BROWNIAN MOTION AND COMPLEX ANALYSIS

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Abstract. In this paper, I would like to cover some important topics, theorems, etc. relating to Brownian motion. We will begin by introducing the basic scientific idea behind Brownian motion, then we'll cover the mathematics of it. I will be using the papers by [Zha13], [Lei12], [Nua16], and [Rud17]. For those less familiar with Markov chains, I will cover the basics needed to understand Brownian Motion from this perspective also.

1. A brief look at the history and science of Brownian motion

In 1827, botanist Robert Brown observed the strange motion of the plant *Clarkia pulchella*'s pollen when placed in water. [Bro28] This fascinating phenomenon, named after Brown, is essentially caused by the random movements of particles in liquid/gas. There are plenty of other examples —coal dust and alcohol, diffusion of calcium through bones, and diffusion of pollutants through the air, to name a few. Around 80 years after Brown first observed this phenomenon, Albert Einstein wrote a paper [1] in which he stated that the particles were being moved by individual water molecules. This was an extremely important paper in science, because Einstein was able to prove the existence of atoms (a debated question originating nearly a century earlier).

Einstein's paper was then used by the physicist Jean Perrin, who proved Dalton's atomic theory[2] using ideas from Einstein's paper and Brownian motion. Brownian motion is occasionally refered to as "pedesis," from Ancient Greek $\pi\eta\delta\eta\sigma\iota\varsigma$, meaning "leaping." This refers to the random movement of the particles. The connection between Brownian motion and random walks should be obvious by now, but we will discuss more about this later in the paper. Let's now turn to the mathematics of Brownian motion.

2. A crash course in Brownian motion

To begin, we will need to know a bit about Markov chains and random walks. Loosely speaking, a Markov chain is a sequence $X_0, X_1, X_2, \ldots$ of random variables where the future depends on the present but not on the past. The change of the state of a Markov chain is called a transition. Similarly, the probabilities for these changes of states are called transition probabilities. When dealing with transition probabilities, we often see the notation $p_{i,j}$, which denotes the probability of moving from i to j in one time step. Now we can give a definition of the random walk:

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[1]Titled "Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen." For those interested in the translation, I'd loosely translate this as "On Molecular Kinetic Heat Theory of the movement demanded of particles suspended in static liquids."

[2]A very famous theory in chemistry, essentially stating what we now see as common facts: all matter is composed of atoms, atoms aren't divisible (Greek α (not) + $\tau\epsilon\mu\nu\omega$ (I cut)="atom"), etc.

Definition 2.1. Suppose we have n vertices on the complete graph K_n, and for each step, we move to another one of the $n-1$ vertices with probability $\frac{1}{n-1}$. We can denote these transition probabilities as follows:

$$\begin{cases} 0 & i = j \\ \frac{1}{n-1} & i \neq j \end{cases}$$

We also find that if $n > 2$, the Markov chain is ergodic (recurrent, has a period of 1, and has finite mean recurrence time), but this is beyond the scope of this paper. The point remains that the expected number of steps to return to the starting state is n. We call this process the *random walk* on K_n.

It may be a bit of a jump from our basic knowledge of Markov chains, but we should now look at the definition of a σ-algebra, and some related definitions (mostly taken from [Lei12]):

Definition 2.2. A σ-algebra on some set S is a nonempty collection of subsets [3] of S, such that

(1) $S \in \Sigma$;
(2) if $A \in \Sigma$, then $A^c \in \Sigma$;
(3) if $A_1, A_2, \ldots \in \Sigma$, then $\bigcup_{i=1}^{\infty} A_i \in \Sigma$.

A *measurable space* consists of a pair (S, Σ), where Σ is a σ-algebra over S. If $\mathcal{C}$ is a collection of subsets of S, then the σ-algebra generated by $\mathcal{C}$, denoted $\sigma(\mathcal{C})$, is the intersection of the σ-algebras on S with $\mathcal{C}$ as a subcollection.

