AN INTRODUCTION TO MULTIVARIABLE COMPLEX ANALYSIS

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1. Preliminaries

1.1. **Confusing Notation.** We introduce the notation that will be used for the remainder of this paper. We define

$$|(z_1,\ldots,z_n)| = \sqrt{|z_1|^2+\cdots+|z_n|^2} \qquad (z_1,\ldots,z_n)\in\mathbb{C}^n,$$

for $(z_1,\ldots,z_n)\in\mathbb{C}^n$. We generally denote this point by a single letter z. Notice that we have $|z-w|\leq|z-x|+|x-w|$ for $z,x,w\in\mathbb{C}^n$ (this is geometrically obvious identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$). If $\alpha=(\alpha_1,\ldots,\alpha_n)\in(\mathbb{Z}^+)^n$, we write

$$z^\alpha=(z_1^{\alpha_1},\ldots,z_n^{\alpha_n}).$$

In the case that $z_i=\alpha_i=0$, we take $z_i^{\alpha_i}=1$. We write $\alpha!=\alpha_1!\cdot\cdots\cdot\alpha_n!$, and $|\alpha|=\alpha_1+\cdots+\alpha_n$. We write $\mathbf{1}=(1,\ldots,1)\in(\mathbb{Z}^+)^n$, and in general $\mathbf{N}=(N,\ldots,N)$. We write $\alpha\in(\mathbb{Z}^+)^n\geq 0$ if $\alpha_i\geq 0$ for all i. For $n\in\mathbb{N}$, we say $\alpha\leq n$ if $\alpha_i\leq n$ for all i.

1.2. **Structure of $\mathbb{C}^n$.** In this paper, we aim to generalize the concepts of complex analysis into multiple dimensions. To do this, we must first generalize concepts such as open sets and limits.

Definition 1.1. Let $z\in\mathbb{C}^n$ and $r>0$. We define the open ball of radius r centered at z to be the set

$$B^n(z,r)=\{w\in\mathbb{C}^n:|z-w|<r\},$$

and the closed ball of radius r centered at z to be the set

$$\overline{B^n(z,r)}=\{w\in\mathbb{C}^n:|z-w|\leq r\}.$$

Definition 1.2. Let $E\subseteq\mathbb{C}^n$. We say that E is open if, for every $z\in E$, there is $r_z\in\mathbb{R}$ such that $B^n(z,r_z)\subseteq E$. We say that E is closed if $\mathbb{C}^n\setminus E$ is open.

Just as in the case of $\mathbb{C}$, $B^n(z,r)$ is open. The proof is identical, so we skip it.

Definition 1.3. Let $E\subseteq\mathbb{C}^n$. We define the interior of E to be

$$\mathring{E}=\{z\in E:\exists\delta>0, B^n(z,\delta)\subseteq E\},$$

the boundary of E to be

$$\mathrm{Bd}(E)=\{z\in\mathbb{C}^n:\forall\delta>0, B^n(z,\delta)\cap E\neq\varnothing, B^n(z,\delta)\cap E^c\neq\varnothing\},$$

and the closure of E to be

$$\overline{E}=\{z\in\mathbb{C}^n:\forall\delta>0, B^n(z,\delta)\cap E\neq\varnothing\}.$$

Proposition 1.4. *Let $E\subseteq\mathbb{C}^n$. The interior of E is the largest open set contained in E, and the closure of E is the smallest closed set containing E.*

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Proof. Suppose that $z \in \mathring{E}$, say with $B^n(z,r) \subseteq E$. Let $w \in B^n(z,r)$. Then, for $x \in B^n(w, r-|z-w|)$, we have that $|x-z| \leq |x-w| + |w-z| < r - |z-w| + |w-z| = r$. Thus, $x \in B^n(z,r) \subseteq E$, so $w \in \mathring{E}$. Hence, $\mathring{E}$ is open. Now, for any open set $A \subsetneq E$, we have that, for each $z \in A$, there is r with $B^n(z,r) \subseteq A \subseteq E$. Thus, $z \in \mathring{E}$, so $A \subseteq \mathring{E}$. Now, let $x \in \overline{E}^c$. Then, by definition, there is some $\delta > 0$ with $B^n(z,\delta) \cap E = \varnothing$. Now suppose that $w \in B^n(z,\delta) \cap \overline{E}$. Then, there is $x \in B^n(w, \delta - |z-w|) \cap E$. Then, $|x-z| \leq |z-w| + |x-w| < |z-w| + \delta - |z-w| = \delta$, which is a contradiction, as $x \in E$. Thus, $B^n(z,\delta) \cap \overline{E} = \varnothing$, so $\overline{E}$ is closed. Notice also that $E \subseteq \overline{E}$. Now, suppose that $A \supseteq E$ is closed, and that $z \in \overline{E} \setminus E$ is not in A. Then, for any $\delta > 0$, $B^n(z,\delta) \cap E \neq \varnothing$, so $B^n(z,\delta) \cap A \neq \varnothing$. Thus, A^c is not open, a contradiction. ■

