JULIA AND FATOU SETS

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1. Abstract

We will state some definitions and then some examples of Julia and Fatou Sets.Julia sets are points in the domain of a rational function where a small perturbation can cause drastic change in the set of iterated values. The Fatou sets are the complement. They are points where a small perturbation causes a small change in the set of iterated values in the function.Then we will prove three theorems about Fatou and Julia Sets. The first one states that if U is a Fatou component and F is a rational map from C to C with degree ≥ 2 then $F(u)$ is also a Fatou component. The second theorem states that The Fatou set of F contains at mos t two simply connected completely invariant components.The last one states that the Fatou set of rational maps has either 0,1,2 or infinitely many parts. We will need to use Montel's Theorem and the Riemann Mapping Theorem to prove that theorem. Finally, we will state but won't prove the No Wandering Domain Theorem.

2. Definitions

Fatou components are defined for rational maps, which are

Definition 2.1. A *rational map* is of the form $f(z) = \frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials with complex coefficients. We may assume without loss of generality that these polynomials are coprime, which means that they have no common factors. The degree of $f(z)$ is defined as

$$d = \deg(f) = \max(\deg(p), \deg(q))$$

. One useful property about degrees is the fact that

$$\deg(f * g) = \deg(f) * \deg(g) \implies \deg(f^n) = (deg(f))^n$$

Definition 2.2. A *Normal family* is a family of pre-compact set(every sequence of functions contains a sequence that converging uniformly on all compact subsets) of the space of continuous functions. Informally this means that the functions don't spread out but stick together in a clustered manner.

Complex dynamics is the study of dynamical systems defined by iteration of functions on the complex number system. The next few definitions discuss the Fatou sets and Fatou components.

Definition 2.3. The *Fatou set* $F(f)$ of f is the set of points $z \subset C$ such that the family F^n is normal in some neighborhood of z_0

Now that we have defined a Fatou component, we will state some definitions that describe the different ways these can behave.

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Definition 2.4. Definitions for Fatou Components

(1) It is a *Periodic Fatou Component* of period ≥ 1 if $f^p(u) = u$ and $f^n(u)! = u$ for all $n < p$
(2) If $f^m(u) = u$ is a *periodic Fatou component* for some $m >= 1$, that is eventually periodic Fatou components then it is a *Preperiodic Fatou Component*.
(3) It is a Fixed Fatou component if $F(u) = u$, that is a *periodic fatou component* of period 1.
(4) It is a *Wandering Domain* if all $f^n(u)$ are distinct

Definition 2.5. .Suppose f is a rational map and that z_0 is a periodic point with multiplier λ. The *multiplier* of f at a fixed point is the complex value $\lambda = f'(z_0)$.

(1) If $\lambda < 1$, then z_0 is said to be attracting. If $\lambda = 0$, then z_0 is superattracting.
(2) If $0 < \lambda < 1$, then z_0 is geometrically attracting.
(3) If $|\lambda| \leq 1$, then z_0 is repelling.
(4) If $\lambda^n \neq 1$ for some n, and f is not the identity, then z_0 is parabolic.
(5) If $|\lambda| < 1$ and $\lambda^n \neq 1$ for any n, then z_0 is indifferent.

Definition 2.6. A point $z_0 \subset C$ is said to be a *periodic point* of F if there exists some k for which z_0 is a fixed point of f^k. The multiplier of such a periodic point of F is defined to be the multiplier of f^k at its fixed point z_0. If k is chosen to be minimal, then k is said to be the period of z_0

3. Theorems for later reference

Theorem 3.1. *The* Riemann Mapping Theorem *states that if U is a non-empty simply connected open subset of the complex number plane C which is not all of C, then there exists a biholomorphic function from u onto the open unit disk.[2]*

Theorem 3.2. *The* Arzeli-Ascoli Theorem *states, Let $\omega \subset C$ be a domain , and let F be a family of holomorphic functions from ω to C. The familly F is normal if and only if it is locally equicontinous on ω. [2]*