We can also review the definitions of measures/measure spaces and probability measure/space:

Definition 2.3. Let (S, Σ) be a measurable space. A map $\mu : \Sigma \to [0, 1]$ is called a *measure* when $\mu(\{\varnothing\}) = 0$ and is countably additive. So, we have

$$\mu\left(\bigcup_{j=1}^{\infty} F_j\right) = \sum_{j=1}^{\infty} \mu(F_j).$$

Additionally, we call the triple (Ω, Σ, μ) a *measure space*. If X is a function $f : \Omega \to \mathbb{R}$, we say that X is Σ-measurable if $X^{-1}(H) \subseteq \Sigma$ for all $H \in \sigma(\mathbb{R})$. Finally, for a measure space (S, Σ, μ), when $\mu(\Sigma) = 1$, we say this map is a *probability measure* and the associated measure space is called a *probability space*.

Note that going forward, we will be using the more common notation $(\Omega, \mathcal{F}, \mathbb{P})$ for measure spaces, rather than (S, Σ, μ). Another common notation is *i.o.*, which stands for "infinitely often," often when dealing with set-theoretic limits. Let's now recall some standard definitions in measure-theoretic probability.

Definition 2.4. We say that the measure space $(\Omega, \mathcal{F}, \mathbb{P})$ is a *probability triple*, where Ω denotes a *sample space* (and $\omega \in \Omega$ is a *sample point*), $\mathcal{F}$ denotes a σ-algebra called a *family of events*. A *measure* on a set can be thought of as a way to assign a number to some subsets of the set.

Finally, we have stochastic processes:

[3] The power set of a set S is the set of all subsets of S, including $\varnothing$ and S itself.

Definition 2.5. A *stochastic process* is a collection of random variables $\{W_t : t \in \mathcal{T}\}$ on a probability space, where $\mathcal{T}$ is a set of times.

Now we can formally define Brownian motion mathematically:

Definition 2.6. *Brownian motion* started at $x \in \mathbb{R}$ refers to a stochastic process such that the following hold:

(1) $W_0 = x$;
(2) For every $0 \leq s \leq t$, $W_t - W_s$ has normal distribution with mean zero and variance $t - s$, and $|W_t - W_s|$ is independent of $\{W_r : r \leq s\}$;
(3) With probability 1, the function $t \to W_t$ is continuous.

If the Brownian motion begins at 0, we call it *standard Brownian motion.*

3. Brownian motion on the dyadic rationals

Before we introduce the definitions, let's briefly recall what a dyadic rational is.[4] A dyadic rational is analogous to a 2-adic rational.

In his paper, mathematician Peter Rudzis tells us that "The first ingredient needed for the mathematical description of Brownian motion is Gaussian distribution." [Rud17] This is indeed true; without the definition of Gaussian distribution (which from here on, we shall refer to as *normal distribution*), it is impossible to understand Brownian motion.

Definition 3.1. A *random variable* is a function $X : \Omega \to E$. A random variable X has *normal distribution* with mean μ and variance σ^2 if

$$\mathbb{P}(X > x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_x^\infty e^{\frac{-(u-\mu)^2}{2\sigma^2}} du.$$

We often denote normal distribution as $N(\mu, \sigma^2)$.

Remark 3.2. It is a good idea to understand the definitions thoroughly, but it's also useful to go by the simpler notation (e.g., the notation in the latter definition above) in following definitions. We will mostly use notations without discussing how the actual (often-complicated) definition fits in; nevertheless, we should try to keep the original meanings in mind.

Definition 3.3. We denote the set of non-negative dyadic rationals as $\mathcal{D} = \bigcup_n \mathcal{D}_n$, where $\mathcal{D}_n = \{\frac{k}{2^n} : k = 0, 1, 2 \ldots\}$. A standard, 1-dimensional Brownian motion on the dyadic rationals $\{W_q : q \in \mathcal{D}\}$ is a random process such that for all n, the random variables $W_{k/2^n} - W(k-1)/2^n$, $k \in \mathbb{N}$ are independent and $N(0, \frac{1}{2^n})$.

We will use the following proposition to prove that Brownian motion exists over the dyadic rationals.