Notice that $\overline{E} = \mathring{E} \cup \mathrm{Bd}(E)$.

Generalizing the notion of open balls, we have the following.

Definition 1.5. Let $z \in \mathbb{C}^n$ and $r \in (\mathbb{Z}^+)^n$. The open polydisc of radius r centered at z is the set

$$D^n(z,r) = \{w \in \mathbb{C}^n : |w_j - z_j| < r_j, 1 \leq j \leq n\}$$

and the closed polydisc of radius r is the set

$$\overline{D^n(z,r)} = \{w \in \mathbb{C}^n : |w_j - z_j| \leq r_j, 1 \leq j \leq n\}.$$

The boundary of the polydisc is the set $\mathrm{Bd}(D^n(z,r))$, consisting of all points in the closed polydisc satisfying $|z_j - a_j| = r_j$ for some j. We define the distinguished boundary $T^n(z,r)$ of the polydisc to be the set $\{w : w_j = z_j + r_j e^{i\theta_j}, 0 \leq \theta_j < 2\pi, 1 \leq j \leq n\}$. These are not the same set, except when $n = 1$. For example, when $n = 2$, $\mathrm{Bd}(D^n(z,r))$ has $\mathbb{R}$-dimension 3 (identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$), while $T^n(z,r)$ has $\mathbb{R}$-dimension 2.

Definition 1.6. Let $f : \mathbb{C}^n \to \mathbb{C}$, and fix $w \in \mathbb{C}^n$. Suppose that there exists an L such that, for any $\varepsilon > 0$, there exists $\delta > 0$ so that

$$z \in B^n(w,\delta) \implies f(z) \in B^1(L,\varepsilon).$$

Then we say that L is the limit of f approaching w, and write $L = \lim_{z \to w} f(z)$.

Notice that such an L must be unique. If L and $L' \neq L$ both satisfy our conditions, we may choose $\varepsilon = |L - L'|/2$. If δ_L and $\delta_{L'}$ are the corresponding values from the definition, and $\delta = \min\{\delta_L, \delta_{L'}\}$, then we have that

$$z \in B^n(w,\delta) \implies f(z) \in B^1(L,\varepsilon), B^1(L',\varepsilon) \implies f(z) \in B^1(L,\varepsilon) \cap B^1(L',\varepsilon).$$

However, if such a value $f(z) = y$ exists, then

$$|L - L'| \leq |L - y| + |L' - y| < 2\varepsilon = |L - L'|.$$

Definition 1.7. Let $f : \mathbb{C}^n \to \mathbb{C}$ and $w \in \mathbb{C}^n$. We say that f is continuous at w if $\lim_{z \to w} f(z)$ exists and is equal to $f(w)$. More generally, for $E \subseteq X$, we say that f is continuous on E if f is continuous at every $w \in E$. If f is continuous on X, we say f is continuous.

It is easy to check that, if $f, g : \mathbb{C}^n \to \mathbb{C}$ are continuous, so are $fg : \mathbb{C}^n \to \mathbb{C}$ and $f + g : \mathbb{C}^n \to \mathbb{C}$.

Definition 1.8. A bounded subset $D \subseteq \mathbb{C}^n$ is called circular if $z \in D$ means that $(e^{i\theta}z_1, \ldots, e^{i\theta}z_n) \in D$ for all $\theta \in \mathbb{R}$. More generally, we say D is multicircular if $(e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n) \in D$ for all $\theta_1, \ldots, \theta_n \in \mathbb{R}$.

