COMPLEX BROWNIAN MOTION

JOSH ZEITLIN

Abstract. We discuss the way we can interpret Complex Analysis through the lens of stochastic processes. First we will develop the language fundamental to stochastic processes, showing some basic ideas from measure theory and statistics. Then we will go into the concept of filtration and talk about martingales. We'll then move on to talking about stochastic calculus and we will finish our discussion there with Itô's formula and Itô Calculus. We will have a solid foundation with stochastic processes and then we will move on to Complex Analysis. Using our background, we'll prove three important and quite profound theorems in Complex Analysis: The Fundamental Theorem of Algebra, Liouville's Theorem and Picard's Little Theorem. The following document is an outline of the paper and each section is split up into subsections according to the definitions and theorems we'll use.

1. Necessary Preliminaries

In this section we will lay out some important background for our discussion of Brownian Motion and its applications to complex analysis.

1.1. Measure and Probability Theory.

Definition 1.1. Metric Space A *metric space* is a pair (X, d) where X is a collection of sets and d is a metric which satisfies the traditional metric properties:

(1) $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) \geq 0$ for all $x, y \in X$.
(3) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(4) $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in X$.

On the other hand a measure space is a more general notion of a metric space. However, to define that we have to define the concept of measures and σ-algebras.

Definition 1.2. A *σ-algebra* on X is a collection of subsets of X which includes X and the empty set and is closed under the complement operation and is closed under countable unions and countable intersections.

One special type of σ-algebra which we will use later is called the Borel σ-algebra, which we define as follows.

Definition 1.3. The *Borel σ-algebra*, denoted $\mathcal{B}$ is the σ-algebra generated by the open sets of X where X is the set of all points in the space.

Definition 1.4. A *measure* on a set X (with the associated σ-algebra $\mathcal{A}$) is a function from

$$\mu : \mathcal{A} \to \mathbb{R}_+.$$

A measure satisfied the following properties:

(1) For all sets $S \in \mathcal{A}$, we have $\mu(S) \geq 0$.
(2) $\mu(\emptyset) = 0$.

(3) For all countable collections ($\{S_k\}_{k=1}^{\infty}$) of pairwise disjoint sets in S,

$$\mu\left(\bigcup_{k=1}^{\infty} S_k\right) = \sum_{k=1}^{\infty} \mu(S_k).$$

Definition 1.5. A *measurable space* is a tuple $(X, \mathcal{A},)$ where X is a set, $\mathcal{A}$ is a σ-algebra.

Definition 1.6. A *measure space* is a triple $(X, \mathcal{A}, \mu)$ where X is a set, $\mathcal{A}$ is a σ-algebra and μ is a measure.

Definition 1.7. A *probability space* is a triple $(X, \mathcal{A}, \mu)$ where X is a set, $\mathcal{A}$ is a σ-algebra and μ is a measure and it satisfies the additional condition that $\mu(X) = 1$.

Now that we know the basics of measure spaces, we are next going to state some properties about measure and probability spaces. After that, we will introduce the concept of the Lebesgue measure and the Lebesgue integral.

Proposition 1.8. *Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.*

(1) Suppose $A \subseteq \Omega$ and $A \in \mathcal{F}$. Then, $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.
(2) Let $A \subseteq B \subseteq \Omega$ and $A, B \in \mathcal{F}$, then, $\mathbb{P}(A) \leq \mathbb{P}(B)$.
(3) If $(A_k)_{k=1}^{n}$ are a finite number of disjoint events, then

$$\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mathbb{P}(A_k).$$

(4) For any $A, B \in \mathcal{F}$, we have that

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

(5) In general, for any family of events $\{S_i\}_{i=1}^{n} \in \mathcal{F}$ we get that

$$\mathbb{P}\left(\bigcup_{i=1}^{n} S_i\right) = \sum_{i=1}^{n} \mathbb{P}(S_i) - \sum_{i<j} \mathbb{P}(S_i \cap S_j) + \sum_{i<j<k} \mathbb{P}(S_i \cap S_j \cap S_k) + \cdots + (-1)^{n+1}$$

(6) If $\{A_i\}_{i\geq 1} \in \mathcal{F}$ then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right)$$

(this is called the continuity of probability measures).
(7) If $\{A_i\}_{i\geq 1}$ is a sequence of decreasing nested events ($A_{i+1} \subseteq A_i$ for all $i \geq 1$) then

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

(8) If $\{A_i\}_{i\geq 1}$ is a sequence of events then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Now we'll talk about the Lebesgue measure and the Lebesgue integral.

Definition 1.9. With respect to an open set $S = \cup_k(a_k, b_k)$, the *Lebesgue measure* is defined as

$$\mu_L(S) = \sum_k (b_k - a_k).$$

And given a closed set $S' = [a, b] - \cup_k(a_k, b_k)$ then

$$\mu_L(S') = (b - a) - \sum_k (b_k - a_k).$$

Now, to define the Lebesgue integral we get that it is defined as follows.

Definition 1.10. The *Lebesgue integral* is defined in terms of the Lebesgue sum which is given as

$$S_n = \sum_i \eta_i \mu(E_i),$$

where $E_i \subseteq X$ and η_i is the value of the function f in the sub-interval i and $\mu(E_i)$ is the Lebesgue measure of E_i and the integral is written as

$$\int_X f \text{ or } \int_X f \, d\mu.$$

We next need to define filtrations and L^p spaces.

Definition 1.11. For an index set I, a *filtration* on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a family $(\mathcal{F}(i) : i \in I)$ of σ-algebras such that $\mathcal{F}(s) \subset \mathcal{F}(t) \subset \mathcal{F}$ for $s < t$.

Definition 1.12. A *filtered probability space* is a probability space with a filtration.

Definition 1.13. A stochastic process $\{X(t)\}_{t \geq 0}$ from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathcal{F}(t)$ is adapted if each $X(t)$ is measurable with respect to $\mathcal{F}(t)$.

Definition 1.14. A *right-continuous filtration* is a family $(\mathcal{F}_i^+)_{i \in I}$ where I is the index set such that $\mathcal{F}_i^+. = \cap_{z>i} \mathcal{F}_z$.

Now, we will lastly define an L^p space.

Definition 1.15. An L^p space is a functional space which is the vector space of all l^p measurable functions, or in other words the Lebesgue integral over some set S is finite, written formally as

$$\left(\int_S |f|^p \, d\mu \right)^{1/p} < \infty.$$

L^p spaces are very important because we are interested in the L^2 space which is the space of all square-integrable measureable functions which can also be thought of as random variables with $\mathbb{E}[X^2] < \infty$ or with finite second moment.

