FOURIER ANALYSIS

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1. The Discrete Fourier Transform

The main idea behind the Fourier transform is simple: to express a function as a sum of sinusoids. The complicated part is how exactly to do it. First, we consider a method for making a "spike" in time using only sinusoids, such as in figure 1. This can be done by simply adding $\frac{1}{n}\left(\cos(x) + \cos(2x) + \cos(3x) + \ldots + \cos(nx)\right)$, to cause constructive interference at $x = 1$. If we have a dataset constructed of points in $\mathbb{R}^2$, we can simply add together a bunch of spikes positioned correctly to match these points. We have these two equations:

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-2\pi ikn/N}$$

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \cdot e^{2\pi ikn/N}$$

which dictate the entire process of a Fourier transform. We have

symbol	quantity
X_k	Phase and amplitude shift for the final result of frequency k
x_n	function value at time n
N	total number of time values, which are $0, 1, \ldots N-1$ individually
n	current time
k	current frequency

We also write $\mathcal{F}(\{x_n\})_k = X_k$. Figure 2 shows a Fourier transform done on a set of random values. Note that the first equation is used to actually do the transform, i.e. find the X_ks, and the second is used to go the other way.

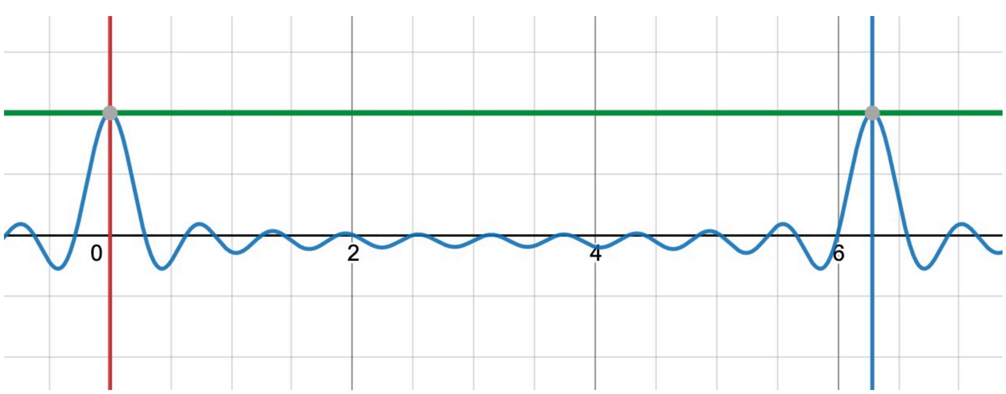


Figure 1. A Spike at $x = 0$, with period 2π and amplitude 1.

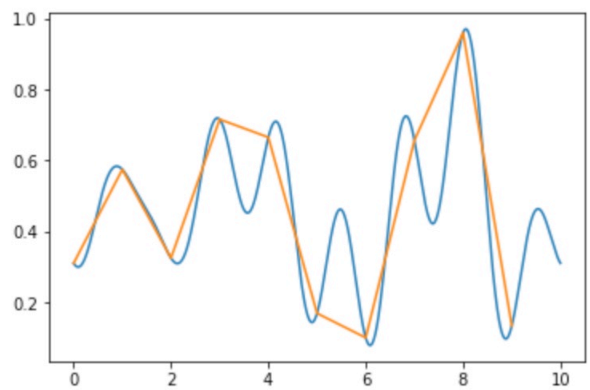


Figure 2. A Fourier transform done on a set of random values.

2. Examples

One way the Fourier transform works is in mp3s for compression. If you've ever been on a crowded wifi network for example, some images might look like this:

This is a higher quality image compressed to an extremely low bit rate. What happens is, a jpeg essentially stores the color value of each pixel in an image with a Fourier transform, and the lower the bitrate, the fewer iterations of the transform will be present. Here's the full image

Something similar happens with mp3s, although it's much more optimized since sound waves are naturally periodic. For example, a 50MB .wav file (which is raw bits) can be compressed to only a few MB using the magic of Fourier transforms, and only a small loss in quality which is barely perceptible to the average listener.

3. Fourier Series

The principle behind the Fourier Series is very similar to a Fourier transform. However, this time we express our function in terms of sin and cos. We have

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

Where

$$a_0 = \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Note that this will produce a periodic function with period 2π, using the value of f from $-\pi$ to π. Let's try an example, with $f(x) = x$. We have

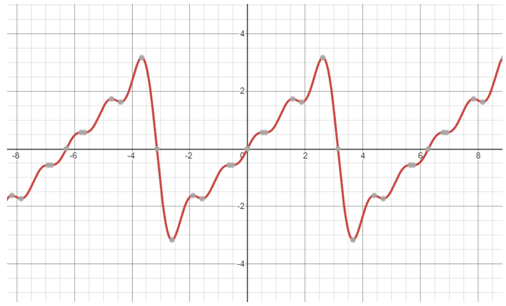
$$a_0 = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)$$

Calculating a few values, we get

$$a_1, a_2, a_3, \ldots = 2, -1, \frac{2}{3} - \frac{1}{2}, \frac{2}{5}, \ldots$$

It's not easy to see exactly how this continues, but if we plot each of these it's easy to see that it does (this is 5 iterations)



