# MODULAR FORMS

SRIRAM ANANTHAKRISHNAN

Abstract. Modular forms are incredibly interesting functions which have a variety of applications. They are especially intertwined with many facets of number theory, such as elliptic curves and the famous monstrous moonshine conjecture. The two fields were unified by the *modularity theorem*, which Wiles used to prove Fermat's last theorem. However, given all of their applications they are still of particular interest individually and can be used in a variety of surprising ways.

## 1. Modular Forms

We must first define the *modular group*, around which modular forms are built. .

**Definition 1.1.** The *modular group* $\mathrm{SL}_2(\mathbb{Z})$ is the set of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries and determinant 1, i.e $ad - bc = 1$.

Notice that $\mathrm{SL}_2(\mathbb{Z})$ is a group action on the space $\mathbb{H}$. By definition,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

It can also be checked that the image of any point $\tau \in \mathbb{H}$ under some matrix $M \in \mathrm{SL}_2(\mathbb{Z})$ remains in $\mathbb{H}$. Now we are ready to define modular forms.

**Definition 1.2.** A *modular form* of weight $k$ is a function $f : \mathbb{H} \to \mathbb{C}$ that is:

(1) Analytic in the upper half plane $\mathbb{H}$.
(2) Satisfies the condition

$$f(\tau)(ck + d)^k = f\left(\frac{a\tau + b}{c\tau + d}\right)$$

for *all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. (*modularity condition*)
(3) $f$ is holomorphic at $\infty$.

A modular form of weight 0 is known as a *modular function*

**Proposition 1.3.** *As a consequence, we also have the following:*

(1) *If $f$ and $g$ are both modular forms of weight $k$, then so is $f + g$.*
(2) *If $f$ and $g$ are modular forms of weight $m$ and $n$, then $fg$ is a modular form of weight $m + n$.*

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*Proof.* (1) Obviously $f+g$ is holomorphic on $\mathbb{H}$ and at $\infty$, so it suffices to check the modularity condition:

$$\begin{aligned}(f+g)(\tau)(ck+d)^k &= f(\tau)(ck+d)^k + g(\tau)(ck+d)^k \\ &= f\left(\frac{a\tau+b}{c\tau+d}\right) + g\left(\frac{a\tau+b}{c\tau+d}\right) \\ &= (f+g)\left(\frac{a\tau+b}{c\tau+d}\right)\end{aligned}$$

i.e $(f+g)(\tau)$ also satisfies the modularity condition as desired.

(2) Similarly all that needs to be checked are the modularity conditions of $fg$. We have

$$\begin{aligned}(fg)(\tau)(ck+d)^{m+n} &= f(\tau)(ck+d)^m \cdot g(\tau)(ck+d)^n \\ &= f\left(\frac{a\tau+b}{c\tau+d}\right) \cdot g\left(\frac{a\tau+b}{c\tau+d}\right) \\ &= (fg)\left(\frac{a\tau+b}{c\tau+d}\right)\end{aligned}$$

from which it follows that $fg$ is also a modular form, moreover of weight $m+n$. ∎

Now this is an undoubtedly strange definition, however it turns out that this is very motivated for the following key reason:

**Theorem 1.4.** *Matrices* $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *generate* $SL_2(\mathbb{Z})$.

This result can be proved either geometrically or noting that since determinants are multiplicative, it suffices to observe what happens when we multiply by each matrix a certain number of times. A formal proof is in [3]

Now, suppose we have two matrices $X$ and $Y$ for which the modularity condition holds. Since $XY \in \mathrm{SL}_2(\mathbb{Z})$, then the modularity condition holds for $XY$ as well. Thus, since all matrices in $\mathrm{SL}_2(\mathbb{Z})$ can be generated from matrices $A$ and $B$, condition (2) reduces to the following:

$$f(\tau) = f(\tau+1) \quad \text{and} \quad f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau).$$

It can also be noted that no nonconstant modular forms of odd weight exist, since doing so would give $f(\tau) = f(\tau)(-1)^k$ with matrix $M = -\mathrm{I}_2$

## 2. Eisenstein Series

Perhaps the nicest example of nonconstant modular forms come as *Eisenstein Series.*