1.2. **Basic Statistics.**

Definition 1.16. A vector of random variables $\mathbf{X} = (X_1, \ldots, X_n)^T$ is called a multivariate d-dimensional standard Gaussian distribution if

1. (1) $X_1, \ldots, X_d$ $\mathbf{N}(0, 1)$, where $\mathbf{N}(\mu, \sigma)$ is the normal distribution of mean μ and standard deviation σ.
2. (2) $X_1, \ldots, X_d$ are independent and identically distributed (iid) random variables (rv).

The expected value and covariance of the distribution of this would be the zero vector and the identity matrix, respectively.

Proposition 1.17. *If* $\mathbf{X}$ *is a* d*-dimensional standard Gaussian random vector and* A *is an orthogonal* $d \times d$ *matrix, then* $A\mathbf{X}$ *is also standard Gaussian.*

Definition 1.18. A random vector $\mathbf{Y}$ is d-dimensional Gaussian if we can express it in the form $A\mathbf{X} + \mu$ given $\mathbf{X} \in \mathbb{R}^m, A \in \mathbb{R}^{d\times m}, \mu \in \mathbb{R}^d$.

For the distribution $\mathbf{Y} = A\mathbf{X} + \mu$ we have $\mathbb{E}[\mathbf{Y}] = \mu$ and $\mathrm{Cov}(\mathbf{Y}) = AA^T$ and the Gaussian property is preserved under Affine transformations.

1.3. **Stochastic Processes.** Now we have defined the basics of measure theory and probability theory and have some of the necessary materials we will need moving forward. Next we need to introduce some basic concepts in stochastic processes.

Definition 1.19. Given a probability space $(\Omega, \Sigma, \mathbb{P})$ equipped with a filtration $\mathcal{F}$, an S-valued *stochastic Process* is a collection of S-valued random variables on Ω, indexed by a totally ordered time set T. That is, a stochastic process B is a collection $\{B_t : t \in T\}$ where each B_i is an S-valued random variable on Ω. The space S is then called the state space of the process.

Now we define a special random variable with respect to a stochastic process called a stopping time.

Definition 1.20. A random variable $T : \Omega \to \mathbb{R}_+$ defined on a filtered probability space is called a *stopping time* with respect to the filtration $\mathcal{F}$ if the set $x \in \Omega : T(x) \leq t \in \mathcal{F}_t$ for all t.

Stopping times are very important in the study of stochastic processes. Now we'll introduce a special stopping time called the hitting time.

Definition 1.21. The *hitting time* of a stochastic process is the first time in which a stochastic process hits a specific value. To define this more formally, let T be an ordered index set. Given a probability space $(\Omega, \Sigma, \mathbb{P})$ and a measurable state space $\mathcal{S}$, let $X : \Omega \times T \to \mathcal{S}$ be a stochastic process, and let A be a measurable subset of the state space. Then the first hit time $\tau_A : \Omega \to \mathbb{R}_+$ is the random variable defined by

$$\tau_A(\omega) = \inf t \in T : X_t(\omega) \in A.$$

One type of very important stochastic process is a martingale.

Definition 1.22. A discrete time *martingale* is a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ such that for all n we get that $\mathbb{E}[|X_n|] < \infty$ and $\mathbb{E}[X_{n+1}|X_1, \ldots, X_n] = X_n$

Martingales are meant to model a fair game as given by the second condition in the definition. More rigorously, a *martingale* can be defined for the continuous case as follows.

Definition 1.23. A stochastic process $X_t : \Omega \to \mathbb{R}$ is a *martingale* with respect to a fixed filtration $\mathcal{F}(t)$ if

(1) $X(t)$ is adapted to $\mathcal{F}(t)$.
(2) $\mathbb{E}[|X(t)|] < \infty$ (integrability).
(3) For any pair of times it is almost surely the case that (with probability 1) $0 \leq s \leq t, \mathbb{E}[X(t)|\mathcal{F}(s)] \leq X(s)$,

Another special type of stochastic process is a nonanticipating process.

Definition 1.24. A *nonanticipating process* is a stochastic process $\{X(t)\}_{t\geq 0}$ defined by the filtration $(\mathcal{F}_t)_{t\geq 0}$ if $X(t)$ is measureable for any $\mathcal{F}(t)$ for $t \geq 0$.

Here are a few more important theorems about stochastic processes.

Theorem 1.25. *Suppose $X(t)$ is a continuous martingale with $0 \leq S \leq T$ being stopping times. If the induced stochastic process of $X(\min\{t,T\})$ is dominated by an integrable random variable Y, then*

$$\mathbb{E}[X(T)|\mathcal{F}(S)] = X(S)$$

with probability 1.

This is the continuous optional stopping theorem and the discrete time analogue is defined as follows.

Theorem 1.26. *Let X_n be a discrete martingale which is uniformly integrable. Then for all stopping times $0 \leq S \leq T$, we have that*

$$\mathbb{E}[X_n(T)|\mathcal{F}(s)] = X_n(S)$$

with probability 1.

Now, one last characterization we need is the idea of a local martingale.

Definition 1.27. A *local martingale* is an adapted stochastic process $\{X(t)\}_{0\leq t\leq T}$ which contains a sequence of stopping times T_n such that $\{X(\min\{t,T_n\})\}_{t\geq 0}$ is a martingale for each n.

2. Brownian Motion

Now we will introduce the idea of Brownian Motion in a rigorous manner.

2.1. Basic Concepts.

Definition 2.1. A d-dimensional *Brownian motion* is a stochastic process $B_t : \Omega \to \mathbb{R}$ from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}^d$ such that the following holds true:

(1) For any finite sequence of times $t_0 < t_1 < \cdots < t_n$, the distributions $B_{t_{i+1}} - B_{t_i}$ are independent.
(2) For all $\omega \in \Omega$, the parametrization function $t \to B_t(\omega)$ is continuous.
(3) For any pair $s, t \geq 0$, let $B_{s+t} - B_s \in A$ we get that

$$\mathbb{P}(B_{s+t} - B_s) = \int_A \frac{1}{(s\pi t)^{d/2}} e^{-|x|^2/2t}\, dx.$$

A standard Brownian motion is a Brownian motion where $B_0(\omega) = 0$.

Definition 2.2. A *standard Brownian motion in one dimension* $(B_t)_{t\geq 0}$ is a real-valued stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which satisfies the following conditions:

(1) $B_0 = 0$.
(2) For any finite sequence of times $t_0 < t_1 < \cdots < t_n$, the distributions $B_{t_{i+1} - B_{t_i}}$ are independent for each $i \in \{1, \ldots, n\}$.
(3) For all $\omega \in \Omega, t \to B_t(\omega)$ is continuous.
(4) For all $s, t \geq 0, B_{s+t} - B_s$ is independent of $(B_u)_{0\leq u\leq s}$ and has distribution $\mathbf{N}(0, t)$.

