UNIVARIATE ASYMPTOTICS IN ANALYTIC COMBINATORICS

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1. Introduction

Analytic combinatorics revolves around finding good approximation to the coefficients of *Generating Functions*. Traditionally, one finds these coefficients by determining a closed form, and then extracting the coefficients from the closed form via clever combinatorial tricks. However, asymptotics presents a systematic method to produce strong approximations instead, also dealing with more complex closed forms.

2. Generating Functions

A generating function is of the form

$$f(x) = \sum_{n \geq 0} a_n x^n.$$

We can represent some particular combinatorial scenario with a recurrence relation, and then solve for the relation's generating function. At this point, we can extract coefficients from the function's *formal* power series. Of course, it helps to be a bit more rigorous.

2.1. Formal Power Series.

Unlike power series, a formal power series has no issues of convergence. If one considers the set of all formal power series with coefficients in a commutative ring (for example, $\mathbb{Z}$ defined with addition and multiplication), the set of all formal power series itself forms a ring. The proof is beyond the scope of this paper, but this fact gives the operation of addition and multiplication the these series, and these, in combination allow for a new broad set of operations. These include:

- Raising a power series to a power
- Finding a multiplicative inverse
- Division
- Extracting Coefficients
- Composition
- Differentiation
- Antidifferentiation

The class of generating functions including the ordinary and expontential generating functions are also collectively known as rational functions.

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2.2. **Algebraic Generating Function.**

Besides the rational functions, algebraic functions arise frequently in combinatorics. In particular, when a combinatorial class solves a convolution equation, its generating function solves an algebraic equation.

Example. Consider a generating function that satisfies the following recurrence:

$$F(z) - 1 = zF(z)^2.$$

Solving for $F(z)$, we find it is equal to $\frac{1 \pm \sqrt{1-4z}}{2z}$. However, only the negative solution is valid because otherwise, division would not be valid at $z = 0$. That is, an example of an algebraic generating function is

$$\frac{1 - \sqrt{1-4z}}{2z}.$$

By expanding out the power series for this, we find that the n-th term of the recurrence is

$$\frac{1}{n+1}\binom{2n}{n}.$$

Of important note is the fact that convergence issues can arise when there are multiple solutions to an algebraic function.

3. Basic Methods

Previously, we considered generating functions as *formal objects*, subject to algebraic manipulation. In either case, the goal is to obtain a method to extract coefficients of a generating function. In the former case, one usually resorts to clever combinatorial tricks to guess coefficients from a closed form. However, when treating these functions as analytic objects, we can utilize methods from complex analysis to obtain strong approximations for the coefficients.

4. Asymptotic Method Outline

A key assumption of the following sections is that the generating function of any particular scenario is a rational function. In general we can guarantee this via a symbolic method. That is, if we can construct the generating function of a scenario with a combination of some finite number of some operations, then the corresponding generating function is rational.

The following are the allowed operations:

- Disjoint Union
- Cartesian Product
- Sequence

That is, consider two rational generating functions $A(x)$ and $B(x)$. Then, the new generating function representing the disjoint union of the objects represented by A and B would be $A(x) + B(x)$. The Cartesian product would be $A(x)B(x)$. Finally, consider just the objects of B. The list of ordered sequences of objects from B (this would be the sequence operation) is of the form $\frac{1}{1-B(x)}$. Therefore, if we begin with any rational generating functions, any other combinatorial objects we can form with a combination of the previous three scenarios would also be rational. This makes it much easier to extract coefficients. For the sake of brevity, we say that $[z^n]A(z)$ refers the the coefficient of z^n in the power series of $A(z)$.

4.1. **Singularity Analysis.** By our previous assumption, the generating function is rational. That is, it is of the form $f(x) = \frac{A(x)}{B(x)}$, where A and B are polynomials. Therefore, after some partial fraction decomposition, we find that $f(x) = g(x) + \sum \frac{c}{(1-x/p)^r}$ for some constants c and p and polynomial $g(x)$. Thus, it suffices to perform singularity analysis on $\frac{1}{(1-x/p)^r}$. Let $[x^n]\frac{1}{(1-x/p)^r}$ refer to the coefficient of x^n in the expansion of function. Then,

$$[x^n]\frac{1}{(1-x/p)^r} = p^{-n}[x^n](1-x)^{-r}$$

$$= p^{-n}\binom{n+r-1}{r-1}$$

$$\approx \frac{p^{-n}n^{r-1}}{(r-1)!}.$$

Then $f(n) = [x^n]f(x) = cp^{-n}n^{r-1}$.

Example. Consider $f(z) = (1-z^2/2)^{-5}(1-z^3)^{-1}(1-2x)^{-5}$. Here, the dominant singularity is $z = 1/2$ and the multiplicity of the singularity is 5, so, from the previous theorem, we have that the coefficient $f(n) \approx c2^n n^4$.

4.2. **Sub exponential Asymptotic Term.** We can generalize our singularity analysis to non-rational generating functions too.