This easy generalization from $\mathbb{C}$ to $\mathbb{C}^n$ is a special case of what are called metric spaces, see [Cop88].

2. Power Series

2.1. **Ambiguity in Ordering.**

When we first consider infinite sums, we consider infinite sums of the form

$$\lim_{k\to\infty} \sum_{a=1}^{k} b_a.$$

This sum implicitly uses the natural ordering on $\mathbb{N}$. When considering a sum over $\mathbb{N}^n$, we have many different ways of summing, none of which are canonical. For instance, we have

$$\lim_{k\to\infty} \sum_{\alpha_1=0}^{k} \cdots \sum_{\alpha_n=0}^{k} b_\alpha,$$

and

$$\lim_{k\to\infty} \sum_{j=0}^{k} \sum_{|\alpha|=j} b_\alpha,$$

and these need not be equal. Because of this, we restrict ourselves to sums converging absolutely, so we may reorder the terms arbitrarily. We write $\sum_{\alpha\geq 0} b_\alpha$ for this value.

2.2. **Domain of Convergence.**

Definition 2.1. The domain of convergence of the power series is the interior of the set of points at which the series converges absolutely.

For example, consider the power series $\sum_{n=1}^{\infty} z_1^n z_2^{n!}$. This converges when

$$(z_1, z_2) \in \{(z_1, z_2) : |z_2| < 1\} \cup \{(z_1, z_2) : z_1 = 0\} \cup \{(z_1, z_2) : |z_1| < 1, |z_2| = 1\}.$$

The latter two sets have no interior points, and the first set is open, so the domain of convergence of this series is $\{(z_1, z_2) : |z_2| < 1\}$.

Suppose that $(z_1, \ldots, z_n)$ is in the domain of the convergence D of some power series. Because we define this by absolute convergence, we have that $(\lambda_1 z_1, \ldots, \lambda_n z_n) \in D$ when $1 = |\lambda_1| = \cdots = |\lambda_n|$. Thus, D is multicircular. Further, using the comparison test for absolute convergence of series $(\lambda_1 z_1, \ldots, \lambda_n z_n) \in D$ when $|\lambda_j| \leq 1$ for each j. In particular, writing $r = (|z_1|, \ldots, |z_n|)$, we have that

$$D = \bigcup_{z\in D} T^n(0, r).$$

We have deduced that every convergence domain is a union of polydiscs centered at the origin. To continue this discussion, we need the following lemma.

Lemma 2.2 (Hölder's Inequality). *For any two sequences $(x_k)_k$ and $(y_k)_k$ in $\mathbb{C}^n$,*

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{\infty} |y_k|^q\right)^{1/q},$$

where $1/p + 1/q = 1$ and all series converge.

Now suppose that both $\sum_\alpha |c_\alpha z^\alpha|$ and $\sum_\alpha |c_\alpha w^\alpha|$ converge. As we require absolute convergence, we can use an enumeration of $(\mathbb{Z}^+)^n$, and write $\sum_{k=1}^{\infty} |c_{\alpha_k} z^{\alpha_k}|$ and $\sum_{k=1}^{\infty} |c_{\alpha_k} w^{\alpha_k}|$ for these series respectively. Now, for $0 \leq t \leq 1$, we have that

$$\sum_\alpha |c_\alpha||z^\alpha|^t|w^\alpha|^{1-t} = \sum_{k=1}^{\infty} |c_{\alpha_k} z^{\alpha_k}|^t |c_{\alpha_k} w^{\alpha_k}|^{1-t} \leq \left(\sum_{k=1}^{\infty} \left(|c_{\alpha_k} z^{\alpha_k}|^t\right)^{1/t}\right)^t \left(\sum_{k=1}^{\infty} \left(|c_{\alpha_k} w^{\alpha_k}|^{1-t}\right)^{1/(1-t)}\right)^{1-t},$$

using Hölder's Inequality on $p = 1/t$ and $q = 1/(1-t)$. Thus, if $z, w \in D$, then $z^t w^{1-t} \in D$ when $0 \leq t \leq 1$. It turns out that these conditions are sufficient for a set to be the domain of convergence of a power series. For a proof, see [Boa12].

We will not use this discussion further in this paper, but will rather use power series to motivate holomorphicity.