Proposition 2.3. *A Brownian motion in $\mathbb{R}^d$ is a d-dimensional vector whose components are independent scalar Brownian motions.*

Now, we present some more theorems and properties about Brownian motion.

2.2. Properties and Theorems.

Definition 2.4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with the filtration $(\mathcal{F}_s, s \in I)$ for a totally ordered index set I; and let $(S, \mathcal{S})$ be a measureable space. A $(S, \mathcal{S})$-valued stochastic process $X = \{X_t : \Omega \to S\}_{t \in I}$ adapted to the filtration is said to possess the *Markov property* if, for each $A \in \mathcal{S}$ and each $s, t \in I$ with $s < t$,

$$\mathbb{P}(X_t \in A|\mathcal{F}) = \mathbb{P}(X_t \in A|X_s).$$

In the case where S is a discrete set with a discrete σ-algebra and $I = \mathbb{N}$, this can be reformulated as

$$\mathbb{P}(X_n = x_n|X_{n-1} = x_{n-1}, \dots, X_0 = x_0) = \mathbb{P}(X_n = x_n|X_{n-1} = x_{n-1}).$$

In other words, the stochastic process of the Brownian motion is "memoryless" and doesn't take into consideration what comes before it.

Theorem 2.5. *Let $\{B_t\}_{t \geq 0}$ is a Brownian motion started at $x \in \mathbb{R}^d$. Fix $t > 0$, then the process $\{B_{t+s} - B_t\}_{s \geq 0}$ is a Brownian motion starting at the origin and independent of $\{B_t\}_{0 \leq t \leq s}$. In other words, Brownian motion satisfies the Markov property.*

Proof. From the definition of Brownian motion, we know that Brownian motion satisfies the independent increments property. That is, for any finite sequence of times $t_0 < t_1 < \cdots < T_n$, the distributions $B_{t_{i+1}} - B_{t_i}$, for $i = 1, \dots, n$ are independent. Since the process

$$B_{t+s} - B_t = \sum_{j=1}^{s} (B_{t+j} - B_{t+j-1})$$

(for $s > 0$), where each term $B_{t+j} - B_{t+j-1}$ is independent, the given process is hence independent. ■

A nice way to interpret Brownian motion is as the limiting behavior of random walks and we can thus use the ideas of Recurrence and Transience from Random Walks and generalize them to a Brownian motion.

Definition 2.6. A Brownian motion $\{B_t\}_{t \geq 0}$ is:

(1) *Transient* if $\lim_{t \to \infty} |B_t| = \infty$.
(2) *Point recurrent* if for every $x \in \mathbb{R}^d$, there is an increasing sequence t_n such that $B_{t_n} = x$ for all $n \in \mathbb{N}$.
(3) *Neighborhood recurrent* if for every $x \in \mathbb{R}^d$ and $\epsilon > 0$, there exists an increasing sequence t_n such that B_{t_n} is in a ball of radius epsilon centered around x for all $n \in \mathbb{N}$ (or more for all $B_{t_n} \in B_\epsilon(x)$).

This gives us some interesting insight onto the shape of Brownian motion in higher dimensions.

Theorem 2.7. *A Brownian motion $\{B_t\}_{t \geq 0}$ is:*

(1) Point recurrent in dimension $d = 1$.
(2) Neighborhood recurrent, but not point recurrent in dimension $d = 2$. Brownian motion in dimension $d = 2$ is called planar Brownian motion.

(3) *Transient in dimension when* $d \geq 3$.

For those who are aware of Poincare's recurrence theorem, this generalizes it to Brownian motion in $\mathbb{R}^d$ as opposed to random walks in $\mathbb{Z}^n$. Now let's prove it. However, to do so we will have to use some techniques from Multivariable Differential Calculus and Ordinary Differential Equations. The proofs of $d = 1$ and $d \geq 3$ are not very useful for our discussion of Complex Analysis, so we'll just prove the case where D. However to do this, we need some more machinery.

2.2.1. *Dirichlet Problem.* The recurrence of Brownian motion is very closely linked with Harmonic functions and the Dirichlet Problem.

Definition 2.8. Let U be a connected open set $U \subset \mathbb{R}^d$ and ∂U be its boundary. A function $u : U \to \mathbb{R}$ is harmonic if u is twice differentiable ($u \in C^2$) and for any $x \in U$

$$\Delta u(x) := \sum_{i=1}^{d} \frac{\partial u}{\partial x_j^2}(x) = 0.$$

Theorem 2.9. *Let* $U \in \mathbb{R}^d$ *be a connected open set and* $u : U \to \mathbb{R}$ *be measurable and locally bounded. Then the following are equivalent:*

(1) u *is harmonic.*
(2) *For any ball* $B_r(x) \subset U$,

$$u(x) = \frac{1}{\mu(B_r(x))} \int_{B_r(x)} u(y)\, d\mu$$

where μ *is the Lebesgue measure on* $\mathbb{R}^d$.

(3) *For any ball* $B_r(x) \subset U$,

$$u(x) = \frac{1}{\sigma_{x,r}(\partial B_r(x))} \int_{\partial B_r(x)} u(y)\, d\sigma_{x,r}$$

where $\sigma_{x,r}$ *is the surface measure.*

Theorem 2.10. *Suppose* $u : U \to \mathbb{R}$ *is a harmonic function on a connected open set* $U \subset \mathbb{R}^d$.

(1) *If* u *attains a maximum on* U, *then it is a constant function.*
(2) *If* u *is continuous on* $\overline{U}$ *and* U *is a bounded set, then* $\sup_{x \in \overline{U}} u(x) = \sup_{x \in \partial U} u(x)$.

Theorem 2.11. *Let* $u_1, u_2 : U \to \mathbb{R}$ *be harmonic functions on a bounded connected open set* $U \subset \mathbb{R}^d$ *and continuous on the closure* $\overline{U}$, *then suppose that* $u_1 = u_2$ *on* ∂U, *then* $u_1 = u_2$ *over* U.

These can be seen as equivalent to the basic results on complex analysis. Now we have an interesting link between harmonic functions and hitting times.