Proposition 3.4. *Suppose X and Y are independent normal random variables, each $N(0, 1)$. Define Z and $\hat{Z}$ as*

$$Z = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}$$

[4] I assume the reader's basic knowledge of the p-adic number system. You may want to see https://www.overleaf.com/project/5d7ff7f3b33b1e0001f7bc02 for a quick introduction.

and similarly

$$\hat{Z} = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}.$$

Then Z and $\hat{Z}$ are independent $N(0,1)$ variables.

Lemma 3.5. *Standard Brownian motion exists over the dyadic rationals.*

Proof. We will prove this using recursive method. Using Definition 3.3., we can define the following:

$$J(k,n) = 2^{n/2}[W_{k/2^n} - W_{(k-1)/2^n}].$$

Let us assume there exist a countable number of independent normal random variables $\{Z_n : n \in \mathbb{N}\}$. We will use a recursive method to define W_q. Now, for $n = 0$, we have $\{J(k,0) = Z_k : k \in \mathbb{N}\}$. Assume there exists n where $\{J(k,n) : k \in \ltimes\}$ was defined using $\{Z_q : q \in \mathbb{D}\}$ and thus are independent $N(0,1)$ variables. This gives us our definition for $J(k, n+1)$:

$$J(2k-1, n+1) = \frac{J(k,n)}{\sqrt{2}} + \frac{Z_{(2k+1)/(2^{n+1}}}{\sqrt{2}}$$
$$J(2k, n+1) = \frac{J(k,n)}{\sqrt{2}} - \frac{Z_{(2k+1)/2^{n+1}}}{\sqrt{2}}.$$

Seeing those $\sqrt{2}$'s in the denominators should remind us of Proposition 3.4; using it repeatedly gives us $\{J(k, n+1) : k \in \mathbb{N}\}$ (independent $N(0,1)$ variables), so we can now define $W_{k/2^n}$ as follows:

$$W_{k/2^n} = 2^{-n/2} \sum_{j=1}^{k} J(j,n).$$

We then have

$$2^{n/2}(W_{k/2^n} - W_{(k-1)/2^n}) = \sum_{j=1}^{k} J(j,n) - \sum_{j=1}^{k-1} J(j,n) = J(k,n),$$

thus proving the existence of standard Brownian motion over the dyadic rationals. [Rud17] □

Here is another lemma, this time about convergence of Brownian motion:

Lemma 3.6. *If W_q, $q \in \mathcal{D}$ is a standard 1-dimensional Brownian motion, then it is almost certain that this function converges uniformly on every closed interval $[a,b]$.*

The proof for this lemma is rather long; it utilizes dyadic rationals, the triangle inequality, the Borel-Cantelli lemma and involves bounding Brownian motion using integrals. See Leiner's paper for a complete proof. [Lei12]

Theorem 3.7. *Standard Brownian motion exists.*

Proof. This is a rather silly statement, for which we can make a fairly straightforward proof. Essentially, this follows from uniform continuity. Let us choose $\frac{\epsilon}{2}$ that yields some δ where $|W_t - W_s| \leq \frac{\epsilon}{2}$ for all $s, t \in \mathcal{D}$. Now pick $n_0 \in \mathbb{N}$ such that $\frac{1}{2^{n_0}} < \delta$. Pick $a \in \mathcal{D}$, and

$n, m > n_0$ and $k_{n_0} = 0, 1, \ldots, 2^{n_0}$ such that $0 < a - \frac{k_{n_0}}{2^n} < \frac{1}{2^n}$. We can pick k_n and k_m similarly. We get

$$|k_n - k_{n_0}| < \frac{1}{2^{n_0}}$$
$$|k_m - k_{n_0} < \frac{1}{2^{n_0}}.$$

This gives us $|W_{k_n} - W_{k_m}| < |W_{k_n} - W_{k_{n_0}}| + |W_{k_m} - W_{k_{n_0}} < \epsilon$. We have a Cauchy sequence W_{k_n}, so there is a convergent subsequence with a unique limit. If we define $\{W_t, t \in \mathbb{R}\}$ to be the limit, we get a unique extension of $\{W_q : q \in \mathcal{D}\}$ to $\{W_t : t \in \mathbb{R}\}$ that is continuous. □

One interesting thing about Brownian motion is that, although it is continuous, it is differentiable nowhere. The reader may find Leiner's section on non-differentiability interesting.