Theorem 4.1. $[z^n]\frac{1}{(z-a)^a}\log^k(\frac{1}{1-z}) \approx \frac{n^{a-1}}{\Gamma a}\log^k(n)$.

Theorem 4.2. *Transfer Theorem WLOG, let* 1 *be the smallest singularity of a particular function* f. *Then, the following holds true:*

- *If* $f(z) \approx g(z)$ *as* $z \to 1$, *then* $f(n) \approx g(n)$
- *If* $f(z) \approx O(g(z))$*as* $z \to 1$, *then* $f(n) \approx O(g(n))$
- *If* $f(z) \approx o(g(z))$*as* $z \to 1$, *then* $f(n) \approx o(g(n))$

That is to say, the coefficients of a generating function depend entirely on the behavior of a function around the neighborhood of its smallest singularity.

5. Unlabelled Objects

Here, we utilize the transfer theorem and sub-exponential asymptotic term theorem to determine the coefficients of common combinatorial objects asymptotically.

5.1. **Binary Trees.** The generating function describing these trees is $\frac{1-\sqrt{1-4z}}{2z}$. The singularity is at $z = 1/4$, and the multiplicity is $-1/2$. Thus, the behavior around the singularity can approximately be modeled by the function $-2\frac{1}{(1-4z)^{-1/2}}$. Applying the sub-exponential asymptotic theorem, we have that the coefficient $f(n) \approx -2\frac{4^n n^{-3/2}}{\sqrt{pi}}$.

5.2. **Unary-Binary Trees.** The generating function for unary-binary trees can be solved in the form $A(z) = z + A(z)^2$. This gives $A(z) = \frac{1-\sqrt{1-4z}}{2z}$ and reapplying the theorem from the earlier example, we have $f(n) \approx -2\frac{4^{n-1}n^{-3/2}}{\sqrt{pi}}$.

6. Labeled Objects

Note that our previous theorems and formulas hinged on the fact that the power series of a generating function converged at a particular neighborhood of a point. However, with labeled objects, the coefficient $n!$ grows much faster than n^r for any value of r, meaning that the radius of convergence is 0. Since the generating functions are no longer analytic, we can no longer use clever techniques from complex analysis. The solution to this is to use exponential generating functions instead.

6.1. Exponential Generating Functions.

An exponential generating function is just like an ordinary generating function, except each term is normalized by $n!$. That is,

$$f(x) = \sum_{n \geq 0} \frac{a_n}{n!} x^n.$$

The following are the allowed operations:

- Disjoint Union
- Cartesian Product
- Sequence

The same arithmetic operations that apply to OGF also apply to EGF. That is, a disjoint union still implies a sum, and so forth. Once again, this implies that the corresponding generating function is rational, and singularity analysis can again be utilized.

6.2. Lagrange Inversion Theorem.

One may observe that many EGFs are of the form $z\phi(A(z))$ for some function ϕ. If so, then we have the formula

$$[z^n]A(z)^k = \frac{k}{n}[y^{n-1}]\phi(y)^n.$$

Proof. This is a direct result of Cauchy's Integral Formula. Define $f(z)$ to be some analytic function of the form $\sum_{n \geq 0} f_n z^n$. Then, from the formula, we have

$$f_n = \frac{1}{2\pi i} \int \frac{f(z)dz}{z^{n+1}}.$$

Define $z = A(z)/\phi(A(z)) = y/\phi(y)$. Differentiating, we have $dz = \frac{dy}{\phi(y)} - \frac{y\phi'(y)}{\phi^2(y)}$. Plugging this back into the integral formula, we have

$$[z^n]A(z) = \frac{1}{2\pi i} \int \frac{\phi(y)^n dy}{y^n} - \frac{1}{2\pi i} \int \frac{\phi(y)^{n-1}\phi' dy}{y^{n-1}} =$$

$$[y^{n-1}]\phi(y)^n - \frac{1}{n}[y^{n-2}](\phi(y)^n)'.$$

The result follows. ■

6.3. Analytic Inversion.

If there exists a function $Y(z) = z\phi(Y(z))$ with a positive radius of convergence R, and there exists a unique τ such that $\phi(\tau) = \tau\phi'(\tau))$, then the radius of convergence $p = 1/\phi'(\tau)$ and $Y(z)$ has an asymptotic expansion at its singularity of the form $Y(z) \approx \tau - \gamma\sqrt{1-z}$, where $\gamma = \sqrt{2\phi(\tau)/\phi''(\tau)}$.

6.4. **Cayley's Formula.** The formula counts the number of trees on n labeled vertices. We can construct the generating function to be $A(z) = z \cdot e^{A(z)}$. Analytic inversion implies that $A(z) \approx 1 - \sqrt{2}\sqrt{1 - ez}$ as z approaches the singularity at e^{-1}. From here, singularity analysis implies $A_n = n![z^n]A(z) \approx \frac{n! e^n n^{-3/2}}{\sqrt{2\pi}}$.

7. Bibliography

References

[1] Phillipe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, Cambridge, 2009.