3. Holomorphic Functions

3.1. **Motivation and Definition.** In $\mathbb{C}$, convergent power series are local models for holomorphic functions. In $\mathbb{C}^n$, power series converge uniformly on compact sets, so they represent continuous functions. Further, these power series can be viewed as power series in one variable, which makes the function of one variable holomorphic. This motivates the following definition.

Definition 3.1. Let $D \subseteq \mathbb{C}^n$ be an open set. We say $f : D \to \mathbb{C}$ is holomorphic on D if it is continuous, and it is holomorphic in each of its variables.

Definition 3.2. Let $K \subseteq \mathbb{C}^n$. A function $f : K \to \mathbb{C}$ is called holomorphic on K if, for each $a \in K$, there is an open neighborhood D such that $D \cap K$ is closed and there is a function f_D which is holomorphic on D, and $(f_D)|_{D\cap K} = (f)|_{D\cap K}$.

It turns out that we do not need to require f to be continuous.

Proposition 3.3. *If f, g are holomorphic on D, then $f + g$, $f - g$, and $f \cdot g$ are holomorphic on D. If $g(z) \neq 0$ for all $z \in D$, then f/g is holomorphic on D.*

Proof. This all follows from the one-dimensional analogue. ■

3.2. **Partial Derivatives.** Suppose that we have a holomorphic function $f : \mathbb{C} \to \mathbb{C}$. Write $f = u + iv$, where $u, v : \mathbb{R}^2 \to \mathbb{R}$. We then have that

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Applying the Cauchy-Riemann Equations, we get that

$$\frac{df}{dz} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right).$$

Thus,

$$\frac{df}{dz} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right] = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right).$$

This value can be computed more generally when f is holomorphic. Further, if f is holomorphic, it coincides with the derivative of f. Suppose that $g : \mathbb{C}^n \to \mathbb{C}$. Write $(z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n)$, so that we can view $g : \mathbb{R}^{2n} \to \mathbb{C}$. We define

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) f.$$

When f is holomorphic on some open set D, this agrees with what we would expect the partial derivative of f to be.

For $m \in (\mathbb{Z}^+)^n$ and $z \in \mathbb{C}^n$, we write

$$\frac{\partial^m f}{(\partial z)^m} = \left(\frac{\partial^{m_1}}{(\partial z_1)^{m_1}} \cdots \frac{\partial^{m_n}}{(\partial z_n)^{m_n}} f\right)(z).$$

4. Integration

4.1. **Definition.** In $\mathbb{R}$, our integrals are of the form

$$\int_a^b f(x)dx.$$

In other words, we integrate over the interval $[a, b]$. Now, in $\mathbb{R}^n$, our integrals are of the form

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \dots, x_n)dx_n \cdots dx_1.$$

Here, we integrate over the set

$$[a_1, b_1] \times \cdots \times [a_n, b_n].$$

Since we integrate over curves in $\mathbb{C}$, it makes sense that we integrate over the product of curves in $\mathbb{C}^n$. Given curves $\gamma_1, \dots, \gamma_n$ and $f : \mathbb{C}^n \to \mathbb{C}$, we define

$$\int_{\gamma_1 \times \cdots \times \gamma_n} f(z)dz = \int_{\gamma_1} \cdots \int_{\gamma_n} f(z_1, \dots, z_n)dz_n \cdots dz_1.$$

For example, we are able to integrate over $T^n(a, r)$, as it is the product of curves, but we cannot integrate over $\mathrm{Bd}(B^n(a, r))$.

4.2. **A General Cauchy Integral Formula.**

Theorem 4.1. *Let f be holomorphic on $\overline{D^n(a, r)}$. Then, for all $z \in D^n(a, r)$,*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{T^n(a,r)} \frac{f(w)}{(w - z)^{\mathbf{1}}} dw.$$

Proof. We induct on n, with $n = 1$ being the usual Cauchy Integral Formula. Suppose that this formula is proven $n - 1$ dimensions. Define $g(z) = f(z, z_2, \dots, z_n)$, which is holomorphic on $\overline{D^1(a_1, r_1)}$. Applying the Cauchy Integral Formula, we have that

$$f(z_1, z_2, \dots, z_n) = \frac{1}{2\pi i} \int_{T^1(a_1,r_1)} \frac{f(w_1, z_2, \dots, z_n)}{w_1 - z_1} dw_1.$$