4. Brownian motion as a Markov process

Even if one knows only the loosest definition of a Markov chain, one will see that Brownian motion is a perfectly intuitive example of a random walk, which we will recall in the following definition:

Definition 4.1. A *random walk*, loosely defined, is a *stochastic/random* process describing a path consisting of random "steps" on some space.

An intuitive observation about random walks is that they are basically Markov chains, because the $(n+1)^{\text{th}}$ step does not depend on the $(n-1)^{\text{th}}$ step.

Remark 4.2. Note that many authors may refer to Markov chains as *Markov processes*, but the two are essentially the same. Note that a Markov process might be a more general term (i.e., with a continuous state space and continuous movements), but a Markov chain may assume discreteness in state space or time steps.

Definition 4.3. Let $W_1, \ldots, W_d$ be independent Brownian motions started in $x_1, \ldots, x_d$, then the random process W_t given by $W_t = (W_1, \ldots, W_d)$ is called a *d-dimensional Brownian motion* started in $(x_1, \ldots, x_d)$. If W_t starts at the origin, we call this a *standard d-dimensional Brownian motion.*

Definition 4.4. A *filtration* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a family $\{\mathcal{F}(t) : t \geq 0\}$ of σ-algebras where $\mathcal{F}(S) \subset \mathcal{F}(t) \subset \mathcal{F}$. We call a probability space with a filtration a *filtered probability space.* A random process $\{X_t : t \geq 0\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is *adapted* if X_t is $\mathcal{F}(t)$ measurable for all $t \geq 0$.

Theorem 4.5 (Simple Markov Property for Brownian motion)**.** *Let $\{W_t : t \geq 0\}$ be a Brownian motion started in $x \in \mathbb{R}^d$. Then the process $\{W_{t+s} - W_s : t, s > 0\}$ is a Brownian motion started at the origin, and it is independent of $\{W_t : 0 \leq t \leq s\}$.*

What this theorem is essentially saying is that we know just as much from the current position as we do from the past positions in Brownian motion. An analogous explanation is that we do not need to know the past positions, we focus entirely on the current one. The latter is the main idea surrounding Markov chains. In addition to the simple Markov property, we also have the strong Markov property. First, let's define the notation $\mathcal{F}^+$:

Definition 4.6. The germ σ-algebra is defined as $\mathcal{F}^+(0)$, where

$$\mathcal{F}^+(t) = \bigcap_{s>t} \mathcal{F}^0(s)$$

and $\{\mathcal{F}^0 : t \geq 0\}$ is the σ-algebra generated by $\{W_t : 0 \leq s \leq t\}$.

Theorem 4.7 (Strong Markov Property for Brownian motion)**.** *For each (almost always) finite stopping time T, we have that the process $\{W_{T+t} - W_T : t \geq 0\}$ is a standard Brownian motion independent of $\mathcal{F}^+T$.*

Proof. In this proof, the general idea is to define a discrete approximation which stops at the first dyadic rational next to the original in terms of stopping times. Using the Markov property, we will find that $\{W_{T+t} - W_T : t \geq 0\}$ is indeed a standard Brownian motion. See Leiner's paper for the complete proof. [Lei12] $\square$

5. Introduction to Martingales

Generally speaking, a *martingale* is a sequence of random variables such that, for a given time, the expected value of the next step in the sequence is equal to that of the current step (regardless of the previous steps). This is also a great example of a Markov chain–a sequence in which the n^{th} step is not determined by the $(n-k)^{\text{th}}$ step. Additionally, we call a process in which we cannot see into the future an *adapted process*. Let's now see the more technical definition:

Definition 5.1. An adapted process $M = \{M_t, t \geq 0\}$ is called a *martingale* with respect to $\mathcal{F}_t$, where $\{\mathcal{F}_t, t \geq 0\}$ denotes a filtration, if the following apply.

