020.

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