If we fix w_1, then $(z_2, \dots, z_n) \mapsto f(w_1, z_2, \dots, z_n)$ is holomorphic on $\overline{D^{n-1}((a_2, \dots, a_n), (r_2, \dots, r_n))}$. Thus,

$$f(w_1, z_2, \dots, z_n) = \frac{1}{(2\pi i)^{n-1}} \int_{T^{n-1}((a_2,\dots,a_n),(r_2,\dots,r_n))} \frac{f(w_1, z_2, \dots, z_n)}{(w_2 - z_2)\cdots(w_n - z_n)} dw_2 \cdots dw_n.$$

Plugging this into the right hand side of our first equation, we get our result. ∎

We once again have a derivative version of the Cauchy Integral Formula

$$\frac{\partial^k f}{(\partial z)^k} = \frac{k!}{(2\pi i)^n} \int_{T^n(a,r)} \frac{f(w)}{(w - z)^{k+\mathbf{1}}} dw.$$

The proof of this is analogous to the proof of the ordinary Cauchy Integral Formula using induction on n. This in particular implies that all partial derivatives of f exist.

We write $f^{(k)}(z)$ for $\frac{\partial^k f}{(\partial z)^k}$.

4.3. Consequences of the Cauchy Integral Formula.

Corollary 4.2. *Suppose that f is holomorphic on $\overline{D^n(a,r)}$ and that $|f(z)| \leq M$ for all $z \in \overline{D^n(a,r)}$. Then, for $w \in D^n(a,r)$, we have*

$$|f^{(k)}(w)| \leq \frac{Mk!}{r^k}.$$

Proof. We have that

$$|f^{(k)}(w)| = \left| \frac{k!}{(2\pi i)^n} \int_{T^n(a,r)} \frac{f(z)}{(z - w)^{k+\mathbf{1}}} dz \right| \leq \frac{k!}{(2\pi)^n} \int_{T^n(a,r)} \left| \frac{f(z)}{(z - w)^{k+\mathbf{1}}} \right| dz.$$

By repeated applications of the ML-Inequality, this is bounded above by $\frac{Mk!}{r^k}$. ∎

Corollary 4.3. *If f is holomorphic on the open polydisc $D^n(a,R)$, then*

$$f(z) = \sum_{k \geq 0} \frac{1}{k!} f^{(k)}(a)(z - a)^k.$$

Proof. Firstly, notice that we still have

$$\frac{1}{(\mathbf{1} - z)^{\mathbf{1}}} = \sum_{0 \leq k \leq N-1} z^k + \frac{z^{\mathbf{N}}}{1 - z}.$$

Pick $z_1 \in B^n(a,r)$, and suppose that $((z_1)_i) = r_i < R_i$. Write $r = (r_1, \dots, r_n)$. Pick $r_i < r_i' < R_i$, and write $r_0 = (r_1', \dots, r_n')$. We therefore have that

$$f(z_1) = \frac{1}{(2\pi i)^n} \int_{T^n(a,r)} \frac{f(z)}{(z - z_1)^{\mathbf{1}}} dz.$$

Now, by our geometric sum, we have that

$$\frac{1}{(z - z_1)^{\mathbf{1}}} = \sum_{0 \leq k \leq N-1} \frac{(z_1 - a)^k}{(z - a)^{k+\mathbf{1}}} + \frac{(z_1 - a)^N}{(z - z_1)^{\mathbf{1}}(z - a)^N}.$$

Multiplying both sides by $f(z)$ and integrating over $T^n(a, r_0)$, we get

$$\int_{T^n(a,r_0)} \frac{f(z)}{z - z_1} dz = \sum_{0 \leq k \leq N-1} \int_{T^n(a,r_0)} \frac{f(z)}{(z - a)^{k+\mathbf{1}}} dz (z_1 - a)^k + (z_1 - a)^{\mathbf{N}} \int_{T^n(a,r_0)} \frac{f(z)}{(z - z_1)^{\mathbf{1}}(z - a)^{\mathbf{N}}}.$$

Dividing both sides by $(2\pi i)^n$ and simplifying, we get

$$f(z_1) = \sum_{0 \le k \le N-1} \frac{f^{(k)}(a)}{k!}(z_1 - a)^k + \frac{(z_1 - a)^{\mathbf{N}}}{(2\pi i)^n} \int_{T^n(a, r_0)} \frac{f(z)}{(z - z_1)^{\mathbf{1}}(z - a)^{\mathbf{N}}} dz.$$