(1) For all $t \geq 0$, we have $E(|M_t|) < \infty$.

(2) For each $s \leq t$, we have $E(M_t|\mathcal{F}_s) = M_s$.

A few things to note about about the second condition–first, we can write it as $E(M_t - M_s|\mathcal{F}_s) = 0$. Second, we call M_t a *supermartingale/submartingale* if the second condition is instead $E(M_t|\mathcal{F}_s) \leq M_s$ or $E(M_t|\mathcal{F}_s) \geq M_s$. We have that for any integrable random variable X, $\{E(X|\mathcal{F}_t), t \geq 0\}$ is a martingale. [Nua16]

Stopping times occur when a given stochastic process exhibits a behavior of interest and we look at the random variable's value at this time. Martingales give us an interesting theorem concerning stopping times.

Theorem 5.2. *Suppose M_t is a continuous martingale. Let $S \leq T \leq K$ be two bounded stopping times. Then we have*

$$E(M_T|\mathcal{F}_S) = M_S.$$

Furthermore, we simply have that

$$E(M_T) = E(M_S).$$

Proof. Let's show that $E(M_T) = E(M_0)$. Suppose T is some value in the set $0 \leq t_1 \leq \ldots \leq t_n \leq K$. By the martingale property, we have

$$\begin{aligned} E(M_T) &= \sum_{i=1}^{n} E(M_T 1_{\{T=t_i\}}) \\ &= \sum_{i=1}^{n} E(M_{t_i} 1_{\{T=t_i\}}) \\ &= \sum_{i=1}^{n} E(M_{t_n} 1_{\{T=t_i\}}) \\ &= E(M_{t_n}) \\ &= E(M_0). \end{aligned}$$

We can approximate T as follows:

$$\tau_n = \sum_{k=1}^{2^n} \frac{kK}{2^n} 1_{\frac{(k-1)K}{2^n} \leq T < \frac{kK}{2^n}}.$$

We have that $M_{\tau_n} \to M_T$ by continuity. We must now show that M_{τ_N} is integrable:

$$\begin{aligned} E(|M_{\tau_N}| 1_{|M_{\tau_n}| \geq A}) &= \sum_{k=1}^{2^n} E\left(|M_{\frac{kK}{2^n}}| 1_{\left\{ |M_{\frac{kK}{2^n}}| \geq A, \tau_n = \frac{kK}{2^n} \right\}} \right) \\ &\leq \sum_{k=1}^{2^n} E\left(|M_K| 1_{\left\{ |M_{\frac{kK}{2^n}}| \geq A, \tau_n = \frac{kK}{2^n} \right\}} \right) \\ &= E(|M_K| 1_{\{|M_{\tau_n}| \geq A\}}) \\ &\leq E(|M_K| 1_{\{\sup_{0 \leq s \leq K} |M_S| \geq A\}}). \end{aligned}$$

This converges to 0 as $A \to \infty$, uniformly in n. This completes the proof. [Nua16] $\square$

Here's another interesting theorem, which we will also give a proof of:

Theorem 5.3. *Let $\{M_t, t \in [0,t]\}$ be a continuous martingale, where $\mathbb{E}(|M_T|^p) < \infty$ for some $p \geq 1$. Then, for all $\lambda > 0$, the following holds:*

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} |M_t| > \lambda \right) \leq \frac{1}{\lambda^p} \mathbb{E}(|M_T|^p).$$

If p is strictly greater than 1, then we have

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |M_t|^p \right) \leq \left(\frac{p}{p-1} \right)^p \mathbb{E}(|M_T|^p).$$

Proof. First we look at

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} |M_t| > \lambda \right) \leq \frac{1}{\lambda^p} \mathbb{E}(|M_T|^p).$$

