NEVANLINNA-PICK INTERPOLATION

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ABSTRACT. In the following paper, we set out to figure out how finite Nevanlinna-Pick Interpolation works through mapping points on the unit disk to itself. We find that such a mapping exists if and only if the Pick matrix is positive semi-definite, and then derive some corollaries and extensions to the original problem, including to Hardy spaces and symmetrized bidisks. Thanks to Simon Rubinstein-Salzedo for his Spring 2021 complex analysis course, without which this paper would not come to be.

1. NEVANLINNA-PICK INTERPOLATION

The Nevanlinna-Pick problem asks, if we have an input sequence $\{z_n\} \subset \mathbb{D}$ and an output sequence $\{w_n\} \subset \mathbb{C}$, then does there exist an analytic function $f\colon \mathbb{D} \to \mathbb{D}$ that maps $\{z_n\}$ to $\{w_n\}$? If there is, we may then further ask: how many functions are there, and what properties they satisfy?

This problem can also be considered in the reals, and we quickly see that although it's easy to find a family of functions that work, it is difficult to find all solutions. Using Gaussian elimination we can find an nth degree polynomial $f(x)$ that interpolates $\{z_n\}$, and then we can just use

$$f(x) + g(x)\prod_i (x - z_i).$$

In this paper, we will restrict ourselves to the complex plane, as the real axis is weirder. We begin with Mobius functions:

Definition 1.1. Mobius transformations are given by $f(z) = \frac{az+b}{cz+d}$ where $ad - bc = 1$.

Because $ad - bc = 1$ then there exists a homormorphism between Mobius functions and $SL_2(\mathbb{R})$; we denote

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

for convenience.

Proposition 1.2. *Mobius functions map $\mathbb{H} \to \mathbb{H}$.*

Proof. Let $z \in \mathbb{H}$, so $\Im(z) > 0$. Then we have

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix} z &= \frac{az+b}{cz+d} \cdot \frac{c\overline{z}+d}{c\overline{z}+d} \\ &= \frac{ac|z|^2 + bd + (ad-bc)z}{|cz+d|^2} \\ &= \frac{ac|z|^2 + bd + \Re(z)}{|cz+d|^2} + \frac{\Im(z)}{|cz+d|^2} i\end{aligned}$$

which lies in $\mathbb{H}$ as desired. ☐

In fact, Mobius functions are the automorphism group on $\mathbb{H}$. Blaschke factors are cousins of Mobius functions, mapping from $\mathbb{D} \to \mathbb{D}$.

Definition 1.3. We denote the Blaschke factor at $a \in \mathbb{D}$ as $b_0(z) = z$ and

$$b_a(z) = \frac{z - a}{1 - \overline{a}z}.$$

Note that the Blaschke factor has a zero at a, so we can define a Blaschke product $B(z)$ as the product of several (maybe infinite) Blaschke factors along with the coefficient $e^{i\alpha}$, and thus get an analytic function (extra condition below) from $\mathbb{D} \to \mathbb{D}$ with those zeroes.

Proposition 1.4. *Given an analytic function $f\colon \mathbb{D} \to \mathbb{D}$ where $f(a) = 0$ then $f(z) = b_a(z)f_1(z)$ for some other analytic function $f_1\colon \mathbb{D} \to \mathbb{D}$.*

This is not hard to see because $b_a'(z)$ must be nonzero as long as $a \neq 1$. Then $f(z)/b_a(z)$ is analytic, and since $b_a(z)$ is an automorphism of $\mathbb{D}$ this function is from $\mathbb{D} \to \mathbb{D}$ as desired.