**Definition 2.1.** For even $k \geq 4$, the *Eisenstein Series* of weight $k$ can be written as

$$G_k(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^k}.$$

A proof of the Eisenstein series being a modular form can be found in [3]. One thing to be noted about modular forms is that they satisfy the periodic condition $f(\tau) = f(\tau+1)$. The following, known as $q$ *expansion* can be motivated by considering other periodic functions which satisfy the same recurrence. The most famous nonconstant example of such a function in the complex plane is the exponential function, namely $e^{2\pi i\cdot\tau}$. Let $q = e^{2\pi i\cdot\tau}$. Then, $q = e^{2\pi i\cdot(\tau+1)}$ as well because $e^{2\pi i} = 1$, so this motivates us to rewrite the sum as a series in $q$, hence the name $q$ *expansion*. But how do we do so? First we must figure out what exactly happens when $\tau = x + yi$. Expanding, we have

$$q = e^{2\pi i\tau} = e^{-2\pi y}\cdot e^{2\pi ix} \implies 0 < |q| < 1$$

since $y$ is positive (recall that $f$ has domain $\mathbb{H}$.). Then, since $f(\tau) = f(\tau+n)$ if and only if $n \in \mathbb{Z}$, we can define a new function $\widetilde{f}\,(q) = f(\tau)$. In other words, we can take a modular form over $\mathbb{H}$ and convert it to a function over the (punctured) open unit disk, which we shall denote $D'$.

First remark that $\widetilde{f}$ is analytic on $D'$ and furthermore bounded (due to holomorphicity at $\infty$). Thus, due to the Riemann Removable Singularity Theorem, there is an analytic continuation of $\widetilde{f}$ from $D'$ to $D$. We are almost there: it is possible to write a power series centered around 0, which will give us a relatively nice $q$ expansion since we are dealing with the open unit disk now.

Now it is known that all modular forms have $q$-expansions centered at 0.

**Theorem 2.2.**

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n.$$

*where $\sigma_{k-1}(n)$ is the sum of the $k$th powers of the divisors of $n$.*

*Proof.* The following proof is from [4].

We will use the identity

$$\frac{1}{\tau} + \sum_{d=1}^{\infty}\left(\frac{1}{\tau-d} + \frac{1}{\tau+d}\right) = \pi\cot(\pi\tau) = \pi i - 2\pi i\sum_{m=0}^{\infty} q^m.$$

This can be shown by integrating the series of $\frac{1}{\sin^2(\pi z)}$ term by term and rearranging. Differentiating $k-1$ times with respect to $\tau$ and rearranging yields

$$\sum_{n\in\mathbb{Z}}\frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty} m^{k-1}q^m.$$

Then,

$$\begin{aligned}\sum_{(c,d)\in\mathbb{Z}}\frac{1}{(c\tau+d)^k} &= 2\sum_{d=1}^{\infty}\frac{1}{d^k} + 2\sum_{c=1}^{\infty}\sum_{d\in\mathbb{Z}} m^{k-1}q^{cm} \\ &= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\end{aligned}$$

as desired.

∎

Euler showed that for general $k$ even,

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = \frac{-(2\pi i)^k B_k}{k! \cdot 2},$$

where $B_k$ are the Bernoulli numbers, the coefficients of the power series of the function $f(x) = \frac{x}{e^x - 1}$. Upon noting now that

$$G_k(\tau) = 2\zeta(k) - \frac{4k\zeta(k)}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

the following definition makes sense:

**Definition 2.3.** The *normalized Eistenstein Series* $E_k(\tau)$ is defined as $E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)}$, i.e

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1} q^n, \quad q = e^{2\pi i \tau}.$$

Remark that we only defined Eisenstein series for even $k \geq 4$. This was due to issues regarding the absolute convergence of $E_2(\tau)$. However, it actually can be defined properly.

**Definition 2.4.** We define $E_2(\tau) = 1 - 24 \sum_{n \geq 1} \sigma(n) q^n$.

This is not actually a modular form. The second modularity condition fails; it actually holds that

$$E_2\left(-\frac{1}{\tau}\right) = \tau^2 E_2(\tau) - \frac{6i}{\pi}\tau.$$

In general, there are no modular forms of weight 2. That being said, $E_2(\tau)$ is still a function of particular interest.

Consider the function $2E_2(2\tau) - E_2(\tau)$. It can be shown that this satisfies the modularity conditions for the subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad N \mid c. \right\}$$

where $N = 2$.