Theorem 2.12. *Suppose* U *is a connected open set and* $\{B(t)\}_{t \geq 0}$ *is a Brownian motion that starts in* U. *Then, define* $\tau = \tau_{\partial U} = \min\{t \geq 0 : B(t) \in \partial U\}$ *be the hitting time of the Brownian motion on the boundary. Then, let* $\phi : \partial U \to \mathbb{R}$ *be a measurable function. Suppose that a function* $u : U \to \mathbb{R}$ *satisfies the property that for every* $x \in U$,

$$u(x) = \mathbb{E}_x[\phi(B(\tau))\mathbb{1}_{\tau < \infty}]$$

is locally bounded, then u *is harmonic.*

Proof. Fix any ball $B_\delta(x) \subset U$. Define $\tilde{\tau} = \inf\{t > 0 : B(t) \notin B_\delta(x)\}$ to be the first exit time. Since ϕ is a measurable function, by a stronger version of the Markov property (which says that $\{B(t+s) - B(s)\}$ is a standard Brownian motion independent of $\mathcal{F}^+(s)$ where s is a stopping time) then

$$\mathbb{E}_x[\mathbb{E}_x[\phi(B(\tau))\mathbb{1}_{\tau<\infty}|\mathcal{F}^+]] = \mathbb{E}_x[u(B(\tilde{\tau}))].$$

The first expression is simply just $u(x)$ by laws of conditional expectation. The second expression can be represented as the expected value taken over the boundary of the ball and hence

$$u(x) = \int_{\partial B_\delta(x)} u(y)\, d\sigma_{x,\delta}.$$

and thus from an above theorem we can see that since u is also locally bounded u is harmonic. ■

Definition 2.13. Let $U \subset \mathbb{R}^d$ be a connected open set. We say that U satisfies the *Poincare cone condition* at $x \in \partial U$ if there exists a cone V with base at x and opening angle $\alpha > 0$ and $h > 0$ such that $V \cap B_h(x) \subset U^c$.

Now we'll show the Dirichlet problem.

Theorem 2.14. *Let $U \subset \mathbb{R}^d$ be a bounded connected open set such that every boundary point satisfies the Poincare cone condition, and suppose that ϕ is a continuous function on ∂U. Define $\tau(\partial U) = \inf\{t > 0 : B(t) \in \partial U\}$ be the first hitting time, which is a surely finite stopping time. Then the function $u : \overline{U} \to \mathbb{R}$ given by*

$$u(x) = \mathbb{E}_x[\phi(B(\tau(\partial U)))]$$

is the unique continuous function that is a harmonic extension of ϕ, that is, $u(x) = \phi(x)$ for all $x \in \partial U$.

Before we prove this we have to show one more lemma.

Lemma 2.15. *Let $0 < \alpha < 2\pi$ and $C_0(\alpha) \subset \mathbb{R}^d$ be a cone based at the origin with opening angle α. Let $\alpha = \sup_{x \in \overline{B_{1/2}(0)}} \mathbb{P}_x\{\tau(\partial B_1(0)) < \tau(C_0(\alpha))\}$. Then $a < 1$ and for any pair of positive integers k, h,*

$$\mathbb{P}_x\{\tau(\partial B_h(z)) < \tau(C_z(\alpha))\} \leq a^k$$

for all x, z such that $|x - z| < s^{-k}h$.

Proof. Suppose that $x \in B_{2^{-k}}(0)$. Then clearly, there is a nontrivial probability for Brownian motion to reach the boundary of the ball before hitting the cone, so $a < 1$. Then by the strong Markov property,

$$\mathbb{P}_x\{\tau(\partial B_0(1)) < \tau(C_0(\alpha))\} \leq \prod_{i=0}^{k-1} \sup_{x \in B_{2^{-k+i}}} \mathbb{P}_x\{\tau(\partial B_{2^{-k+i+1}}(0)) < \tau(C_0(\alpha))\}.$$

For any positive integer k and j, by scaling, we get

$$\mathbb{P}_x\{\tau(\partial B_h(z)) < \tau(C_z(\alpha))\} \leq a^k$$

for all x where $|x - z| < 2^{-k}h$. ■

Now, we will prove Dirichlet's theorem.

Proof. First off, the uniqueness follows obviously from harmonic continuation. Moreover, since the stopping time is finite, u is locally bounded and hence harmonic on U. Now, fix $z \in \partial U$. Then there is a cone at z with angle α such that $C_z(\alpha) \cap B_h(z) \subset U^c$. Now, by the previous lemma, for positive integers k, h we get

$$\mathbb{P}_x\{\tau(B_h(z)) < \tau(C_z(\alpha))\} \leq a^k$$

with $|x - z| < 2^{-k}h$. In particular, given $\epsilon > 0$ and $\delta < h$ we get $|y - z| < \delta$ and then $|\phi(y) - \phi(z)| < \epsilon$. Thus for all $x \in \overline{U}$ such that $|z - x| < 2^{-k}\delta < 2^{-k}h$,

$$|u(x) - u(z)| = |\mathbb{E}_x[\phi(B(\tau(\partial U)))] - \phi(z)| \leq \mathbb{E}_x[|\phi(B(\tau(\partial U))) - \phi(z)|].$$

Thus, if the Brownian motion hits the cone before $\partial B_\delta(z)$, then $|z - B(\tau(\partial U))| < \delta$ and $\phi(B(\tau(\partial U)))$ is close to $\phi(z)$ which implies that

$$2\left\|\phi\right\|_\infty \mathbb{P}_x\{\tau(\partial B_\delta(z)) < \tau(C_z(\alpha))\} + \epsilon \mathbb{P}_x\{\tau(\partial U) < \tau(\partial B_\delta(z))\} \leq 2\left\|\phi\right\|_\infty a^k + \epsilon.$$

Because the bound can be made arbitrarily small, we have thus proved continuity on the boundary. ■

Now, we can go back and prove that planar Brownian motion is recurrent (well... neighborhood recurrent but not point recurrent.)

2.3. **Proof of Planar Brownian Motion.** So far we have developed the machinery to solve the Dirichlet problem and prove the theorem. Now, we can show that planar Brownian motion is recurrent but only with the help of a few more lemmas. For the following lemmas we will let A denote an annulus of $A = \{x \in \mathbb{R}^d | r < |x| < R\}$ and to prove recurrent we will look at the exit probabilities from the annulus. Here, we will let $r = (r, 0, \ldots, 0)$ and

$$T_r = \tau(\partial B_r(0)) = \inf\{t > 0 | |B(t)| = r\}$$

for $r > 0$, which are the first time the Brownian motion hits the ball of radius r. Then the first exit time from the annulus A is $\min\{T_r, T_R\}$ where T_R is defined in the same way as T_r was defined as, above and we'll use the same u as defined in the Dirichlet theorem.