Proposition 1.5. *An infinite Blaschke product converges on all compact subsets of $\mathbb{D}$ if and only if the sum of coefficients $\sum_n (1 - |a_n|)$ converges.*

The above proposition is proved as Theorem 1.5 in Chapter 9 of the book. For convenience, we state it here:

Proof. Since for a Blaschke factor $b_a(z)$ we have $|a|, |z| < 1$, then

$$\begin{aligned} |1 - b_a(z)| &= \left| \frac{(a + |a|z)(1 - |a|)}{a(1 - \overline{a}z)} \right| \\ &\leq \frac{|a|(1 + |z|)}{|a|(1 - |a||z|}(1 - |a|) \\ &\leq \frac{2}{1 - |z|}(1 - |a|). \end{aligned}$$

Then the product of these $b_{a_n}(z)$ terms converges uniformly as long as the sum of $1 - |a_n|$ terms converges.

Now suppose that the product converges, so

$$\sum_{n=1}^{\infty} 1 - b_{a_n}(z)$$

also converges, and by reversing the steps shown above we prove that $\sum_n (1 - |a|)$ must converge. ☐

Now let's construct the analytic function given by the Nevanlinna-Pick theorem. We define a series of functions $B_n(z)$ inductively as $B_0(z) = 1$ and

$$B_n(z) = (b_{w_n})^{-1} \circ (zB_{n-1}) \circ b_{z_n}(z).$$

For convenience, we define $A_n(z) = b_{z_n}(z)$ and $C_n(z) = b_{w_n}(z)$. I claim that this function works as long as the Pick matrix M, defined by

$$M_{i,j} = \frac{1 - w_i\overline{w_j}}{1 - z_i\overline{z_j}}$$

is positive semi-definite. We use the following lemma:

Lemma 1.6. *Suppose the Pick matrix is positive semi-definite. Then the matrix N defined by*

$$N_{i,j} = \frac{1 - \frac{C_n(w_i)\overline{C_n(w_j}}{A_n(z_i)\overline{A_n(z_j)}}}{1 - A_n(z_i)\overline{A_n(z_j)}}$$

for $1 \le i, j < n$ is positive semi-definite.

Proof. Rearranging and simplifying, we have

$$\begin{aligned} N_{i,j} &= \frac{1}{A_n(z_i)\overline{A_n(z_j)}} \left(\frac{1 - C_n(w_i)\overline{C_n(w_j)}}{1 - A_n(z_i)\overline{A_n(z_j)}} - 1 \right) \\ &= \frac{1}{A_n(z_i)\overline{A_n(z_j)}} \left(\frac{(1 - z_i\overline{z_n})(1 - z_n\overline{z_j})(1 - |w_n|^2)(1 - w_i\overline{w_j})}{(1 - w_i\overline{w_n})(1 - w_n\overline{w_j})(1 - |z_n|^2)(1 - z_i\overline{z_j})} - 1 \right). \end{aligned}$$

Now we define a n by n diagonal matrix D_1 with the ith diagonal element as

$$\left(\frac{1 - z_i\overline{z_n}}{1 - w_i\overline{w_n}} \right) \sqrt{\frac{1 - |w_n|^2}{1 - |z_n|^2}}.$$

This matrix is nice because conjugating it with M gives us that nasty fraction in the indices of N:

$$(D_1 M D_1^*)_{i,j} = \frac{(1 - z_i\overline{z_n})(1 - z_n\overline{z_j})(1 - |w_n|^2)(1 - w_i\overline{w_j})}{(1 - w_i\overline{w_n})(1 - w_n\overline{w_j})(1 - |z_n|^2)(1 - z_i\overline{z_j})}.$$

When $i = n$ or $j = n$, then this fraction immediately collapses down to 1. Now define another n by n matrix I_1 as

$$(I_1)_{i,j} = \begin{cases} 1 & i = j \\ -1 & i \neq n,\ j = n \\ 0 & \text{otherwise} \end{cases}.$$

In other words, I_1 is the $n-1$ by $n-1$ identity matrix plus a column of -1s and a row of 0s. Conjugating this, we get that

$$(I_1 D_1 M D_1^* I_1^*)_{i,j} = \begin{cases} \frac{(1-z_i\overline{z_n})(1-z_n\overline{z_j})(1-|w_n|^2)(1-w_i\overline{w_j})}{(1-w_i\overline{w_n})(1-w_n\overline{w_j})(1-|z_n|^2)(1-z_i\overline{z_j})} - 1 & (i,j) \neq (n,n) \\ 1 & i = j = n \end{cases}.$$