In general, it is useful to define modular forms for finite-index *subgroups* of $\mathrm{SL}_2$.

**Definition 2.5.** A modular form for some subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a function $f : \mathbb{H} \to \mathbb{C}$. satisfying

(1) $f$ is holomorphic.
(2) Satisfies the condition

$$f(\tau)(ck + d)^k = f\left(\frac{a\tau + b}{c\tau + d}\right)$$

for *all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

(3)

$$\frac{1}{(c\tau + d)^k} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

is bounded as $\tau \to i\infty$ for *all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. This condition is known as being *holomorphic at the cusps.*

## 3. Application to Sum of Four Squares and Other Interesting Facts

Now that we have lots of background in place, we are ready for some applications. It may be surprising that many of the following can be proved using applications of modular forms.

**Theorem 3.1.** *(Lagrange) Every integer $n \geq 0$ is the sum of four positive integer squares (not necessarily distinct)*

Jacobi was able to actually find the number of such quadruples. Namely,

**Theorem 3.2.** *(Jacobi) For positive integers $n \geq 0$, number of quadruples $(a, b, c, d)$ of integers $r_4(n)$ satisfying*

$$a^2 + b^2 + c^2 + d^2 = n \quad \textit{is} \quad \begin{cases} 8\sigma(n) & n \textit{ odd} \\ 24\sigma(n_{odd}) & n \textit{ even.} \end{cases}$$

*where $n_{odd} = n/(2^{\nu_2(n)})$, i.e the largest odd number that divides $n$.*

*Proof.* To count the number of quadruples, we introduce the generating function

$$\theta(\tau) = \sum_{m \in \mathbb{Z}} e^{2\pi i m^2 \tau} = \sum_{m \in \mathbb{Z}} q^{m^2},$$

known as a *theta function*. The key is that when $\theta(\tau)$ is raised to the fourth power, we get the following relation:

$$\theta(\tau)^4 = \sum_{n \geq 0} r_4(n) q^n$$

where each coefficient counts what we want. Due to the Poisson Summation formula, it can be shown that $\theta(\tau)^4$ is a modular form of weight 2 under subgroup $\Gamma_0(4)$. Recall that $2E_2(2\tau) - E_2(\tau) \in M_2(\Gamma_0(2)) \subset M_2(\Gamma_0(2))$. Similarly, $2E_2(4\tau) - E_2(2\tau) \in M_2(\Gamma_0(4))$.

It can be separately shown the space $M_2(\Gamma_0(4))$ has only 2 dimensions. If we let

$$f_1 = E_2(2\tau) - E_2(\tau) \quad \text{and} \quad f_2 = 2E_2(4\tau) - E_2(2\tau),$$

since $f_2$ has no linear coefficient and $f_1$ does, $f_1$ and $f_2$ are not scalar multiples of each other, i.e they can be used as the basis for $M_2(\Gamma_0(4))$. Knowing that $\theta_4(\tau) \in M_2(\Gamma_0(4))$, we can freely write $\theta_4(\tau) = af_1 + bf_2$, i.e as some linear combination of $f_1$ and $f_2$. Now,

$$f_1 = 1 + 24 \sum_{n \geq 1} \sigma(n_{\text{odd}}) q^n$$

$$f_2 = 1 + 24 \sum_{n \geq 1} \sigma(n_{\text{odd}}) q^{2n}$$

Comparing coefficients, it follows that

$$\theta(\tau)^4 = \frac{1}{3} f_1 + \frac{2}{3} f_2 = 8 \sum_{2|n} \sigma(n) q^n + 24 \sum_{2\nmid n} \sigma(n_{\text{odd}}) q^n$$

as desired. ∎

We stated in the above proof that $M_2(\Gamma_0(4))$ has only two dimensions. Let $M_k$ denote the modular forms of weight $k$. It's also interesting to consider $\dim M_k$, and there's a way to explicitly find the dimension for all even $k$.

**Theorem 3.3.** *For even* $k$, $\dim M_k = \begin{cases} \lfloor \frac{k}{12} \rfloor & k \equiv 2 \pmod{12} \\ 1 + \lfloor \frac{k}{12} \rfloor & k \not\equiv 2 \pmod{12} \end{cases}$

The proof of this can be found in [3], but it's too long to include here. Nonetheless, it is interesting.