It suffices to show that the left integral tends to 0 as N tends to ∞. If M is the maximum of $|f(z)|$ over $T^n(a, r_0)$, then repeated use of the ML-Inequality gives that this integral is at most

$$\frac{r^{\mathbf{N}}}{(2\pi)^n} \cdot \frac{M}{(r_0 - r)^{\mathbf{1}} r_0^{\mathbf{N}}} \cdot (2\pi)^n r_0^{\mathbf{1}} = \frac{r^{\mathbf{N}}}{r_0^{\mathbf{N}}} \frac{M r_0^{\mathbf{1}}}{(r_0 - r)^{\mathbf{1}}}.$$

By construction of r_0, this tends to zero as N tends to infinity. ■

Corollary 4.4. *Suppose that f is holomorphic on an open connected subset of $\mathbb{C}^n$. If f vanishes on an open subset of D, then $f = 0$ on D.*

Proof. Consider the set $E = \{z \in D : \forall n \in (\mathbb{Z}^+)^n, f^{(n)}(z) = 0\}$. This is the intersection of $E_n = \{z \in D : f^{(n)}(z) = 0\}$. Now, each the E_n are closed, as the $f^{(n)}$ are continuous. Thus, E is closed. Now, if $z \in E$, and U is a neighborhood of z in which the Taylor Series Expansion holds, then $f(z) = 0$ in U, so E is open. Thus, E is clopen so $E = D$ or $E = \varnothing$. Now, if f vanishes on an open subset of D, then $E \neq \varnothing$, so $E = D$ and $f = 0$ on D. ■

5. A Difference From $\mathbb{C}$

5.1. **Statement.** Recall the Riemann Mapping Theorem in $\mathbb{C}$, which states that every nonempty simply connected open set that is not all of $\mathbb{C}$ is conformally equivalent to the unit disc. We show that (an analog of) this is not true in higher dimensions.

Definition 5.1. Given $D \subseteq \mathbb{C}^n$, $f = (f_1, \ldots, f_m) : D \to \mathbb{C}^m$ is called holomorphic if each of the coordinate functions $f_k : D \to \mathbb{C}$ is holomorphic. A map $f : D_1 \to D_2$ is called biholomorphic if it is holomorphic and has an inverse which is holomorphic. We say that D_1 is biholomorphic to D_2.

We prove the following theorem.

Theorem 5.2 (Poincaré)**.** *The ball $B^n(0, 1)$ is not biholomorphic to the polydisc $D^n(0, \mathbf{1}\mathbf{1})$ for $n \geq 2$.*

We write B^n and D^n for these sets.

5.2. **Basic Group Theory.**

Definition 5.3. A group is a set G with a binary operation $\cdot : G \times G \to G$ such that the following are true.

- $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$ for all $g_1, g_2, g_3 \in G$.
- There is $e \in G$ such that $e \cdot g = g \cdot e = g$ for all $g \in G$.
- For each $g \in G$, there is $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$.

Example. The integers $\mathbb{Z}$ are a group under $a \cdot b = a + b$.

We write gh for $g \cdot h$.

Proposition 5.4. *The identity element $e \in G$ is unique. Further, for $g \in G$, the inverse g^{-1} is unique.*

Proof. Suppose that e and e' are identity elements. Then,

$$e = ee' = e'.$$

Now, suppose that h and k are both inverses of g. We then have

$$h = he = h(gk) = (hg)k = ek = k.$$

■

Definition 5.5. A group G is said to be abelian if $gh = hg$ for all $g, h \in G$.

Definition 5.6. Let G be a group. A subset $\varnothing \neq H \subseteq G$ is called a subgroup of G if, for all $g, h \in H$, $gh \in H$ and $g^{-1} \in H$.

Subgroups are essentially smaller groups that are contained in the larger one.

Aside from groups as objects, we also study certain maps between these groups.

Definition 5.7. Let G and H be groups. A homomorphism from G to H is a map $\phi : G \to H$ such that

$$\phi(gh) = \phi(g)\phi(h)$$

for all $g, h \in G$. A bijective homomorphism is called an isomorphism. If there exists an isomorphism between G and H, we say that G and H are isomorphic.