The question is now, why does this work? To start, let's start with a Fourier series...

$$f(x) = \frac{1}{2}a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \dots$$

Now let's integrate both sides from from $-\pi$ to π. We get

$$\int_{-\pi}^{\pi} \Big(f(x)\,dx = \int_{-\pi}^{\pi} \frac{1}{2}a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \dots\Big)\,dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2}a_0\,dx + a_1\int_{-\pi}^{\pi}\cos(x)\,dx + b_1\int_{-\pi}^{\pi}\sin(x)\,dx + a_2\int_{-\pi}^{\pi}\cos(2x)\,dx + b_2\int_{-\pi}^{\pi}\sin(2x)\,dx + \dots$$

Note that $\int_{-\pi}^{\pi}\sin(mx)\,dx = 0$ for any nonzero integer m, and the same for cosine. This leaves us with

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2}a_0\int_{-\pi}^{\pi} dx$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = a_0$$

as desired. Now for the other coefficients. Let's lay down a few simple identities, which we can use later. If m, n are different nonzero integer,

$$\int_{-\pi}^{\pi}\sin(mx)\sin(nx) = 0$$

$$\int_{-\pi}^{\pi}\cos(mx)\cos(nx) = 0$$

$$\int_{-\pi}^{\pi}\sin(mx)\cos(nx) = 0$$

$$\int_{-\pi}^{\pi}\sin^2(mx) = \pi$$

$$\int_{-\pi}^{\pi}\cos^2(mx) = \pi$$

These are all pretty easy and boring to check using simple calculus and trig identities. The main takeaway is, if we integrate a product of two trig functions from $-\pi$ to π of the form $\cos(mx)$ or $\sin(mx)$ where m is a nonzero integer, the result is 0 if they are different functions and π if they are the same. Now, say we're looking for b_n for some n. We have

$$f(x) = \frac{1}{2}a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \dots$$

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$$\int_{-\pi}^{\pi}\sin(nx)f(x)\,dx = \int_{-\pi}^{\pi}\sin(nx)\left(\frac{1}{2}a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \dots\right)dx$$

$$= \int_{-\pi}^{\pi}\frac{1}{2}a_0\sin(nx)\,dx + a_1\int_{-\pi}^{\pi}\cos(x)\sin(nx)\,dx + b_1\int_{-\pi}^{\pi}\sin(x)\sin(nx)\,dx +$$

$$a_2\int_{-\pi}^{\pi}\cos(2x)\sin(nx)\,dx + b_2\int_{-\pi}^{\pi}\sin(2x)\sin(nx)\,dx + \dots$$

$$= \pi b_n$$

as desired. There are quite a few use cases of Fourier series, but one of the main ones is a solution to the Basel problem.

Lemma 1. Parseval's Identity

$$\sum_{n=-\infty}^{\infty} a_n^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2\,dx$$

where

$$a_n = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-inx}\,dx$$

is a modified form of the Fourier series of f.

Note that there are a few possible notations for Fourier series, and the one shown here has been adapted to work for complex numbers.

Proof. We have

$$\int_{-\pi}^{\pi} |f(x)|^2\,dx = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} a_n e^{inx} \sum_{m=-\infty}^{\infty} a_m e^{imx}\,dx$$

If the integrand is of the form e^{inx}, the integral will be zero, so the only nonzero contributions to this are when $n = -m$ and in that case the integral is $2\pi a_n^2$. Thus, we have

$$\sum_{n=-\infty}^{\infty} 2\pi a_n^2 = \int_{-\pi}^{\pi} |f(x)|^2\,dx$$

as desired. ∎

Now, to the Basel Problem. For $f(x) = x$, we have

$$a_n = \frac{1}{2\pi}\int_{-\infty}^{\infty} xe^{-inx}\,dx = \frac{(-1)^{n+1}}{in}$$

Plugging in,

$$2\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$2\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{2\pi}\frac{2\pi^3}{3}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

$$= \zeta(2)$$

as desired.

4. Continuous Fourier Transforms

The Fourier series model is great for somethings, but it's still lacking in a few areas. First of all, it only works for periodic functions, and second of all, the frequencies are quantized. However, we can extend the Fourier transform to fit these conditions. This time we only have one function, the Fourier transform of f, defined by:

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$

and

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\,dx$$

So this time instead of doing an infinite sum to get back the original function, we do an integral. First, let's prove another version of Parseval's Identity for Fourier transforms.

Theorem 2. Plancherel
$$\int_{-\infty}^{\infty} f(x)^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(x)|^2$$

Proof.
$$\begin{aligned}\int_{-\infty}^{\infty} f(x)^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{\infty}\hat{f}(k)e^{ikx}\,dk\right)dx\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(x)\left(\int_{-\infty}^{\infty} f(k)e^{ikx}\,dk\right)dx\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(x)\hat{f}^*(x)\,dx\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(x)|^2\,dx\end{aligned}$$

as desired. ■

5. Sources

(1) https://brilliant.org/wiki/fourier-series/#applications-and-generalizations
(2) https://www.jezzamon.com/fourier/
(3) https://secure.math.ubc.ca/~yhkim/yhkim-home/teaching/Math257/M257-316Notes_Lecture16.pdf