If we let

$$\tau = \inf\{s \geq 0 : |M_S| \geq \lambda\} \wedge T.$$

Notice that τ is a bounded stopping time, and $|M_t|^p$ is a martingale, so we have

$$\mathbb{E}(|M_\tau|^p) \leq \mathbb{E}(|M_T|^p).$$

Next, we consider

$$\mathbb{E}\left(\sup_{0\le t\le T}|M_t|^p\right) \le \left(\frac{p}{p-1}\right)^p \mathbb{E}(|M_T|^p),$$

for $p > 1$. We have

$$|M_\tau|^p \ge 1_{\{\sup_{0\le t\le T}|M_t|\ge\lambda\}}\lambda^p + 1_{\{\sup_{0\le t\le T}|M_t|<\lambda\}}|M_T|^p$$

by definition of τ. This implies that

$$\mathbb{P}\left(\sup_{0\le t\le T}|M_t| > \lambda\right) \le \frac{1}{\lambda^p}\mathbb{E}(|M_\tau|^p) \le \frac{1}{\lambda^p}\mathbb{E}(|M_T|^p).$$

□

6. Martingales and Brownian motion

Here are some applications of martingales to Brownian motion, in a series of short propositions with quick proofs.

Proposition 6.1. *Let B_t be a Brownian motion. Consider $a \in \mathbb{R}$ and the hitting time*

$$\tau_a = \inf\{t \ge 0 : B_t = a\}.$$

If $a < 0 < b$, then we have

$$P(\tau_a < \tau_b) = \frac{b}{b-a}.$$

Proof. A stopping theorem for martingales states that the expected value of a martingale at a stopping time is equal to its initial expected value. By this theorem, we have

$$E(B_{t\wedge\tau_a}) = E(B_0) = 0.$$

If we let $t \to \infty$, we have

$$aP(\tau_a < \tau_b) + b(1 - P(\tau_a < \tau_b)) = 0.$$

□

Proposition 6.2. *Let $T = \inf\{t \ge 0 : B_t \notin (a,b)\}$, where $a < 0 < b$. Then we have*

$$E(T) = -ab.$$

Proof. We can use the fact that $B_t^2 - t$ is a martingale to see that

$$E(B_{T\wedge t}^2) = E(T \wedge t).$$

Furthermore, we have

$$E(T) = \lim_{t\to\infty} E(B_{T\wedge t}^2) = E(B_T^2) = -ab.$$

□

Here is one more interesting proposition:

Proposition 6.3. *Let $a > 0$. Then the hitting time*

$$\tau_a = \inf\{t \ge 0 : B_t = a\}$$

satisfies (assuming $\alpha > 0$)

$$E[\exp(-\alpha\tau_a)] = e^{-\sqrt{2\alpha}a}.$$

Now, we will finally define the strong Markov property in terms of martingales:

Theorem 6.4. *Let B be a Brownian motion and let T be a finite stopping time where the filtration $\mathcal{F}_t^B$ is generated by B. Then the process*

$$\{B_{T+t} - B_T, t \geq 0\}$$

is a Brownian motion independent of B_T.

Proof. Consider the process $\tilde{B}_t = B_{T+t} - B_T$, and suppose T is bounded. Let $\lambda \in \mathbb{R}$ and $0 \leq s \leq t$. We have

$$E[e^{i\lambda B_{T+t}+\frac{\lambda^2}{2}(T+t)}|\mathcal{F}_{T+s}] = e^{i\lambda B_{T+s}+\frac{\lambda^2}{2}(T+s)}$$

by the optional stopping theorem for the martingale

$$\exp\left(i\lambda\tilde{B}_t + \frac{\lambda^2 t}{2}\right).$$

Therefore, we have

$$E[e^{i\lambda(B_{T+t}-B_{T+s})}|\mathcal{F}_{T+s}] = e^{-\frac{\lambda^2}{2}(t-s)}.$$

[Nua16] □

7. Ito's Process/Formula and Levy's Theorem

The end goal of this paper is to prove Liouville's Theorem, a famous theorem from complex analysis, using Brownian Motion. To do so, we first need to discuss the Ito Process and Ito's Formula, then Levy's Theorem.