For example, for any groups G and H, there is the trivial homomorphism $\phi(g) = e_H$. An important result we will use is the following.

Proposition 5.8. *Suppose that G and H are isomorphic groups, and that G is abelian. Then H is abelian.*

Proof. Let $\phi : G \to H$ be an isomorphism, and let $h_1, h_2 \in H$. Write $h_1 = \phi(g_1)$ and $h_2 = \phi(g_2)$. Then,

$$h_1h_2 = \phi(g_1)\phi(g_2) = \phi(g_1g_2) = \phi(g_2g_1) = \phi(g_2)\phi(g_1) = h_2h_1.$$

■

We will also need the notion of a topological group. However, delving into this takes too much time, so we leave the reader to get acquainted with the subject on their own.

5.3. **Proof.** We denote the group of biholomorphic maps from D to D by $\mathrm{Aut}(D)$, and the subgroup of maps fixing $a \in D$ by $\mathrm{Aut}_a(D)$. When D is bounded, we make $\mathrm{Aut}(D)$ into a topological group by defining $d(\sigma_1, \sigma_2) = \sup_{z\in D} |\sigma_1(z) - \sigma_2(z)|$. We write $\mathrm{Aut}^{\mathrm{Id}}(D)$ for the group of automorphisms connected to the identity.

Lemma 5.9. *If D_1 is biholomorphic to D_2, then the groups $\mathrm{Aut}(D_1)$ and $\mathrm{Aut}(D_2)$ are isomorphic. Given $a_1 \in D_1$ and $a_2 \in D_2$ for which there is a biholomorphic map $f : D_1 \to D_2$ with $f(a_1) = a_2$, then $\mathrm{Aut}_{a_1}(D_1)$ and $\mathrm{Aut}_{a_2}(D_2)$ are isomorphic. Further, $\mathrm{Aut}^{\mathrm{Id}}(D_1)$ and $\mathrm{Aut}^{\mathrm{Id}}(D_2)$ are isomorphic, as well as $\mathrm{Aut}^{\mathrm{Id}}_{a_1}(D_1)$ and $\mathrm{Aut}^{\mathrm{Id}}_{a_2}(D_2)$.*

Proof. Let $f : D_1 \to D_2$ be biholomorphic. Then,

$$\sigma \mapsto f \circ \sigma \circ f^{-1}$$

is a group homomorphism from $\mathrm{Aut}(D_1)$ to $\mathrm{Aut}(D_2)$. Because it is invertible, it is an isomorphism. It is also clearly an isomorphism between $\mathrm{Aut}_{a_1}(D_1)$ and $\mathrm{Aut}_{a_2}(D_2)$. This also serves as an isomorphism between $\mathrm{Aut}^{\mathrm{Id}}(D_1)$ and $\mathrm{Aut}^{\mathrm{Id}}(D_2)$, along with $\mathrm{Aut}^{\mathrm{Id}}_{a_1}(D_1)$ and $\mathrm{Aut}^{\mathrm{Id}}_{a_2}(D_2)$. ■

Given an $n \times n$ matrix, we write A^* for the conjugate transpose of A. We say a matrix is unitary if $AA^* = I$. We write $\mathrm{SU}(n)$ for the group of unitary $n \times n$ matrices with determinant 1. We leave it as an exercise to verify that $\mathrm{SU}(n)$ is not abelian.

Proposition 5.10. *$\mathrm{Aut}_0^{\mathrm{Id}}(B^n)$ is not abelian.*

Proof. We have that $\mathrm{SU}(n)$ is a subgroup of $\mathrm{Aut}_0^{\mathrm{Id}}(B^n)$, as $A \in \mathrm{SU}(n)$ defines a biholomorphic map given by $z \mapsto Az$. This map leaves 0 invariant. ■

The following proposition establishes our theorem.

Proposition 5.11. *For every $a \in D^n$, the group $\mathrm{Aut}_a^{\mathrm{Id}}(D^n)$ is abelian.*

However, proving this requires a bit more work.

Definition 5.12. A function $f(z) = \sum_{|k|=N} a_k z^k$ is called a homogeneous polynomial of degree N.