Lemma 2.16.

$$\mathbb{P}_x\{T_r < T_R\} = \frac{u(R) - u(x)}{u(R) - u(r)}.$$

Lemma 2.17. *Let $u(r), u(R)$ be the fixed and constant on the boundary of the annulus. Then the Dirichlet solution to this boundary is given by*

$$u(x) = \begin{cases} |x| : d = 1 \\ 2\log|x| : d = 2 \\ |x|^{2-d} : d \geq 3 \end{cases}$$

Lemma 2.18. *Suppose that $\{B(t)\}_{t \geq 0}$ is Brownian motion started at some x in the annulus A and then*

$$\mathbb{P}_x\{T_r < R_R\} = \begin{cases} \frac{R - |x|}{R - r} : d = 1 \\ \frac{\log R - \log|x|}{\log R - \log r} : d = 2 \\ \frac{R^{2-d} - |x|^{2-d}}{R^{2-d} - r^{2-d}} : d \geq 3 \end{cases}$$

Lemma 2.19. *For any $x \in B_r(0)$, we have*

$$\mathbb{P}_x\{T_r < \infty\} = \begin{cases} 1 : d = 1, 2 \\ \frac{r^{d-2}}{|x|^{d-2}} : d \geq 3 \end{cases} .$$

Now, here is the proof that Brownian motion is recurrent in $\mathbb{R}^2$:

Proof. Fix $\epsilon > 0$ and $x \in \mathbb{R}^2$. Then by the shift invariance property of Brownian motion, we get the first stopping time

$$t_1 = \inf\{t > 0 : B(t) \in B_\epsilon(x)\}.$$

By the just above lemma and because $d = 2$ this is finite with probability 1. Now consider the time

$$t_2 = \{t > t_1 + 1 | B(t) \in B_\epsilon(x)\},$$

which is again finite. Repeating this process we get an increasing sequence of stopping times such that $B(t_n) \in B_\epsilon(x)$. ■

Corollary 2.20. *Neighborhood recurrence implies the path of planar Brownian motion is dense in the plane.*

3. stochastic Calculus

3.1. **The stochastic Integral.** stochastic calculus is a very important tool for dealing with stochastic processes. This is a form of calculus which helps with models of random systems. Here we will introduce some concepts in stochastic calculus which will help us with our proofs in complex analysis later on. We will introduce the integral for stochastic process called Itô's formula.

One thing we know from Brownian motion is that it behaves in a random manner so we cannot use traditional methods for integration and differentiation. We will have to integrate based on the following differential equation which takes in some randomness:

$$\frac{dB}{dt} = v(t, X_t) + w(t, X_t)W_t.$$

The last variable W_t is the randomness and it is a variable that has the following properties:

(1) For unique t, s we get that W_t, W_s are independent of each other.
(2) The set $\{W_t\}_{t \geq 0}$ is stationary.
(3) $\mathbb{E}[W_t] = 0$ for every t.

The solution to this is the stochastic integral which is defined as follows.

Definition 3.1. *The stochastic integral with respect to Brownian motion* $(B_t)_{t \in \mathbb{R}^+}$ of any measurable function $f \in L^2(\mathbb{R}^+)$ is defined as

$$\int_0^\infty f(t)\, dB_t = \mathbf{N}(0, \int_0^\infty |f(t)|^2\, dt).$$

The independent increments property of Brownian motion helps ensure that ever little increment in dB_t is an independent process and furthermore, from the stationary property, each of these independent processes have the normal distribution. Thus if $X_1, \ldots, X_n$ are independent Gaussian random variables then the sum is a random variable normal distribution of mean (summed from the individual means) and standard deviation (summed from the individual standard deviations).

3.2. **Itô Process and Itô's Formula.** Now we will talk about the most important result in stochastic calculus called Itô's formula. It is analogously like the chain rule.

Definition 3.2. An *Itô process* is a stochastic process I_t on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$I_T = I_0 + \int_0^t u(s,w)\,ds + \int_0^t v(s,w)\,dB_s$$

where

$$\mathbb{P}\left(\int_0^t v^2(s,w)\,ds < \infty \forall t \geq 0 \right) = 1$$

and with u being adapted to $\mathcal{F}(t)$ and

$$\mathbb{P}\left(\int_0^t |u(s,w)|\,ds < \infty \forall t \geq 0 \right) = 1.$$

Here, v satisfies the following properties:

(1) The function mapping $(t,\omega) \to v(t,\omega)$ is measurable with respect to $\mathcal{B}[0,\infty] \times \mathcal{F}^+$ measurable with $\mathcal{B}[0,\infty]$ being the Borel σ-algebra on $[0,\infty]$

More succinctly, we can write the first equation in the definition as

$$dI_t = u\,dt + v\,dB_t.$$

Now we will move on to show Itô's formula which we will prove in one dimension and then state in d dimensions (but won't prove because we don't use any new techniques out of the proof in d dimensions.)

Theorem 3.3. *Let I_t be an Itô process, let $g(t,I) \in C^2(\mathbb{R}^+ \times \mathbb{R})$ (twice differentiable over $\mathbb{R}^+ \times \mathbb{R}$). Then $J_t = g(t,I_t)$ will be another Itô process with*

$$dJ_t = \frac{\partial g}{dt}(t,I_t)\,dt + \frac{\partial g}{dx}(t,I_t)\,dI_t + \frac{1}{2}\frac{\partial^2 g}{dx^2}(t,I_t)(\,dI_t)^2.$$

(Here, $(dI_t)^2 = dI_t \cdot dI_t$ is computed by $dt\,dt = 0$, $dt\,dB_t = dB_t\,dt = 0$ and $dB_t\,dB_t = dt$.)

Proof. First we have to show that J_t is an Itô process. Expanding $(dI_t)^2$ we get

$$(dI_t)^2 = (u\,dt + v\,dB_t)^2 = v^2\,dt$$

by the above identities. Then through substitution we get

$$\begin{aligned} dJ_t &= \frac{\partial g}{dt}(t,I_t)\,dt + \frac{\partial g}{dx}(t,I_t)\,dI_t + \frac{1}{2}\frac{\partial^2 g}{dx^2}(t,I_t)(\,dI_t)^2 \\ &= \frac{\partial g}{dt}(t,I_t)\,dt + \frac{\partial g}{dx}(t,I_t)(u\,dt + v\,dB_t) + \frac{1}{2}\frac{\partial^2 g}{dx^2}(t,I_t)v^2\,dt \\ &= \left(\frac{\partial g}{dt}(t,I_t) + \frac{\partial g}{dx}(t,I_t)u + \frac{1}{2}\frac{\partial^2 g}{dx^2}(t,I_t)v^2 \right) dt + \frac{\partial g}{dx}(t,I_t)v\,dB_t. \end{aligned}$$