We are getting closer and closer to what we want in N. Lastly, define D_2 as the n by n diagonal matrix with entries $1/A_n(z_i)$ for $1 \le i < n$ and $(D_2)_{n,n} = 1$. Conjugating this as well, we get that

$$(D_2 I_1 D_1 M D_1^* I_1^* D_2^*)_{i,j} = \begin{cases} N_{i,j} & i, j < n \\ 1 & i = j = n \\ 0 & \text{otherwise} \end{cases}.$$

□

Because $|w_n| < 1$, we see that the matrix $D_2 I_1 D_1$ is actually invertible, so this actually proves the stronger statement that $N \ge 0$ if and only if $M \ge 0$. With this, we can prove one side of the theorem:

Theorem 1.7. *If the Pick matrix is positive semi-definite, then there exists an analytic function $f\colon \mathbb{D} \to \mathbb{D}$ that maps $\{z_n\}$ to $\{w_n\}$.*

Proof. I claim that the aforementioned construction works, and that for all positive integral n there exists a function $B_{n-1}(z)\colon \mathbb{D} \to \mathbb{D}$ satisfying

$$B_{n-1}(A_n(z_i)) = \frac{C_n(w_i)}{A_n(z_i)}$$

for all $1 \le i \le n-1$. One can quickly check that if such a function exists, then by induction we have found a function that interpolates $\{z_n\}$ and $\{w_n\}$. This is because taking $i = n$ quickly yields equality as well.

Clearly when $n = 1$ then $B_0(z) = 1$ exists. Now using strong induction, suppose for $1 \le i \le n-1$ that B_{i-1} exists. It now suffices to show that when M is positive semi-definite that $B_{n-1}(z)$ exists, but this is analogous to showing that the matrix

$$M'_{i,j} = \frac{1 - \frac{C_n(w_i)\overline{C_n(w_j}}{A_n(z_i)\overline{A_n(z_j)}}}{1 - A_n(z_i)\overline{A_n(z_j)}}$$

is also positive semi-definite. This is exactly N from Lemma 1.6, so we're done. ☐

We now go over an example to showcase Nevanlinna-Pick interpolation. Let's first consider the sequences

$$\{z_n\} = \frac{\zeta^n}{2} \qquad \{w_n\} = \frac{1}{n+1}$$

where ζ is a primitive 3rd root of unity for $1 \le n \le 3$. Clearly both sequences lie in $\mathbb{D}$. Then

$$b_{w_1} \circ B_1(z) = \frac{B_1(z) - \frac{1}{2}}{1 - \frac{1}{2}B_1(z)} = b_{z_1}(z) = \frac{z - \zeta}{1 - \zeta^2 z} = -\zeta,$$

which implies $B_1(z) = \frac{1-2\zeta}{2-\zeta}$. We can then continue the inductive process to find $B_3(z)$, our answer.

On the next page is a figure on a Nevanlinna-Pick algorithm in progress:

Extensions

Corollary 1.8. *If the Pick matrix has determinant zero, then the function guaranteed by the Nevanlinna-Pick Theorem is unique.*

This corollary will be left as an exercise to the reader, though due to Lemma 1.6 we know that the rank of N is only one less than that of M.

Many others have taken Nevanlinna-Pick Interpolation and tried to find an analytic function on the Euclidean sphere, Riemann surfaces, or subalgebras of H^∞. It also has applications in operator theory. If we let $\mathfrak{J}_E H^2$ be the ideal of the functions that disappear on $E = \{z_1, \ldots, z_n\}$ and $M(E) = H^2 \backslash \overline{\mathfrak{J}_E}$, then