Theorem 3.3. *The* Riemann-Hurwitz Formula *states, let F be a proper(holomorphic) map of degree d of some n_1 connected domain D_1 onto some n_2-connected domain D_2. Suppose f has exactly n_c critical points in D_1 including/counting multiplicity. Then*

$$n_1 - 2 = d(n_2 - 2) + n_c.$$

This proof requires the use of Euler's Formula. [2]

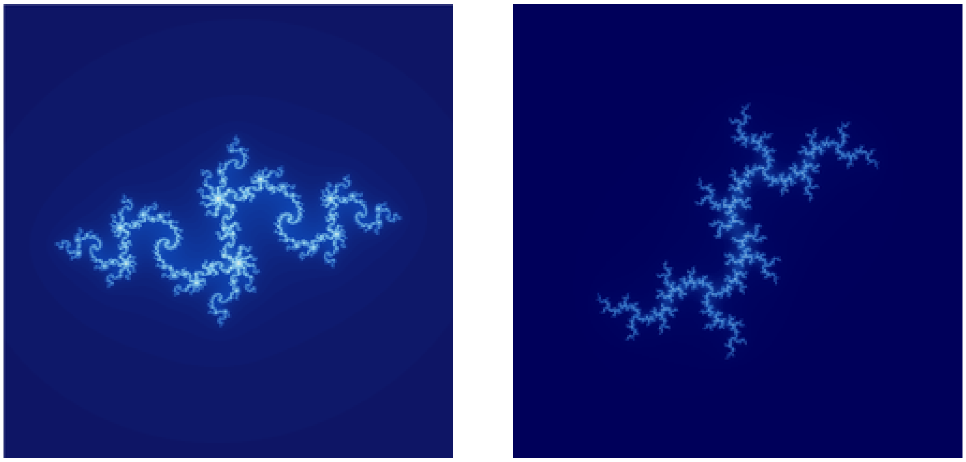
Theorem 3.4. *The Fatou and Julia sets are completely invariant under f . [2]*

The completely invariance of the Fatou set follows from the definition of it as an open set, the continuity of F and the fact that rational maps are open. The proof is not given here.

Theorem 3.5. *Montel's Theorem states, Let ω in C be a domain and let F be a family of holomorphic functions from ω to C. The family is f is equicontinous at $z + 0$ in ω if given any ϵ¿0 there exists δ s.t.*

$$d_c(v_0, v) < \delta \implies d_c(v_o, v) < \epsilon$$

for all z in ω and every f in F.[2]



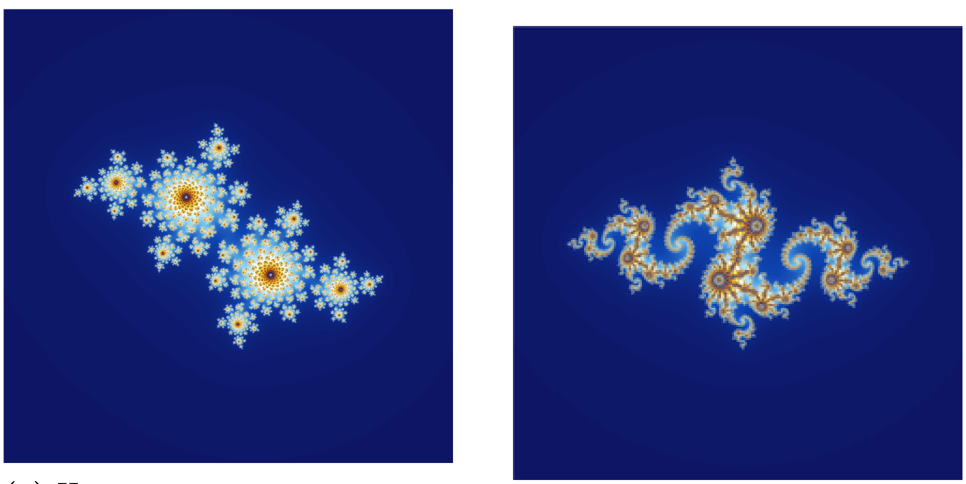
(a) Here $c = 0.8350.2321i$

(b) Here $c = -0.8i$

Figure 1. these are both showing the Julia sets for the rational map $x^2 + c$, for different values of c

4. Examples

Example. Example of a Julia set: For $f(z) = z^2$ the Julia set is the unit circle and on this the iteration is given by doubling the angle. There are two Fatou domains the interior and the exterior of the circle, with iteration towards 0 and ∞, respectively.