Proposition 5.13. *Let $D \subseteq \mathbb{C}^n$ be bounded, and let $a \in D$. If $f \in \mathrm{Aut}_a(D)$ satisfies $f'(a) = 1$, then $f(z) = z$ for all $z \in D$.*

Proof. Assume without loss of generality that $a = 0$. We have that $\overline{D} \subseteq D^n(0, R)$ for some $R > 0$. For all $f \in \mathrm{Aut}_0(D)$, if we write $f(z) = \sum_n a_n z^n$, we have that $|a_n| \leq M r^{-n}$, where r is so that $D^n(0, r) \subseteq D$. Now, write f's Taylor Expansion as

$$f(z) = z + f_N(z) + \cdots ,$$

where f_k denotes an n-tuple of homogeneous polynomials of degree k, and N is chosen to be the smallest possible k. We then have that f^k has Taylor Expansion

$$f^k(z) = z + k \cdot f_N(z) + \cdots .$$

However, for large k, this contradicts our bound. We thus have that $f_N = 0$. As we chose N to be minimal, $f(z) = z$ in $D^n(0, r)$. By our analog of analytic continuation, $f(z) = z$ on D. ■

We write $k_\theta(z) = (e^{i\theta} z_1, \ldots, e^{i\theta} z_n)$. If $\{0\} \subseteq D \subseteq \mathbb{C}^n$ is circular, then k_θ is an element of $\mathrm{Aut}_0(D)$.

Corollary 5.14. *Let $D \subseteq \mathbb{C}^n$ be bounded and circular, and suppose $0 \in D$ and $f \in \mathrm{Aut}_0(D)$. Then f is linear.*

Proof. Let

$$g = k_{-\theta} \circ f^{-1} \circ k_\theta \circ f.$$

Then,

$$g'(0) = k'_{-\theta}(0) \cdot (f^{-1})'(0) \cdot k'_\theta(0) \cdot f'(0) = 1,$$

so that $g(z) = z$. Thus,

$$k_\theta \circ f = f \circ k_\theta.$$

Write $f = (f_1, \ldots, f_n)$, so that $f_j(e^{i\theta} z) = e^{i\theta} f_j(z)$. Let $f_j(z) = \sum_{k \geq 0} a_k z^k$. Then,

$$e^{i\theta} a_k = e^{i|k|\theta} a_k,$$

meaning that $a_k = 0$ for all $|k| > 1$. ■

Now, our result follows from this next corollary.

Corollary 5.15. *Every* $f = (f_1, \dots, f_n) \in \mathrm{Aut}(D^n)$ *has the form*

$$f_j(z) = e^{i\theta_j} \frac{z_{p(j)} - a_j}{1 - \overline{a_j}} z_{p(j)},$$

where $\theta_j \in \mathbb{R}$, $a \in D^n$, *and* p *is a permutation of the multi-index* $j = (j_1, \dots, j_n)$.

Proof. Clearly this map is an automorphism. Write σ_a for the map given by

$$f_j = \frac{z - a_j}{1 - \overline{a_j} z}.$$

Then the inverse of σ_a is given by σ_{-a}. If $f \in \mathrm{Aut}(D^n)$, then $\sigma_{f(0)} \circ f$ leaves 0 invariant. We show that all elements of $\mathrm{Aut}_0(D^n)$ permute the variables and multiply by some number with norm 1. Suppose that $f \in \mathrm{Aut}_0(D^n)$. As D^n is circular, f is linear, say $f_k = \sum_{j=1}^n A_{k,j} z_j$. We therefore have that $\sum_{k=1}^n |A_{k,j}| \leq 1$, as $f(D^n) \subseteq D^n$. Now, consider the sequence $z^{(n)} = (0, \dots, 0, 1 - 1/n, 0, \dots, 0)$, converging to the boundary of D^n. We then have that $f(z^{(n)}) = \big((1 - 1/n)A_{1,j}, \dots, (1 - 1/n)A_{n,j}\big)$ converges to the boundary of D^n. Thus, $|A_{k,j}| = 1$ for some k. Write $q(j)$ for this value. As $\sum_{k=1}^n |A_{k,j}| \leq 1$, q is a permutation. If p is the inverse permutation of q, then $f_k(z) = A_{k,p(k)} z_{p(k)}$, and $|A_{k,p(k)}| = 1$. ■

References

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