Another interesting application of modular forms comes when looking at the zeta function $\zeta(k)$.

**Proposition 3.4.** *For all integers* $k > 0$, $\frac{\zeta(k)}{\pi^k} \in \mathbb{Q}$.

The solution to this relies on the fact that if some modular form $f$ has any rational coefficient in it's $q$ expansion, then its constant term is also rational. Since modular forms are also modular forms up to scaling (since constants are also modular forms of weight 0). Then, dividing $G_k(\tau)$ by $\frac{(2\pi i)^k}{(k-1)!}$ yields the given solution.

## 4. A special type of Modular Form - The j function

The $j$-function is a modular function (i.e it is a modular form of weight 0) which satisfies many interesting properties. It was initially studied due to its connection with the monster group, a conjecture known as monstrous moonshine.

**Definition 4.1.** The *modular invariants* are defined to be

$$g_2(\tau) = 60G_4(\tau) \quad \text{and} \quad g_3(\tau) = 140G_6(\tau)$$

**Definition 4.2.** The *modular discriminant* $\Delta(\tau)$ is defined as

$$\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2.$$

Remark that this is a modular form of weight 12. Furthermore, this function has many interesting properties such as being a *cusp form*, in where its Fourier series has constant term 0. Finally, the modular discriminant is nonvanishing on $\mathbb{H}$, as proved in [3].

Now we are ready to define the $j$ function.

**Definition 4.3.** The *j-function* (or *j-invariant*) can be defined as a function $j : \mathbb{H} \to \mathbb{C}$ satisfying

$$j(\tau) = 1728\frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2} = 1728\frac{(g_2(\tau))^3}{\Delta(\tau)}.$$

Clearly, this is a modular form of weight 0. Furthermore, it can be shown that the $q$-expansion of $j(\tau)$ is

$$j(\tau) = \frac{1}{q} + 744 + 196844q + 21493760q^2 + \cdots .$$

These coefficients are surprisingly related to the *monster group*, which is the largest sporadic simple group.

However, there are more interesting properties to be unearthed which are unrelated to monstrous moonshine, such as the closeness of $e^{\sqrt{163}\pi}$ to an integer. This constant, known as *Ramanujan's constant* has approximation

$$e^{\pi\sqrt{163}} = 262537412640768743.999999999999\ldots$$

which is extremely close to an integer! The reason behind this lies behind the following:

**Theorem 4.4.** $j\left(\frac{1+i\sqrt{163}}{2}\right) = -640320^3.$

The proof behind this requires introducing elliptic curves and complex multiplication which will not be discussed in this paper, however the initial claim of $e^{\pi\sqrt{163}}$ being very close to an integer can be justified as follows:

When $\tau = \frac{1+i\sqrt{163}}{2}$, the corresponding value of $q$ can be computed as

$$q = e^{2\pi i\tau} = -e^{-\pi\sqrt{163}}.$$

Plugging this into the $q$-expansion of $j(\tau)$ gives

$$j(\tau) = \frac{1}{q} + 744 + O(-e^{-\pi\sqrt{163}})$$
$$\implies e^{\pi\sqrt{163}} \approx 640320^3 - 744$$

where in the above expression we omit the error terms and simply stick to the approximation. In any case, since $O(-e^{-\pi\sqrt{163}})$ is very small, the resulting value will be very close to an integer.

There also exist other values which have similar properties, known as *Heegner numbers*, which are, as written in [6], "The values of $-d$ for which imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$ are uniquely factorable into factors of the form $a + b\sqrt{-d}$. Here, $a$ and $b$ are half-integers, except for $d = 1$ and 2, in which case they are integers. The Heegner numbers therefore correspond to binary quadratic form discriminants $-d$ which have class number $h(-d)$ equal to 1, except for Heegner numbers $-1$ and $-2$, which correspond to $d = -4$ and $-8$, respectively."

It can be shown that there exist finitely many Heegner numbers, and they are:

$$-1, -2, -3, -7, -11, -19, -43, -67, \text{and} -163$$

These exhibit similar properties. For example,

$$e^{\pi\sqrt{43}} = 884736743.999\ldots \quad \text{and} \quad e^{\pi\sqrt{67}} = 147197952743.99999\ldots$$

## References


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