Fourier Transforms

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Chapter 1

Introduction

A Fourier transform, abbreviated as FT, is a transformation that composes and decomposes functions. First conceived by French mathematician Jean-Baptiste Joseph Fourier in 1822, the Fourier transform is an integral part of modern world. This paper will analyze the applications of the Fourier transform alongside its mathematical background.

Chapter 2

Background

The Fourier transform can be best understood through studying Fourier series. When studying Fourier series, complex, periodic functions can be represented as the sum of trigonometric functions. The transform builds off the Fourier series that is generated when the period of the function is extended towards infinity. This allows for compression and limited storage of information through a mathematical method.

2.1 Definitions

The most common definition of the Fourier transform is as follows:

$$f(\omega) = \frac{1}{\sqrt{m}} \int_{-\infty}^{\infty} \exp(\sigma q i \omega x) f(x) dx.$$

Here, m and q can take the values of 1 or $2pi$, while σ can take the values of -1 or 1. Therefore, only a limited number of Fourier transforms exist. The eight possible ones are listed as follows:

$$F_{+11},\ F_{+1\tau},\ F_{+\tau 1},\ ...,\ F_{-\tau\tau}.$$

The most common definition for $(\mathcal{F}_{-\tau 1} f)$ is further defined:

$$(\mathcal{F}_{-\tau 1} f)(\omega) = \int_{-\infty}^{\infty} \exp(-2\pi i \omega x) f(x)\, dx.$$

This paper will primarily focus on the applications of the Fourier transform, along with the specific transforms used within each setting.

2.2 Applications: JPEG

The JPEG is a common application of the Fourier transform. Known as a lossy compression technique, the JPEG compresses large images into a small file, allowing for easy distribution of files through telecommunications. The JPEG specifically works by applying a discrete cosine transform, a transform that builds off the Fourier transform and was invented by electrical engineer Nasir Ahmed in 1972.

The use of cosine within the function allows for impressive compression ratios. Due to a variety of properties which will be discussed further, cosine compresses encoded information in an efficient manner.

Chapter 3

Discrete Cosine Transform

3.1 Definitions

The DCT is an invertible linear function of the form $f : \mathbb{R}^N \to \mathbb{R}^N$. There are four major variants of the DCT which will be further discussed.

3.2 DCT-1

The DCT-1 cosine transform matrix, which is rarely used today, is defined below.

$$X_k = \tfrac{1}{2}(x_0 + (-1)^k x_{N-1}) + \textstyle\sum_{n=1}^{N-2} x_n \cos\left[\tfrac{\pi}{N-1} nk\right] \qquad k = 0, \ldots, N-1.$$

The DCT-1 is equivalent to a Fourier transform (discrete) of 2N-2 real numbers, all of which have even symmetry. This does not hold for N less than two. We further see that x_n is even at both n = 0 and n = N-1, which holds for X_k.

3.3 DCT-2

This is the most common form of the DCT. It is defined below:

$$X_k = \textstyle\sum_{n=0}^{N-1} x_n \cos\left[\tfrac{\pi}{N}\left(n + \tfrac{1}{2}\right) k\right] \qquad k = 0, \ldots, N-1.$$

This DCT, which is oftentimes denoted as "the DCT," is equivalent to a discrete Fourier transform with 4N real inputs, all with even symmetry.

Matlab uses a varied version of the DCT-2, multiplying the X_0 by $\frac{1}{\sqrt{2}}$. This resulting matrix is multiplied by a total scale factor of $\sqrt{\frac{2}{N}}$.

JPEG, however, uses an arbitrary scaling, because the computational process optimizes for a scaling that utilizes the least computer time.

3.4 DCT-3

This is the inverse of the DCT-2. Is is defined as:

$$X_k = \frac{1}{2} x_0 + \sum_{n=1}^{N-1} x_n \cos\left[\frac{\pi}{N} n \left(k + \frac{1}{2}\right)\right] \qquad k = 0, \ldots, N-1.$$

This is commonly referred to as the inverse DCT, denoted as the IDCT. This also can also be defined by dividing x_0 by $\sqrt{2}$ instead of its 2, as it is currently defined.

3.5 DCT-4

This is another iteration of the DCT-4, which has yet to be adopted. Is is defined as:

$$X_k = \sum_{n=0}^{N-1} x_n \cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)\left(k+\frac{1}{2}\right)\right] \qquad k = 0, \ldots, N-1.$$

Further research has yielded the modified discrete cosine transform. Additional information regarding this process can be found at the end of this paper.