(a) Here $c = (\phi - 2) + (\phi - 1)i = 0.4 + 0.6i$

(b) Here $c = 0.8 + 0.156i$

Figure 2. these are both showing the Julia sets for the rational map $x^2 + c$, for different values of c

5. Theorems

In the following theorems, F is a rational map from C to C with degree ≥ 2

Proposition 5.1. *If U is a Fatou component, then $F(u)$ is also a Fatou component. [2]*

Proof. Suppose two points z_1, z_2 in a Fatou component U are mapped by f to $f(z_1)$ and $f(z_2)$ in two distinct Fatou components U_1 and u_2 Since U is connect there is a smooth path between z_1 and z_2 that is entirely contained in U. Then the image of this path should intersect the boundary of each component as the ration map f is continuous. But this is a contradiction since the path should lie in $F(f)$ but the boundary point is in $F(f)$ since Fatou components are connected open subsets. therefore we have $F(u)$ is contained in only one Fatou component U'
At this point suppose $F(u) \subset U'$. Let $W \subset U'$ be a point in the boundary of $F(u)$. Consider a sequence w_n of points in $F(u)$ converging to w... and the sequence v_n of the preimages of w_n in U. If v is any accumulation point of the sequence of v_n then $f(v) = w$. In that case, $v \subset \partial u \subset F(f)$ which contradicts with the fact that $w \subset U' \subset F(f)$. In conclusion, we have that $F(U) = U'$ is another fatou component.[2] ∎

Theorem 5.2. *Let F be rational map of degree greater than or equal to 2. The Fatou set of F contains at most two simply connected completely invariant components. [2]*

Proof. By the Riemann Mapping Theorem, such components are conformally equivalent to the unit disc D. Then the restriction of F to D is a d to 1 mapping. From the Riemann-Hurwitz Formula[2] we have that f has

$$n_c = d - 1$$

critical points in D, counting multiplicity. We will the fact that a common upper bound of $2d - 2$ critical points of rational maps, we may have at most two such Fatou components, i.e at msot two simply connected completely invariant components of $F(f)$.[2] ∎

Theorem 5.3. *The Fatou set of a rational map f has either 0, 1, 2 or infinitely many components.[2]*

Proof. For $d = 1$ where the function has degree 1 and the Fatou set has infinitely many components. If the degree is greater than or equal to 2. we have to use the previous proposition for the case that $F(f)$ has finitely many components $U_1, U_2, U_3, ...U_N$ Here F must act as a permutation of the components, so there is an integer m such that $g = f^m$ maps each U_i to itself. Since $F(f) = F(f^m)$ we have that each U_i is completely invariant for g. Moreover, since $F(f)$ is the smallest closed completely invariant closed set under g, we have $\partial U_i = F(f)$, so the sequence f^n omits the open set U_i on C_i. Then f^n is normal there by Montel's theorem and hence $C/U_i = F(f)$ Since U_i is connected each other component of C_i is simply connected. In a similar way U_i is also simply connected. Then using the proposition there are at most two components. [2] ∎

Theorem 5.4. *The No Wandering Domain Theorem states that every Fatou component of a rational map is eventually periodic. [2]*

The proof of this theorem is beyond the scope of this paper because it requires more advanced topics to prove

References

[1] Wikipedia contributors. (2021, April 11). Julia set. In Wikipedia, The Free Encyclopedia. Retrieved 23:57, April 12, 2021, from **Julia Set**

[2] Memoria.pdf **Dynamics of Rational Functions**

[3] dozier.pdf **Classification and Strucuture of Periodic Fatou Components**

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