$$f + \mathfrak{J}_E \to P_M(e) + M_f P_{M(E)}$$

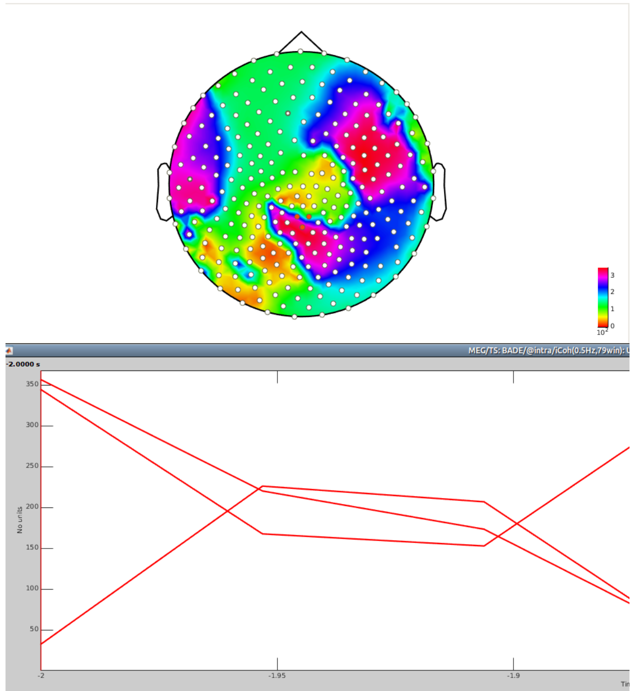

Figure 1

means that being isometric is equivalent to the existence of an analytic function interpolating these values.

If we constrain our work to $f'(0) = 0$, then we can find a function interpolating our values for the Hardy space

$$H_1^\infty = \{f \in H^\infty(\mathbb{D})\colon f'(0) = 0\}.$$

In this case, the Pick matrix would be of the form

$$M_{i,j} = (1 - w_i\overline{w_j})\langle P_L k_{z_j}^S, k_{z_i}^S \rangle.$$

Interestingly, it turns out that the condition for having a Nevanlinna-Pick function is not strict in this case as well. If H is an reproducing kernel Hilbert space, then any weak-multiplication-closed algebra that obeys the strong factorization property will admit a Nevanlinna-Pick family of functions.

We can also study Nevanlinna-Pick interpolation in other interesting subsets of $\mathbb{C}$, such as the symmetrized bidisk $\mathbb{G} = \{(z + w, zw) \mid z, w \in \mathbb{D}\}$. Given a solvable Pick problem (that the inputs and outputs satisfy the Pick matrix), we define the uniqueness set:

Definition 1.9. The *uniqueness set* is the largest subset contained in all possible functions.

Interestingly, all solutions to a solvable Pick problem on $\mathbb{G}$ are rational functions. The proof is dependent on a representative formula for Hilbert spaces; a function $f\colon \mathbb{D} \to \mathcal{B}(E, F)$ is contractive analytic (analytic and obeys dilation property) if and only if there exists an auxiliary Hilbert space H and unitary operator

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} E \\ H \end{bmatrix} \to \begin{bmatrix} F \\ H \end{bmatrix}$$

such that

$$f = A + zB(I - zD)^{-1}C.$$

Here, I denotes the identity matrix. For example, given the inputs $\{(0,0),(1,\frac{1}{4})\}$ and the outputs $\{0,\frac{1}{2}\}$, there clearly is a solution: $f(z,w) = \frac{z}{2}$. Now given any other solution g, we can construct

$$g'(z,w) = (z+w, zw)$$

which solve another problem in $\mathbb{G}$, namely the inputs $\{(0,0),(\frac{1}{2},\frac{1}{2})\}$ and outputs $\{0,\frac{1}{2}\}$. By Schwarz's lemma, we have

$$g'(z,z) = g(2z, z^2) = z \qquad \forall z \in \mathbb{D},$$

which implies that the suprenum norm of g over $\mathbb{G}$ is actually one. But since we also have

$$g'(z,w) = g(z+w, zw) = \frac{z+w}{2} \qquad \forall z, w \in \mathbb{D}$$

as a solution, it must be unique.

References

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