Now writing out the shorthand, we get

$$g(t,I_t) = g(0,I_0) + \int_0^t \left(\frac{\partial g}{ds}(s,I_s) + \frac{\partial g}{dx}(s,I_s)u + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(s,I_s)v^2 \right) ds + \int_0^t \frac{\partial g}{dx} v\,dB_s$$

This is the expression of the form required by Itô's formula; however, we still have to show that the first expression was correct. We do this by approximation and assuming that g and both its first partials and second partial and mixed partials are bounded. Then we can use Taylor's theorem on multivariable functions to see

$$g(t, I_t) = g(0, I_0) + \sum_j \Delta g(t_j, I_j) = g(0, I_0) + \sum_j \frac{\partial g}{\partial t}\Delta t_j$$

$$+ \sum_j \frac{\partial g}{\partial x}\Delta I_j + \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial t^2}(\Delta t_j)^2 + \frac{1}{2}\sum_j \frac{\partial^2}{\partial t \partial x}(\Delta t_j)(\Delta I_j) + \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta I_j)^2 + \sum_j R_j.$$

Here, R_j is the remainder term and $R_j = o(|\Delta t_j|^2 + |\Delta I_j|^2)$. Now we take the limit of Δt_j as it goes to 0. Now, the first sum goes to:

$$\sum_j \frac{\partial g}{\partial t}(t_j, I_j)\Delta t_j \to \int_0^t \frac{\partial g}{\partial t}(s, I_s)\, ds = \frac{\partial g}{\partial t}\, dt.$$

The second sum becomes:

$$\sum_j \frac{\partial g}{\partial x}(t_j, I_,)\Delta I_j \to \int_0^t \frac{\partial g}{\partial x}(x, I_s)\, dI_s = \frac{\partial g}{\partial x}\, dI_t.$$

The third sum becomes:

$$\sum_j \frac{\partial^2 g}{\partial t^2}(t_j, I_j)(\Delta t_j)^2 \to \Delta t_j \int_0^t \frac{\partial^2 g}{\partial t^2}(s, I_s)\, ds \to 0.$$

The fourth sum becomes:

$$\sum_j \frac{\partial^2 g}{\partial t \partial x}(t_j, I_j)(\Delta t_j)(\Delta I_j) \to \Delta t_j \int_0^t \frac{\partial^2 g}{\partial t \partial x}(s, I_s)\, dI_s \to 0.$$

It remains for us to show that the fifth sum converges to

$$\frac{1}{2}\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta I_j)^2 \to \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial x^2}(s, I_s)(\, dI_s)^2$$

because it is easy to see from above that

$$\frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial x^2}(s, I_s)(\, dI_s)^2 = \frac{1}{2}\frac{\partial^2 g}{dx^2}(s, I_s)v^2\, ds.$$

Now when we expand the sum we get that

$$\frac{1}{2}\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta I_j)^2 = \frac{1}{2}\left(\sum_j \frac{\partial^2 g}{\partial x^2}u_j^2(\Delta t_j)^2 + \sum_j \frac{\partial^2 g}{\partial x^2}u_j v_j(\Delta t_j)(\Delta B_j) + \sum_j \frac{\partial^2 g}{\partial x^2}v_j^2(\Delta B_j)^2\right).$$

From the previous sums converging we see that the first two sums converge to 0 so all we have to do is show

$$\sum_j \frac{\partial^2 g}{\partial x^2}v_j^2(\Delta B_j)^2 \to \int_0^t \frac{\partial^2 g}{\partial x^2}v^2\, dB_s.$$

and then it is obvious that the remainder term goes to 0 as $\Delta t_j \to 0$ and then we're done. The proof of

$$\sum_j \frac{\partial^2 g}{\partial x^2}v_j^2(\Delta B_j)^2 \to \int_0^t \frac{\partial^2 g}{\partial x^2}v^2\, dB_s$$

is a bit tedious so we leave that up to the reader which can be found in [1]. ∎

Now, we will state the theorem for the multidimensional Itô formula.

Definition 3.4. A d-dimensional Itô process I_t is a d-dimensional stochastic process given by $dI_t(\omega) = u(t,\omega)\,dt + v(t,\omega)\,dB_t$, (which is the analogous shorthand from the 1 dimensional case). Here, we let u be a d dimensional vector and B_t be a d dimensional Brownian motion and v is a $d \times d$ matrix where each u_i, v_j satisfies the requirements of a 1-dimensional Itô process. Writing it out in long form we get that a d-dimensional Itô process takes the form:

$$d\begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_d \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \end{pmatrix} dt + \begin{pmatrix} v_{1,1} & \cdots & \cdots & v_{1,d} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ v_{d,1} & \cdots & \cdots & v_{d,d} \end{pmatrix} d\begin{pmatrix} B_1(t) \\ B_2(t) \\ \vdots B_d(t) \end{pmatrix}$$

Now we can discuss the Multidimensional Itô formula.

Theorem 3.5. *Let $dI_t = u\,dt + v\,dB_t$ be a d-dimensional Itô process. Let $f(t,\omega) : [0,\infty] \times \mathbb{R}^D \to \mathbb{R}^p$ be C^2 (twice differentiable) such that $f(t,x) = (f_1(t,x), \ldots, f_p(t,x))$. Then $J(t,\omega) = f(t, I(t,\omega))$ is a p-dimensional Itô process where for $1 \le k \le p$, the k^{th} component of $J(t,\omega)$ will be given by*

$$dJ_k = \frac{\partial f_k}{\partial t}\,dt + \sum_{i=1}^{d} \frac{\partial f_k}{\partial x_i}\,dX_i + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f_k}{\partial x_j \partial x_j}\,dX_i\,dX_j$$

and again $dt\,dB_i = dB_j\,dt = 0$ and $dB_i\,dB_j = dt$.

3.3. **Some more background.** In this section we'll introduce some more theorems relating to stochastic calculus which will be useful later on.