Definition 7.1 (Ito Process)**.** An Ito process is a stochastic process I_t on $(\Omega, \mathcal{F}, P)$ of the form

$$I_t = I_0 + \int_0^t u(s,w)ds + \int_0^t v(s,w)dB_s,$$

where $v \in V$, $P(\int_0^t v^2(s,w)ds < \infty, \forall t \geq 0) = 1$, u is $\mathcal{F}_t$-adapted and $P(\int_0^t |u(s,w)|ds < \infty, \forall t \geq 0) = 1$. The above equation can be written as follows

$$dI_t = udt + vdB_t,$$

where the first term is called *drift* and the second is called *volatility*.

Now that we know about the Ito Process, we can looks at Ito's Formula:

Theorem 7.2 (Ito's Formula)**.** *Let I_t be an Ito process. Let $g(t,I) \in C^2([0,\infty) \times \mathbb{R})$. Then $Y_t = g(t, I_t)$ is an Ito process and we have*

$$dY_t = g_t'(t,I_t)dt + g_i'(t,I_t)dI_t + \frac{1}{2}g_{ii}''(t,I_t)(dI_t)^2,$$

where

$$(dI_t)^2 = (udt + vdB_t)^2 = v^2 dt.$$

Now we substitute dI_t and get

$$dY_t = (g_t' + g_i'u + \frac{1}{2}g_{ii}''v^2)dt + g_x'vdB_t.$$

Here, the first term is drift and the second term is volatility as in the Ito Process definition.

Before we get into the next part, we should define *harmonic*. We say a function is harmonic if two variables have the property that its value at any point is equal to the average of its values along any circle around that point, provided the function is defined within the circle. Now, the next theorem comes from Ito's formula:

Theorem 7.3. *Let $D \subset \mathbb{R}^d$ be a connected open set and $f : D \to \mathbb{R}$ be harmonic on D. Let B_t with $0 \leq t \leq T$ be a Brownian motion which starts in D and stops at T, then the process $f(B_t) : 0 \leq t \leq T$ is a local martingale.*

Theorem 7.4 (Levy's Theorem)**.** *Suppose that both M and $(M_t^2 - t)_{t\geq 0}$ are local martingales*[5]*. Assume $M_0 = 0$. Then M is a Brownian Motion with respect to $(\mathcal{F}_t)$.*

Proof. Let $f(x) = e^{ivx}$, where $v \in \mathbb{R}$. We have $f \in C^2(\mathbb{R})$, so by Ito's formula we have the following:

$$f(M_t) = f(0) + \int_0^t f'(M_s)dM_s + \frac{1}{2}\int_0^t f''(M_s)ds,$$

where $M_t^f := \int_0^t f'(M_s)dM_s$ is a local martingale. Also, if f' and f'' are bounded, then M_t^f is a martingale. So we have the expected value $\mathbb{E}$:

$$\mathbb{E}[f(M_t)] = f(0) + \frac{1}{2}\int_0^t \mathbb{E}[f'(M_s)]ds.$$

Now, let $g(t) = \mathbb{E}[f(M_t)]$. Then, by substituting $f(M_t)$ in the integral, we have

$$g(t) = 1 - \frac{v^2}{2}\int_0^t g(s)ds.$$

So $g(t)$ is the solution to a differential equation satisfying the original conditions

$$g'(t) = -\frac{v^2}{2}g(t) \text{ and } g(0) = 1.$$

We have a unique solution to g, which is

$$g(t) = e^{-\frac{tv^2}{2}},$$

and

$$\mathbb{E}[e^{ivM_t} = e^{-\frac{tv^2}{2}}.$$

This means that M_t is normally distributed with mean 0 and variance $t^{\frac{1}{2}}$.