Proposition 3.6. *For every continuous local martingale, there exists a unique continuous adapted nondecreasing process, denoted $[M]_t$ and called the quadratic variation $(M)_{t\ge 0}$, for which $(M^2 - [M]_t)_{t\ge 0}$ is a continuous local martingale.*

Proposition 3.7. *The quadratic variation of a Brownian motion $(B_t)_{t\ge 0}$ is t.*

Proposition 3.8. *Let $0 < a < b$, let B be a Brownian motion started at a, and let $\tau_r = \inf\{t \ge 0 : B_t = r\}$. Then*

$$\mathbb{P}(\tau_0 > \tau_b) = \frac{a}{b}.$$

Lemma 3.9. *Let $f : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ be continuously differentiable in the first coordinate and twice continuously differentiable in the second coordinate. Suppose that there exists $K > 0$ for which*

$$|f(t,x)| + \left|\frac{\partial f}{\partial t}(t,x)\right| + \sum_{i=1}^{d}\left|\frac{\partial f}{\partial x_i}(t,x)\right| + \sum_{i,j=1}^{d}\left|\frac{\partial^2 f}{\partial x_i \partial x_j}(t,x)\right| \le Ke^{K(t+|x|)},$$

for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. Then

$$C_t = f(t,B_t) - f(0,B_0) = \int_0^t \left(\frac{\partial}{\partial t} + \frac{1}{2}\Delta\right) f(s,B_s)\,ds$$

is a martingale and Δ is the Laplacian operator and if f is harmonic then $f(t,B_t)$ will be a martingale.

Lemma 3.10. *(Dubins Schwarz) Let M be a continuous local martingale for $M_0 = 0$ almost surely and $\lim_{t\to\infty}[M]_t = \infty$ almost surely. Let $\sigma(t) = \inf\{s : [M]_s > t\}$. Then for all $t \geq 0, \sigma(t)$ is an $(\mathcal{F}_s)_{s\geq 0}$ stopping time, $(\mathcal{F}_{\sigma(t)})_{t\geq 0}$ is a filtration and $M_{\sigma(t)}$ is a Brownian motion adapted to $(\mathcal{F}_{\sigma(t)})_{t\geq 0}$.*

3.4. **Transition.** Hooray! We've officially established and proved the main ideas in stochastic calculus and Brownian motion. Brownian motion is quite amazing and can be thought of and applied in so many different ways. Now one of the most amazing ways in which Brownian motion can be applied is to complex analysis. Now, we're going to use Brownian motion to prove some interesting results in complex analysis which most people have seen in a complex analysis class.

4. Brownian Motion in Complex Analysis

Here are a couple rules in complex analysis we want to recall.

4.1. Conformal Invariance of Brownian motion.

Theorem 4.1. *(Conformal invariance of Brownian motion) Let D be a domain and let $(B_t)_{t\geq 0}$ be a Brownian motion started at $z \in D$. If $f : D \to \mathbb{C}$ is analytic, then there exists a Brownian motion $\tilde{B}_t$ in $f(D)$ started at $f(z)$ for which $f(B_t) = \tilde{B}_{\int_0^t |f'(B_s)|^2\, ds}$.*

Proof. Let $z = x_i y$ and $f(z) = u(x, y) + iv(x, y)$ and $B_t = X_t + iY_t$ where X_t and Y_t are independent scalar Brownian motions. Recall the Cauchy-Riemann equations state that with respect to f,

$$u_x = v_y, u_y = -v_x.$$

We know also that u and v are also harmonic functions and therefor we can use Itô's formula to show

$$du(X_t, Y_t) = \frac{\partial u}{\partial x}(X_t, Y_t)\, dX_t + \frac{\partial u}{\partial y}(X_t, Y_t)\, dY_t$$

$$+\frac{1}{2}\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) + \frac{1}{2}\frac{\partial^2 u}{\partial x \partial y}(X_t, Y_t)\, d[X, Y]_t$$

$$= \frac{\partial u}{\partial x}(X_t, Y_t)\, dX_t + \frac{\partial u}{\partial y}(X_t, Y_t)\, dY_t.$$

Similarly,

$$dv(X_t, Y_t) = \frac{\partial v}{\partial x}(X_t, Y_t)\, dX_t + \frac{\partial v}{\partial y}(X_t, Y_t)\, dY_t.$$

From these expressions we get that we can compute the quadratic variation to get

$$[u(B)]_t = [v(B)]_t = \int_0^t |f'(B_t)|^2\, ds$$

and the covariation is 0 between $u(B)$ and $v(B)$. Letting $\sigma(t) = \inf\{s \geq 0 : \int_0^s |f'(B_u)|^2\, du > t\}$ and define $\tilde{B}_t = f(B_{\sigma(t)}) = u(B_{\sigma(t)}) + iv(B_{\sigma(t)})$. Then by Dubins Schwarz $\tilde{B}_t$ is a Brownian motion with respect to the filtration $\mathcal{F}_{\sigma(t)}$. ∎

4.2. **Maximum Modulus Principle.**

Theorem 4.2. *(Maximum Modulus Principle) If $U \subset \mathbb{C}$ is a domain and $f : U \to \mathbb{C}$ is a nonconstant analytic function, then $|f|$ has no local maxima in U.*

Lemma 4.3. *If h is harmonic on D, that is $h_{xx} + h_{yy} = 0$, where $D = \{w \in \mathbb{C} : |w - z| < R\}$ then for all $r < R$, we have that*

$$h(z_0) = \int_0^{2\pi} h(z + re^{i\theta})\, d\theta.$$

Proof. Let τ be the exit time of a planar Brownian motion starting at z_0 from the disk of radius r centered at z_0. Then, by Lemma 3.3.1, $(h(B_t))_{t\geq 0}$ is a martingale. Thus applying the optional stopping theorem

$$h(z_0) = \mathbb{P}(h(B_\tau)) = \int_0^{2\pi} h(z + re^{i\theta})\, d\theta$$

which is what we sought out to show. ■

Now it's just a basic complex analysis technique to show that the Maximum Modulus Principle holds.

Proof. (Proof of the Maximum Modulus Principle) Suppose for the sake of contradiction that there exists $z_0 \in U$ and $\epsilon > 0$ for which $|f(z_0)| \geq |f(z_0 + re^{i\theta})|$ for all $0 < r < \epsilon$. By adding a suitable constant to f, we may assume that the image of $\{z : |z - z_0| \leq \epsilon\}$ under f is contained in the right half-plane. Therefore, $\log f$ is analytic on $\{z : |z - z_0| < \epsilon\}$, from which it follows that the real part of $\log|f|$ is harmonic. By the previous lemma, for any $0 < r < \epsilon$ we have that $\log|f(z_0)|$ is an average of the values of $\log|f(z_0 + re^{i\theta})|$. Therefore, $\log|f|$ is constant on $\Delta = \{z : |z - z_0| < \epsilon\}$. This implies that $|f|$ is constant on Δ which implies that f is constant on Δ by the Cauchy-Riemann equations. Since U is connected, f is constant on U. ■

4.3. **Fundamental Theorem of Algebra.**

Theorem 4.4. *(Fundamental Theorem of Algebra) If p is nonconstant polynomial, then there exists $z \in \mathbb{C}$ for which $p(z) = 0$.*

Proof. Suppose that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then $f = \frac{1}{p}$ is an analytic function on $\mathbb{C}$, and since $p \to \infty$ as $z \to \infty$, f is bounded. Let B_t be a Brownian motion started at the origin. By Lemma 3.3.1 we get that $\Re f(B_t)$ is a martingale. Since $\Re f$ is bounded, the martingale convergence theorem implies that $\Re f(B_t)$ converges almost surely as $t \to \infty$. On the other hand, $\Re f(\mathbb{C})$ contains more than one element, since f is nonconstant. Choose $\alpha < \beta$ so that $\inf \Re f(\mathbb{C}) < \alpha < \beta < \sup \Re f(\mathbb{C})$. The sets $U_1 = \{z : \Re f(z) < \alpha\}$ and $U_2 = \{z : \Re f(z) < \beta\}$, are nonempty, disjoint open sets in $\mathbb{C}$. By the recurrence of of Brownian motion, the Brownian motion visits each of the sets U_1 and U_2 at arbitrarily large times, so $\liminf \Re f(B_t) < \alpha < \beta < \limsup \Re f(B_t)$, almost surely. This thus contradicts the convergence of $\Re f(B_t)$ thus showing that the fundamental theorem of algebra is true. ■

4.4. **Schwarz's Lemma.**

Theorem 4.5. *(Schwarz's Lemma) Let $\mathbb{D} = \{z : |z| < 1\}$ denote the unit disk. If $f : \mathbb{D} \to \mathbb{D}$ is an analytic function with $f(0) = 0$, then $|f(z)| \leq |z|$ for all z. Moreover, if there exists $z \neq 0$ for which $|f(z)| = |z|$, then there exists $\theta \in \mathbb{R}$ for which $f(z) = e^{i\theta} z$ for all $z \in \mathbb{D}$.*

Proof. Let $z_0 \in \mathbb{D}$ and choose $0 < r < 1$ so that z_0 is contained in the open disk centered at the origin with radius f. Let S denote the hitting time of the circles of radius r. The function $g(z) = \frac{f(z)}{z}$ is continuous at the origin since $\lim_{z\to 0} g(z) = f'(z)$. Thus $g(z)$ is analytic in $\mathbb{D}$. Now, let B_t be a Brownian motion started at z_0, and then applying the optional stopping theorem to $g(B_t)$ to find

$$g(z_0) = \mathbb{E}[g(B_s)].$$

Then, since $|f(z)| \leq 1$ for all $z \in \mathbb{D}$, we have $|g(B_s)| = \frac{|f(B_s)|}{|B_s|} \leq \frac{1}{r}$. Letting $r \to 1$ gives $|g(z_0)| \leq 1$. Moreover, if $|g(z_0)| = 1$, then z_0 is a local maximum for g. By the above theorem, this implies that g is constant and $|g| = 1$. Therefore, there exists $\theta \in \mathbb{R}$ for which $g(z) = e^{i\theta}$, so $f(z) = ze^{i\theta}$. ■

4.5. **Liouville's Theorem.** Now we will show Liouville's theorem where we will use the conformal invariance of Brownian motion.

Theorem 4.6. *If $f : \mathbb{C} \to \mathbb{C}$ is a bounded analytic function, then f is constant.*

Proof. Let B_t be a Brownian motion in $\mathbb{C}$ started at the origin. By the previous theorem, $f(B_t)$ is a time change of Brownian motion (which is a change of variable). Since Brownian motion visits every neighborhood of ∞, the boundedness of f requires that the time change does not go to ∞. That is almost surely

$$\int_0^\infty |f'(B_s)|^2 \, ds < \infty.$$

We claim that this holds only if f is constant. For if f is nonconstant, then we may choose a disk $D \in \mathbb{C}$ such that $\overline{D}$ (it's closure) contains no zeros of f. By the open mapping theorem (holomorphic functions maps open sets to open sets), there exists $\delta > 0$ so that $\underline{f(z)} \geq \delta$ for all $z \in D$. Define S_n and T_n to be the n^{th} entrance and exit times, respectively, of $\overline{D}$. Then S_n and T_b are almost surely finite for all n. By the Markov Property of Brownian Motion, the variables $T_n - S_n$ are independent and identically distributed. Also, each has finite expectation and the strong law of large numbers show that almost surely

$$\int_0^\infty |f'(B_s)|^2 \, ds \geq \sum_{n=1}^{\infty} \delta^2 (T_n - S_n) = \infty.$$

■

5. Conclusion

This paper discussed a large variety of topics and showed the way in which we could interpret Complex Analysis through the lens of stochastic processes. We first went and showed some fundamental measure theory. We then went and explored some ideas in probability theory and stochastic processes with the idea of Martingales and Brownian Motion. After developing that theory, we went on to discuss stochastic Calculus and wound up discussing Itô's formula and Itô Calculus which was imperative for our discussion of Complex Analysis with Brownian Motion. Overall, using our background, we proved three profound theorems in Complex Analysis.

6. Bibliography

References

[1] Watson, Sam. *"Brownian motion, complex analysis, and the dimension of the Brownian frontier"* Trinity College, Cambridge University. 30, April, 2010. http://math.mit.edu/ sswatson/pdfs/partiiiessay.pdf

[2] Shalizi, Cosma. *"stochastic Differential Equations"* Carnegie Mellon University. https://www.stat.cmu.edu/ cshalizi/754/2006/notes/lecture-19.pdf

[3] Hui, Chen. Teo, George. *"Brownian Motion and Liouville's Theorem"* University of Chicago. 2012. http://math.uchicago.edu/ may/REU2012/REUPapers/Teo.pdf

[4] Zhang, Yuxuan. *"Complex Analysis and Brownian Motion"* Washington University. 5, June, 2013. https://sites.math.washington.edu/ morrow/33613/papers/yuxuan.pdf

[5] Jani, Kishan. *"Brownian Motion"* Euler Circle. 2020. http://simonrs.com/eulercircle/markovchains/kishan-brownian.pdf

[6] Ellis, Mehana. *"Brownian Motion"* Euler Circle. 2020. http://simonrs.com/eulercircle/markovchains/mehana-brownian.pdf

[7] *"Conditional expectations, filtration and martingales"* Massachusetts Institute of Technology. 2, October, 2013. https://ocw.mit.edu/courses/sloan-school-of-management/15-070j-advanced-stochastic-processes-fall-2013/lecture-notes/MIT15070JF13Lec9.pdf

Email address: `jzeitlin36@gmail.com`