Let $s > 0$ and $A \in \mathcal{F}_s$, with $P(A) > 0$. Now let $P^*B = P(B|A)$, $\mathcal{F}_t^* = \mathcal{F}_{t+s}$ and $M_t^* = M_{t+s} - M_s$ for $t \geq 0$. Then with respect to $\mathcal{F}_t^*$ over probability space $(\Omega, \mathcal{F}, P^*)$, we have that $(M_t^*)_{t\geq 0}$ is continuous local martingale with $M_0^* = 0$, such that $[M_t^*]^2 - t$ is also a local martingale. Then we have

$$\mathbb{E}[e^{ivM_t^*}] = e^{-\frac{tv^2}{2}}.$$

We're almost done, now we substitue $M_t^* = M_{t+s} - M_t$ and let A vary in $\mathcal{F}_s$, so

$$\mathbb{E}[e^{iv(M_{t+s}-M_t)}|\mathcal{F}_s| = e^{-\frac{tv^2}{2}},$$

which shows that $M_{t+s} - M_s$ is independent of $\mathcal{F}_s$, so it has a normal distribution. Therefore, we have what we wanted to show, which is that $M_{t+s} - M_s$ is a Brownian motion. $\square$

[5]The definition of "local" is a bit difficult given that we haven't gone in-depth with stopping times, I would suggest reading a bit about it on Wikipedia.

Now we get an interesting corollary from Dublins and Schwarz:

Corollary 7.5 (Dublins and Schwarz). *Let M be a continuous local martingale which is null at 0 such that $[M]_t$ is increasing as $t \to \infty$. For $t \geq 0$, define stopping times as $\tau_t := \inf u : [M]_u > t$ and a shifted filtration $\mathfrak{G}(t) = \mathcal{F}(\tau_t)$. Then $X(t) = M(\tau_t)$ is a standard Brownian motion.*

8. A Complex Analysis/Brownian Motion proof of Liouville's Theorem

At this point, we have seen a lot about Brownian motion from a Markov chains perspective, so we should be a bit more comfortable introducing lots of complex analysis into our understanding of it as well. Let's now turn to Liouville's Theorem and its background.

Definition 8.1. We say that a function which is holomorphic on all of $\mathbb{C}$ is an *entire function.*

Theorem 8.2 (Liouville's Theorem). *A bounded entire function is constant.*

First we will present the complex analysis proof of Liouville's Theorem:

Proof. Suppose f is an entire holomorphic function such that $|f(z)| \leq M$ for $z \in \mathbb{C}$. Then, by Cauchy's estimates, we have

$$|f'(z_0)| \leq \frac{M}{R}$$

for $R > 0$. Note that M is independent of R, so $\frac{M}{R}$ can be as small as possible by choosing some large value of R. Also, nonnegative numbers less than A for $A > 0$ are equal to 0. So $|f'(z_0)| = 0$ for all $z_0 \in \mathbb{C}$, or $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$. Hence, any function whose derivative is zero everywhere is a constant. □

One interesting thing to note is that Liouville's theorem gives us the fundamental theorem of algebra:

Theorem 8.3 (Fundamental Theorem of Algebra). *Let $p(z)$ be a nonconstant polynomial with complex coefficients. Then, there is some $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.*

Proof. Suppose p is a polynomial with no complex roots. Let's consider the function $\frac{1}{p(z)}$, which is an entire function. Now let's show that $\frac{1}{p(z)}$ is bounded. We have seen before that if $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$, where $a_n \neq 0$, then there is $R > 0$ where if $|z| > R$, then $|p(z)| > \frac{|a_n|}{2} R^n$. We have

$$\left| \frac{1}{p(z)} \right| < \frac{2}{|a_n| R^n}.$$

So $\frac{1}{p(z)}$ is bounded when $|z| > R$. This means we should just focus on the disk $|x| \leq R$. It is compact, which tells us the function is bounded on the disk, since a continuous function takes a maximum value on the disk. It is bounded, so by Liouville's theorem, $\frac{1}{p(z)}$ is constant and so is $p(z)$. □

Now it is finally time to look at the Brownian Motion version of the proof for Liouville's Theorem:

Proof of Liouville's Theorem with Brownian Motion. Suppose f is an entire function, but that it is not constant. By Theorem 7.3, we know that for a Brownian motion B_t, then $f(B_t)$ will be a local martingale. By the Dublins and Schwarz corollary, we have that $f(B_t)$

is also a Brownian motion. So f is dense in the entire complex plane and it is therefore not bounded. This contradicts the assumption that the function is not constant, so the entire function must be constant and therefore Liouville's Theorem must hold true. $\square$

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